Exercise sheet #8
Set Theory 2023

This exercise sheet is intended as preparation for the exam. I recommend solving it under exam conditions: without using your notes and with a stopwatch. You should be able to solve these problems in 2 hours.

Exercise 1. Let a be any set. For every $b \in \mathcal{P}(a)$ there exists a unique $c \in \mathcal{P}(a)$ such that $d \setminus b = d \cap c$ for all $d \in \mathcal{P}(a)$.

Exercise 2. Show that every countable linear order can be embedded into $(\mathbb{Q}, <)$.

Exercise 3.

1. Let $\varepsilon_0 = \lim_{n \to \omega} \alpha_n$ where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$ for all n. Show that ε_0 is the least ordinal ε such that $\omega^\varepsilon = \varepsilon$.
2. ε_0 is countable.
3. Find a subset of $\mathbb{Q}$ isomorphic to ε_0.

Exercise 4. Let $(A, <)$ be a linear order. For any $a \in A$, let I_a denote the set $\{x \in A : x < a\}$.

1. If $(A, <)$ is a well-ordered set and X is an initial segment of A, then there exists $a \in A$ such that $X = I_a$.
2. If $(A, <)$ is a complete linear order and X is an initial segment of A, then there exists $a \in A$ such that $X = I_a$ or $X = I_a \cup \{a\}$.

Exercise 5. Let α and β be ordinals. The following are equivalent:

1. β is isomorphic to an initial segment of α
2. β is an initial segment of α
3. $\beta \subset \alpha$
4. $\beta \in \alpha$

Exercise 6.

1. Every element of an ordinal is an ordinal.
2. Consider the following statement as an alternative definition for an ordinal: An ordinal is a transitive set of ordinals. Show that this definition is equivalent to the one given in class.