Exercise sheet #7
Set Theory 2023

In this exercise sheet, we use $\oplus$ and $\otimes$ to denote the sum and lexicographic product of orders. That is, if $(A, <_A)$ and $(B, <_B)$ are disjoint linear orders, then $A \oplus B$ denotes the pair $(A \cup B, <_+)$ and $A \otimes B$ denotes $(A \times B, <_\times)$, where

$$<_+ = \{(a,b) : a,b \in A \wedge a <_A b\} \cup \{(a,b) : a,b \in B \wedge a <_B b\} \cup \{(a,b) : a \in A \wedge b \in B\}, \text{ and}$$
$$<_\times = \{((a_1,b_1),(a_2,b_2)) : a_1 <_A a_2 \vee (a_1 = a_2 \wedge b_1 <_B b_2)\}$$

Exercise 1. Find two disjoint linear orders $(a, <_a)$ and $(b, <_b)$ such that $a \oplus b$ is not isomorphic to $b \oplus a$.

Exercise 2. Find two disjoint linear orders $(a, <_a)$ and $(b, <_b)$ such that $a \otimes b$ is not isomorphic to $b \otimes a$.

Exercise 3. Suppose that $(a, <_a)$ and $(b, <_b)$ are disjoint well-ordered sets. Show that $a \oplus b$ is well-ordered.

Exercise 4. Suppose that $(a, <_a)$ and $(b, <_b)$ are disjoint well-ordered sets. Show that $a \otimes b$ is well-ordered.

Exercise 5. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be partially ordered sets. A function $f \colon P \to Q$ is an *embedding of partial orders* if it satisfies $\forall x, y \in P(x \leq_P y \Leftrightarrow f(x) \leq_Q f(y))$. Show that an order embedding is always injective.

Exercise 6. Let $(P, \leq_P)$ be a partially ordered set. For each subset A of P, define

$$A^\uparrow := \{x \in P : \forall a \in A(a \leq_P x)\} \text{ and}$$
$$A^\downarrow := \{x \in P : \forall a \in A(x \leq_P a)\}$$

A *cut* of $(P \leq_P)$ is a pair $(A, B) \in \mathcal{P}(P)^2$ such that $A^\uparrow = B$ and $B^\downarrow = A$. Let $\mathcal{C}(P)$ denote the set of all cuts of P.

1. Show that $(\mathcal{C}(P), \leq)$ is a partially ordered set, where $(A_1, B_1) \leq (A_2, B_2)$ iff $A_1 \subseteq A_2$.
2. Show that if (A, B) is a cut, then $(A^\uparrow)^\downarrow = A$.
3. Show that if $(A^\uparrow)^\downarrow = A$, then $(A, A^\uparrow)$ is a cut.
4. Show that the function $\ell \colon P \to \mathcal{C}(P)$ given by $\ell(p) = (\{x \in P : x \leq p\}, \{x \in P : x \leq p\}^\uparrow)$ is an embedding.
5. Show that for all $Q \subseteq P$, the set $\ell[Q]$ has a supremum and an infimum in $(\mathcal{C}(P), \leq)$.