Exercise sheet #8
Set Theory 2022

Definition 1.

1. For any $\alpha, \varepsilon, \nu \in \mathrm{ORD}$ with $\nu < \alpha$, $\alpha^\varepsilon \cdot \nu$ is a *monomial in* α.
2. For any $n \in \omega$ and ordinals $\varepsilon_{n-1} > \varepsilon_{n-2} > \ldots > \varepsilon_0$ and $\nu_0, \ldots, \nu_{n-1} < \alpha$, $\sum_{i=n-1}^{0} \alpha^{\varepsilon_i} \cdot \nu_i$ is a *polynomial* in α. If all the ν_i are greater than 0, then $\sum_{i=n-1}^{0} \alpha^{\varepsilon_i} \cdot \nu_i$ is a *proper polynomial* in α.

Exercise 1. Let $n > 0$ be a natural number and suppose that $p \coloneqq \sum_{i=n-1}^{0} \alpha^{\varepsilon_i} \cdot \nu_i$ is a polynomial in α. Prove $\alpha^{\varepsilon_{n-1}+1} > p$.

Exercise 2. Let $n > 0$ be a natural number and suppose that $p \coloneqq \sum_{i=n-1}^{0} \alpha^{\varepsilon_i} \cdot \nu_i$ is a polynomial in α. Prove $\alpha^{\varepsilon_{n-1}} \cdot (\nu_{n-1} + 1) > p$.

Exercise 3. Suppose that M is a set of monomials in α. Prove that $\sup M$ is a monomial in α.

Exercise 4. For all $\alpha, \beta \in \mathrm{ORD}$ with $\alpha \geq 2$, there exist $n \in \omega$, a decreasing sequence $\varepsilon_{n-1} > \ldots > \varepsilon_0$ and $\nu_0, \ldots, \nu_{n-1} < \alpha$ such that $\beta = \sum_{i=n-1}^{0} \alpha^{\varepsilon_i} \cdot \nu_i$.
Hint: Prove it by induction on β. Identify the largest monomial $m = \alpha^\varepsilon \cdot \nu \leq \beta$. If $m < \beta$, then there exists $\gamma \in \mathrm{ORD}$ such that $m + \gamma = \beta$.

Exercise 5. If $p = \sum_{i=n-1}^{0} \alpha^{\varepsilon_i} \cdot \nu_i$ and $q = \sum_{i=m-1}^{0} \alpha^{\xi_i} \cdot \mu_i$ are two proper polynomial representations of β, then $m = n$ and for all $i < m$ $\varepsilon_i = \xi_i$, $\nu_i = \mu_i$.

Definition 2. An ordinal α is a *gamma number* if $\beta + \gamma < \alpha$ for all $\beta, \gamma < \alpha$.

Exercise 6. Let α be an ordinal. The following are equivalent:

1. α is a gamma number.
2. $\beta + \alpha = \alpha$ for all $\beta < \alpha$.
3. $\alpha = 0$ or there exists $\varepsilon \in \mathrm{ORD}$ such that $\alpha = \omega^\varepsilon$.