Exercise sheet #1
Set Theory 2022

Exercise 1. Let $\text{tv}(P)$ denote the truth value of P. Show that in arithmetic modulo 2,

1. $\text{tv}(\neg P) = 1 + \text{tv}(P)$
2. $\text{tv}(P \wedge Q) = \text{tv}(P)\text{tv}(Q)$
3. $\text{tv}(P \vee Q) = \text{tv}(P) + \text{tv}(Q) + \text{tv}(P)\text{tv}(Q)$
4. $\text{tv}(P \Rightarrow Q) = 1 + \text{tv}(P) + \text{tv}(P)\text{tv}(Q)$
5. *$\text{tv}(P \Rightarrow Q) = \text{tv}(\neg P \vee Q)$
6. *Find similar expressions for $\text{tv}(\neg(\neg P \vee \neg Q))$ and $\text{tv}(\neg(\neg P \wedge \neg Q))$.

Exercise 2. Compute the following truth tables:

1. $P \vee \neg P$
2. $\neg(P \wedge \neg P)$
3. $(P \Rightarrow \neg\neg P) \wedge (\neg\neg P \Rightarrow P)$
4. $(\neg P \vee Q) \wedge (\neg R \vee P)$
5. $P \Rightarrow (P \vee Q)$
6. $(P \vee Q) \Rightarrow P$
7. $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$

Exercise 3. Prove the following equivalences:

Proof by contradiction: $(A \Rightarrow B) \Leftrightarrow ((A \wedge \neg B) \Rightarrow C)$, where C is a contradiction (i.e., only calculate the tables for C false).

Proof by contraposition: $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

Exercise 4. *Let $|$ be a logical operator such that $P|Q$ is false only when both P and Q are true. Show that $\neg P$, $P \wedge Q$, $P \vee Q$, and $P \Rightarrow Q$ are equivalent to statements using only $P, Q, |$.

Exercise 5. If w is a term and w' is a proper initial segment of w then w' is not a term.

Exercise 6. For this exercise, write terms without parentheses or commas. For example, if f is a ternary function symbol and x_0, x_1, x_2 are variables then write $fx_0x_1x_2$ instead of $f(x_0, x_1, x_2)$.

1. Convince yourself that eliminating parentheses and commas does not affect terms in the sense that they are still readable in a unique way.
2. *Define λ as the following function on finite strings

$$\lambda(w) = \begin{cases} -1 & \text{if } w \text{ is a variable} \\ r - 1 & \text{if } w \text{ is an } r\text{-ary function symbol} \\ \sum_{i<n} \lambda(u_i) & \text{if } w = u_0 \dots u_{n-1}, n > 1 \end{cases}$$

Prove that w is a term if an only if $\lambda(w) = -1$ and $\lambda(w') \geq 0$ for all nonempty proper initial segments w' of w.