

## Exercise sheet #2

### Set Theory 2022

**Exercise 1.** Let  $a, b, c$  be sets. Prove that there exists a unique set  $d$  satisfying

$$\forall x(x \in d \leftrightarrow x = a \vee x = b \vee x = c).$$

**Exercise 2.** Consider the following weak versions of axioms seen in class:

**Weak axiom of existence** There exists a set.

**Weak axiom of pair** If  $a$  and  $b$  are sets, then there exists a set  $c$  satisfying  $a \in c \wedge b \in c$ .

**Weak power set axiom** If  $a$  is a set, then there exists a set  $p$  satisfying  $\forall x(x \subset a \rightarrow x \in p)$ .

Use each weak axiom above and the axiom schema of comprehension to prove the corresponding non-weakened axiom.

**Exercise 3.** Let  $a = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$  and  $b = \{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ .

1. How many elements do  $a$  and  $b$  contain?

2. Calculate  $\bigcup a, \bigcap b$ .

**Exercise 4.** Prove or give a counterexample.

1.  $a \subseteq b \leftrightarrow a \cap b = a \leftrightarrow a \cup b = b \leftrightarrow a \setminus b = \emptyset$ .

2.  $a \setminus b = (a \cup b) \setminus b = a \setminus (a \cup b)$ .

3.  $a \cap b = a \setminus (a \setminus b)$ .

4.  $a \Delta (b \cup c) = (a \Delta b) \cup c$ .

5.  $a \setminus (b \setminus c) = (a \setminus b) \cup (a \cap c)$ .

**Exercise 5.** Let  $s$  and  $a$  be sets and suppose  $s \neq \emptyset$ . Define

$$T_1 = \{y \subseteq a : \exists x(x \in s \wedge y = a \cap x)\}$$

$$T_2 = \{y \subseteq a : \exists x(x \in s \wedge y = a \setminus x)\}$$

Prove  $a \cap \bigcup s = \bigcap T_1$  and  $a \setminus \bigcap s = \bigcup T_2$ .

**Exercise 6.** Prove that  $\bigcap s$  is defined for all  $s \neq \emptyset$ . What is wrong with  $\bigcap \emptyset$ ?