Exercise sheet #3
Set Theory 2022

This exercise sheet will be marked. Please hand in your solutions by Friday 11 March 2022 at 2pm CET. You can send your solutions by email to aranda@kam.mff.cuni.cz, hand them in personally at the next exercise session (11 March 2022, 13:10 to 14:40 in S10), or leave them on my desk (S324, first desk on the right).

In this Exercise Sheet, (a, b) denotes the set $\{\{a\}, \{a, b\}\}$.

Exercise 1. (5 points) Let a and b be sets. Prove that $(a, b) \in \mathcal{P}(\mathcal{P}(\{a, b\}))$. More generally, prove that if $a \in A$ and $b \in A$ then $(a, b) \in \mathcal{P}(\mathcal{P}(A))$.

Exercise 2. A *binary relation* on a set X is a subset of $X \times X \coloneqq \{(x, y) : x \in X \wedge y \in X\}$. Let R be a binary relation on X, and define

$$\mathcal{D}_R \coloneqq \{x : \exists y((x, y) \in R)\}$$
$$\mathcal{R}_R \coloneqq \{y : \exists x((x, y) \in R)\}$$

1. (5 points) Prove that $\mathcal{D}_R$ and $\mathcal{R}_R$ are sets.
2. (10 points) Prove that the collection $\{(a, b) : a, b \text{ are sets and } a \in b\}$ is not a set.

Exercise 3. A set X is *transitive* if $a \in b \in X$ implies $a \in X$.

1. (10 points) Prove that X is transitive if and only if for all $x \in X$ we have $x \subseteq X$.
2. (10 points) Prove that if X is a transitive set, then so is $\mathcal{P}(X)$.

Exercise 4. If x is a set, we define $s(x)$ as $x \cup \{x\}$. A set I is *inductive* if $\varnothing \in I$ and whenever $c \in I$ then $s(c) \in I$. (One version of) The Axiom of Infinity simply states "There exists an inductive set".

1. (5 points) Prove that whenever x is a set, then so is $s(x)$.
2. (5 points) Define $\omega \coloneqq \{x : x \in I \text{ for all inductive sets } I\}$. Is ω a set? Prove it.
3. (10 points) Prove that ω is transitive and inductive.
4. (10 points) Prove that ϵ is a total order on ω.

A function $f \colon X \to Y$ is a subset of $X \times Y$ such that for all $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in f$. A function $f \colon X \to Y$ is *injective* if for all $x, y \in X$ the conditions $(x, a) \in f$ and $(y, a) \in f$ imply $x = y$, and *surjective* if for all $y \in Y$ there exists $x \in X$ with $(x, y) \in f$.

Exercise 5. (5 points) Prove the (finite) Pigeonhole Principle: for all $n \in \omega$, there is no injective function $f \colon n + 1 \to n$.

Exercise 6. Let $f \colon X \to Y$ be a function.

1. (10 points) Prove that f is injective if and only if there exists $g \colon \mathcal{R}_f \to X$ such that for all $x \in X$ we have $(x, x) \in g \circ f$ (g is called a *left inverse* of f).
2. (10 points) Prove that f is surjective if and only if there exists $g \colon \mathcal{R}_f \to X$ such that for all $y \in Y$ we have $(y, y) \in f \circ g$ (g is called a *right inverse* of f).
3. (5 points) Prove that f is injective and surjective (*bijective*) if and only if there exists $g \colon Y \to X$ which is a left and right inverse of f.