Exercise sheet #2
Set Theory 2023

This exercise sheet will be marked. Please hand in your solutions by Friday March 3 2023 at 4pm CET. You can send your solutions by email to aranda@kam.mff.cuni.cz, hand them in personally at the next exercise session (7 March 2022, 10:40 to 12:10 in S10), or leave them on my desk (S324, first desk on the right).

Exercise 1. Find sets a, b, c that satisfy each list of requirements.

1. (5 points) $a \subseteq b$, $a \in c$, $c \subseteq b$, $c \neq b$.
2. (5 points) $a \in b, b \in c, a \notin c$.
3. (5 points) $a \in b, b \subseteq c, a \nsubseteq c$.
4. (5 points) $a \cap b \subseteq c, a \nsubseteq c, b \nsubseteq c$.

In the exercises below, (a, b) denotes $\{\{a\}, \{a, b\}\}$.

Exercise 2. (10 points) Suppose that a and b are sets. Prove that $a \times b \coloneqq \{(u, v) : u \in a \wedge v \in b\}$ is a set.

Exercise 3. This exercise uses the notation from Exercise 2. Supposing that a, b, c, d are sets, prove or give a counterexample (5 points each):

1. $\varnothing \times a = a \times \varnothing = \varnothing$.
2. $a \times b = b \times a$.
3. $a \times b \subseteq c \times d$ if and only if $a \subseteq c$ an $b \subseteq d$.
4. $a \times (b \cap c) = (a \times b) \cap (a \times c)$.
5. $a \times (b \cup c) = (a \times b) \cup (a \times c)$.
6. $(a \times b) \cup (c \times d) = (a \cup c) \times (b \cup d)$.

Exercise 4. Given sets a, b, c, define (a, b, c) as $((a, b), c)$.

1. (5 points) Show that (a, b, c) is a set.
2. (5 points) Find an example where $(a, b, c) \neq (a, (b, c))$.
3. (5 points) Show that $(a, b, c) = (a', b', c')$ if and only if $a = a', b = b', c = c'$.

Exercise 5. A *binary relation* on a set X is a subset of $X \times X \coloneqq \{(x, y) : x \in X \wedge y \in X\}$. Let R be a binary relation on X, and define

$$\mathcal{D}_R \coloneqq \{x : \exists y((x, y) \in R)\}$$
$$\mathcal{R}_R \coloneqq \{y : \exists x((x, y) \in R)\}$$

1. (5 points) Prove that $\mathcal{D}_R$ and $\mathcal{R}_R$ are sets.
2. (10 points) Prove that the collection $\{(a, b) : a, b \text{ are sets and } a \in b\}$ is not a set.