Exercise sheet #3
Set Theory 2024

Definition 1.

1. If a is a set, we define $s(a)$ as $a \cup \{a\}$.
2. A set a is said to be *inductive* if $\varnothing \in a$ and $\forall b \in a (s(b) \in a)$.
3. A set a is said to be *transitive* if $b \in c \in a$ implies $b \in a$.
4. The Axiom of Infinity is the statement "There exists an inductive set".

Exercise 1. Prove:

1. a set a is transitive if and only if $\bigcup a \subseteq a$.
2. if a and b are inductive sets, then $a \cap b$ is also inductive.
3. if t is a transitive set, then $\bigcup s(t) = t$.
4. if a is an inductive set, then $b \coloneqq \{c \in a : c \subseteq a\}$ is inductive.
5. if a is inductive, then $\{x \in a : x \text{ is transitive}\}$ is inductive.
6. if a is inductive, then $\{x \in a : x \text{ is transitive and } x \notin x\}$ is inductive.
7. if a is inductive then so is $\{x \in a : x = \varnothing \text{ or } x = s(y) \text{ for some } y\}$.

Exercise 2. Let ω denote the class of all sets x such that, for all inductive sets I, $x \in I$. Prove:

1. ω is a set.
2. ω is transitive and inductive.
3. $\in$ is a total order on ω.
4. There is no function $f\colon \omega \to \omega$ such that $f(s(a)) \in f(a)$ for all $a \in \omega$.
5. Let a be a subset of ω such that $\varnothing \in a$ and if $b \in a$ then $s(b) \in a$. Prove $a = \omega$.