Exercise sheet #4
Set Theory 2024

Exercise 1. Suppose that $(A, \leq)$ is a partially ordered set. Show that there exists $S \subset \mathcal{P}(A)$ such that $(A, \leq \cap A \times A)$ is isomorphic to $(S, \subseteq)$.

Exercise 2. Show that a partial order in which every subset has a minimum element is a well-order.

Exercise 3. If $(P, \leq)$ is a well-ordered set and $f \colon P \to P$ is order-preserving, then $x \leq f(x)$ for all $x \in P$.

Exercise 4. A total order $\leq$ on a set X is a well-ordering if and only if it has no infinite descending chains.

Exercise 5. Suppose that $(P, \leq_P)$ is a totally ordered set. An ordered set $(S, \leq_S)$ is an *initial segment* of $(P, \leq_P)$ if S is a proper subset of P, $\leq_S = \leq_P \cap (S \times S)$, and for all $p \in P$ and $s \in S$, if $p \leq_P s$ then $p \in S$.

Show that if $(P, \leq)$ is a well-ordered set, then $(P, \leq)$ is not isomorphic to any of its initial segments.

Exercise 6. Let $(X, \leq)$ be a totally ordered set. We say that y is the successor of x (notation: $y = s(x)$) if $y > x$ and $z > x$ imply $z \geq y$. If y is not the successor of any element in X, then y is called a limit.

Let Λ denote the set of limits in X. Prove:

1. Any $x \in X$ is either a maximum or has a successor.
2. $X = \{y \in X : \exists n \exists \ell \in \Lambda (y = s^n(x))\}$
3. If $y \in \Lambda$ and $x < y$, then $s^n(x) < y$ for all n.
4. If X has a maximum element, then so does Λ, and the maximum element of X is some $s^n(\ell)$, where ℓ is the maximum element of Λ.

Exercise 7. Prove that multiplication is distributive over addition, that is, for all natural numbers k, m, n the following equality holds:

$$k \cdot (m + n) = (k \cdot m) + (k \cdot n)$$

Exercise 8. Prove that multiplication is associative: for all natural k, m, n, $(k \cdot m) \cdot n = k \cdot (m \cdot n)$.

Exercise 9. Prove that for all natural numbers m, n such that $m \in n + 1$, there exists a unique natural k such that $m + k = n$.