Exercise sheet #4
Set Theory 2023

Definition 1.

1. A function $f\colon X \to Y$ is a subset of $X \times Y$ such that for all $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in f$. A function $f\colon X \to Y$ is
   - *injective* if $\forall x, y \in X((x, a) \in f \wedge (y, a) \in f \to x = y$, and
   - *surjective* if $\forall y \in Y \exists x \in X((x, y) \in f)$.
2. A binary relation $\leq$ on a set X is a *partial order* on X if
   (a) $\forall a \in X((a, a) \in \leq)$,
   (b) $\forall a, b \in X(((a, b) \in \leq \wedge (b, a) \in \leq) \to a = b)$, and
   (c) $\forall a, b, c \in X(((a, b) \in \leq \wedge (b, c) \in \leq) \to (a, c) \in \leq)$.

   The notation $x \leq y$ means $(x, y) \in \leq$. The pair $(X, \leq)$ is calles a *partially-ordered set*.
3. A partial order $\leq$ is called a *total order* on X if $\forall a, b \in X((a, b) \in \leq \vee (b, a) \in \leq)$.
4. A total order on a set X is a *well-ordering* of X if $\forall S \subset X(S \neq \varnothing \to \exists m \in S \forall u \in S(m \leq u))$.
5. If $(P, \leq_P)$ and $(Q, \leq_Q)$ are partially ordered sets, then a function $f\colon P \to Q$ is *order-preserving* if $\forall p, p' \in P(p \leq_P p' \to f(p) \leq_Q f(p'))$. An injective order-preserving function is called an *order embedding* and a bijective order-preserving function is called an *order isomorphism*.

Exercise 1. Let $f\colon X \to Y$ be a function.

1. Prove that f is injective if and only if there exists $g\colon \mathcal{R}_f \to X$ such that for all $x \in X$ we have $(x, x) \in g \circ f$ (g is called a *left inverse* of f).
2. Assume that for every set of pairwise disjoint nonempty sets $\{U_i : i \in I\}$ there exists a set V such that for all $i \in I$, $|V \cap U_i| = 1$. Prove that, under this assumption, f is surjective if and only if there exists $g\colon \mathcal{R}_f \to X$ such that for all $y \in Y$ we have $(y, y) \in f \circ g$ (g is called a *right inverse* of f).
3. Prove that f is injective and surjective (*bijective*) if and only if there exists $g\colon Y \to X$ which is a left and right inverse of f.

Exercise 2.

1. Show that if $(P, \leq)$ is a partially ordered set, then $(P, \geq)$ is also a partially ordered set, where $\geq = \{(u, v) \in P \times P : (v, u) \in \leq\}$.
2. Suppose that $(P, \leq)$ is a partially ordered set, and define $> = \{(u, v) \in P \times P : (v, u) \in \leq\} \setminus \{(u, u) : u \in P\}$. Show that $>$ is the complement of $\leq$ in $P \times P$ if and only if $\leq$ is a total order on P.
3. Show that $\subseteq$ is a partial order on $\mathcal{P}(X)$ for any set X.

Exercise 3. Suppose that $(A, \leq)$ is a partially ordered set. Show that there exists $S \subset \mathcal{P}(A)$ such that $(A, \leq \cap A \times A)$ is isomorphic to $(S, \subseteq)$.

Exercise 4. If $(P, \leq)$ is a well-ordered set and $f\colon P \to P$ is order-preserving, then $x \leq f(x)$ for all $x \in P$.

Exercise 5. Suppose that $(P, \leq_P)$ is a totally ordered set. An ordered set $(S, \leq_S)$ is an *initial segment* of $(P, \leq_P)$ if S is a proper subset of P, $\leq_S = \leq_P \cap (S \times S)$, and for all $p \in P$ and $s \in S$, if $p \leq_P s$ then $p \in S$.

Show that if $(P, \leq)$ is a well-ordered set, then $(P, \leq)$ is not isomorphic to any of its initial segments.