Exercise sheet #7
Set Theory 2022

Definition 1. A binary relation $R \subseteq X \times X$ is

1. *total* if $\forall a \in X \exists b \in X (R(a, b))$ and
2. *well-founded* if $\forall S \subset X (S \neq \varnothing \Rightarrow \exists m \in S \forall s \in S (\neg R(s, m))$.

The following statement is known as the Axiom of Dependent Choice (DC): For every nonempty X and every total binary relation $R \subset X \times X$ there exists a sequence of elements of X $(x_n)_{n\in\omega}$ such that for all $n \in \omega$ $R(x_n, x_{n1})$.

Exercise 1. Use DC to prove that a binary relation $R \subseteq X \times X$ is well-founded if and only if there is no infinite sequence $(x_n)_{n\in\omega}$ such that $R(x_{n+1}, x_n)$ for all $n \in \omega$.

Exercise 2. In a well-founded partial order, every nonempty subset has a minimal element.

Exercise 3.

1. If $(A, <)$ is a well-ordered set, then all its proper nonempty initial segments are of the form $I_a = \{x \in A : x < a\}$.
2. If $(A, <)$ is a complete linear order, then its proper nonempty initial segments are of the form I_a (as above) or $I_a \cup \{a\}$.

Exercise 4. Every element of an ordinal is an ordinal.

Exercise 5. Let α be an ordinal. Then $c \in \alpha$ if and only if c is an initial segment of α.

Exercise 6. For all $\alpha, \beta \in \text{ORD}$, $\beta \subset \alpha$ if and only if $\beta \in \alpha$.

Exercise 7. Let α, β be ordinals and $\delta = \alpha \cap \beta$. Then $\delta = \alpha$ or $\delta = \beta$.

Exercise 8. In view of the exercises above, is ORD a set?