Exercise sheet #5
Set Theory 2024

Definition 1.

1. For sets a and b, the expression $|a| \leq |b|$ means that there exists an injective function $f\colon a \to b$; $|a| = |b|$ means $|a| \leq |b| \wedge |b| \leq |a|$.

2. A set a is *finite* if there exists a natural number n such that $|a| = |n|$. If a is not finite, then we say that a is *infinite*.

3. A set a is *D-finite* if for all $b \in \mathcal{P}(a) \setminus \{a\}$ there is no injective mapping $a \to b$. If a is not D-finite, then we say that a is *D-infinite*.

4. A set a is *S-finite* if there exists a total order $\leq$ on a such that every nonempty $b \subseteq a$ has a $\leq$-least and a $\leq$-greatest element. If a is not S-finite, then we say that a is *S-infinite*.

5. A set a is *T-finite* if every nonempty $x \subset \mathcal{P}(a)$ has a $\subseteq$-maximal element. If a is not T-finite, then we say that a is *T-infinite*.

6. A set a is *P-finite* if every partial ordering of a contains a maximal element.

7. A set a is *L-finite* if every linear ordering of a contains a maximal element.

8. A set a is *Decomposition-finite* if it is empty, a singleton, or there exists a partition $a = a_1 \cup a_2$ where $|a_1|, |a_2| < |a|$.

Exercise 1. Let a be a set. Prove that the following are equivalent:

1. a is finite.

2. a is S-finite.

3. $\mathcal{P}(\mathcal{P}(a))$ is D-finite.

4. a is T-finite.

Exercise 2. Let a be a set such that for every set b, if there exists a surjective function $a \to b$, then b is P-finite. Show that if there exists an injective function $c \to a$, then c is P-finite, and that a itself is P-finite.

Exercise 3. Let x be an infinite set, and suppose that $(x, \leq)$ is a linear order. Then there exists $y \subseteq x$ such that either $(y, \leq)$ or $(y, \geq)$ has no maximal element.

Exercise 4. A set x is L-finite if and only if it is finite or not linearly orderable.

Exercise 5. A set x is P-finite if and only if for each partial order $\leq$ on x there exists a positive integer M such that every $\leq$-chain of X has length at most M.

Exercise 6. Every D-finite set is decomposition-finite.

Exercise 7. Every decomposition-finite set is either T-finite or not well-orderable.

Exercise 8. If s is D-infinite, then there exists $p \subset s$ such that $|p| = |\omega|$.

Exercise 9. If a is infinite, then $\mathcal{P}(\mathcal{P}(a))$ is D-infinite.

Exercise 10. If a is T-finite, then for every partition of a into two sets at least one part is T-finite.

Exercise 11. If a is T-finite, then $\mathcal{P}(a)$ is D-finite.