Exercise sheet #2
Set Theory 2024

Exercise 1. If R, S are binary relations (in particular, sets) then dom R, ran R, $R \circ S$, R^{-1} are sets.

Exercise 2. Suppose that R is a binary relation and A, B are sets. Prove or give a counterexample:

1. $R[A \cup B] = R[A] \cup R[B]$
2. $R[A \cap B] = R[A] \cap R[B]$
3. $R[A \setminus B] = R[A] \setminus R[B]$
4. $\text{dom } R = \text{ran } R^{-1}$
5. $R^{-1} \circ R = \text{Id}_{\text{dom } R}$
6. $R \circ R^{-1} = \text{Id}_{\text{ran } R}$

Exercise 3. Let $f\colon X \to Y$ be a function.

1. Prove that f is injective if and only if there exists $g\colon \mathcal{R}_f \to X$ such that for all $x \in X$ we have $(x, x) \in g \circ f$ (g is called a *left inverse* of f).
2. (This is a trick question) Prove that f is surjective if and only if there exists $g\colon \mathcal{R}_f \to X$ such that for all $y \in Y$ we have $(y, y) \in f \circ g$ (g is called a *right inverse* of f).
3. Prove that f is injective and surjective (*bijective*) if and only if there exists $g\colon Y \to X$ which is a left and right inverse of f.

Exercise 4. Suppose that R is a reflexive and transitive binary relation on a set X. Define a relation E on X by $E(a, b)$ if and only if $R(a, b) \wedge R(b, a)$.

1. Prove that E is an equivalence relation.
2. Let R/E be the following relation on X/E: $R/E([a]_E, [b]_E) \leftrightarrow R(a, b)$.
    (a) Prove that the definition of R/E does not depend on the choice of representatives.
    (b) Prove that R/E is a partial order on X/E.

Exercise 5. Let X be a nonempty set. Show that any $S \subseteq P(X)$ has a supremum and an infimum in $(P(X), \subseteq_{P(X)})$.

Exercise 6. Let $\text{Fn}(X, Y)$ denote the set $\bigcup\{Y^Z : Z \subseteq X\}$, and define $\leq$ on $\text{Fn}(X, Y)$ by

$$f \leq g \leftrightarrow f \subseteq g.$$

Prove that $\leq$ is an ordering. Under what conditions does $\sup F$ exist for $F \subseteq \text{Fn}(X, Y)$?