~ Selected Chapters in Combinatorics ~

Lecture 11

* Homomorphism preservation theorem for infinite structures

* Finish proofs from last time

* Cores of ω-cat strs.

Andrés Aranda
aranda@kam.mff.cuni.cz

Reminders

- κ-saturated : types over parameter sets A, $|A|<\kappa$ are realised
- saturated $\rightsquigarrow$ $|M|$-saturated.

<u>Thm ①</u>: L: a language $\lambda \geq |L|$. Then every L-str M has a λ^+-saturated elementary ext of card at most $|M|^\lambda$.

Existential positive n-type : a set p of e.p. formulas on free vars $x_1 \cdots x_n$ s.t. $p \cup T$ is satisfiable

Maximal $\exists^+$ n-type: for all $\exists^+$ φ
$p \cup T \cup \{\varphi\}$ is unsatisfiable.

Lemma (Bodirsky 2.2.6) 2:

M, N L-structures. Suppose N ep $|M|$-saturated

If every pp sentence true in M is true in N,

then there is a homomorphism $M \to N$.

(Well-order M, use transfinite induction)

Lemma 3: T a f.o. theory

φ f.o fmla on $x_1 \cdots x_n$.

If $c_1 \cdots c_n$ are new constant symbols,

$$T \models \varphi(c_1 \cdots c_n) \iff T \models \forall x_1 \cdots x_n \; \varphi(x_1 \cdots x_n)$$

(In every model, φ is satisfied.)

Homomorphism preservation theorem:

T : a first-order theory with infinite models.

A first-order fmla φ is equivalent mod T to an $\exists^+$ fmla iff φ is preserved by all homomorphisms between models of T.

(We can assume that the language includes $\perp$)

$x \neq x$

Proof: One direction is clear: $\exists^+$ fmlas are preserved by homomorphisms.

Let $\varphi(x_1 \dots x_n)$ be a fmla that is preserved by all homs. between models of T.

Let $c_1 \dots c_n$ be new constants (not in L)
$\bar{c} = (c_1, \dots, c_n)$.

If $T \cup \{\varphi(\bar{c})\}$ is not satisfiable, we are done $\varphi \underset{T}{\approx} \bot$, so assume
$T \cup \{\varphi(\bar{c})\}$ is satisfiable.

Ψ all e.p. $L \cup \{c_1, \dots, c_n\}$ – sentences ψ s.t.

$T \cup \{\varphi(\bar{c})\} \models \psi$.

Let M be a model of $T \cup \Psi$.

$\mathcal{U}$: all pp sentences ϑ s.t. $M \models \neg\vartheta$.

<u>Claim</u>: $\boxed{T \cup \{\neg\vartheta : \vartheta \in \mathcal{U}\} \cup \{\varphi(c)\}}$ is satisfiable.

<u>Pf</u>: Suppose not. By compactness, there is finite $\mathcal{U}' \subseteq \mathcal{U}$ s.t. $T \cup \{\neg\vartheta : \vartheta \in \mathcal{U}'\} \cup \{\varphi(\bar{c})\}$ is unsatisfiable.

Then $\psi := \bigvee_{\vartheta \in \mathcal{U}'} \vartheta$ is e.p sentence and $T \cup \{\varphi(\bar{c})\} \models \psi$, so $\psi \in \Psi$. Contradiction

N : a model of $T \cup \{\neg\vartheta : \vartheta \in \mathcal{U}\} \cup \{\varphi(\bar{c})\}$.

Apply Thm ① to find an elementary extension M' which is $|N|$-saturated.

<u>Claim</u> : every pp $L \cup \{c_1 \ldots c_n\}$ - sentence ϑ true in N is true in M'

<u>Pf</u>: Suppose not.
If ϑ is false in M', then it is false in M, so $\vartheta \in \mathcal{U}$.
This contradicts $N \models \{\neg\vartheta : \vartheta \in \mathcal{U}\}$.

Apply Lemma 2 : There is a homomorphism $N \longrightarrow M'$. $N \models \varphi(\bar{c})$ and φ preserved by homs, so $M' \models \varphi(\bar{c})$

From the above, $T \cup \Psi \cup \{\neg \varphi(\bar{c})\}$ is not satisfiable.

Apply compactness: there is finite $\Psi' \subseteq \Psi$ s.t.

$T \cup \Psi' \cup \{\neg \varphi(\bar{c})\}$ is unsatisfiable.

$\xi := \bigwedge_{\psi \in \Psi'} \psi$ is an existential positive sentence $\mathcal{L} \cup \{c_1 \cdots c_n\}$

Change c_i to x_i in ξ to obtain $\mathcal{L}$-fmla ξ'

Apply Lemma 3: $T \models \forall x \, (\varphi(x) \leftrightarrow \xi'(\bar{x}))$

(φ is eq. to e.p. fmla) $\blacksquare$
mod T

$T \models \varphi(\bar{c})$ is eq to $T \models \forall \bar{x} \, \varphi(\bar{x})$

Thm from last time: TFAE:

(1) T is model-complete

(2) fmlas are eq. to $\exists$ fmlas.

(3) for every embedding $e: A \to B$, every $\bar{a} \in A^n$ and every $\exists$ fmla φ, if $B \models \varphi(\bar{a})$ then $A \models \varphi(\bar{a})$

(4) every $\exists$ fmla is eq. to $\forall$ fmla

(5) every fmla is eq. to $\forall$ fmla.

$(1) \to (2) \to (3)$: last time.

$(3) \to (4)$: Let φ be existential. We prove $\neg\varphi$ is eq. to $\exists$ fmla. By (3) $\neg\varphi$ is preserved by embeddings. By Łoś–Tarski $\neg\varphi$ is eq. to $\exists$ fmla; and φ is eq. to $\forall$ fmla.

(4) - (5)

Every $\exists$ fmla is eq. to $\forall$ $\Rightarrow$ every fmla is eq. to $\forall$.

φ: a fmla. Assume φ is in prenex normal form $Q_1 x_1 \dots Q_n x_n \psi$ with ψ q free.

let i be the smallest index s.t.

$$Q_i = \dots = Q_n.$$

<u>Case $i=1$</u>: either universal or equivalent to universal by (4)

<u>Case $i>1$</u>:

Case $i > 1$:

- If $Q_i = \dots = Q_n = \exists$, then by (4)
  $\exists x_i \dots \exists x_n \varphi$ is eq. to universal
  fmla ξ

  $Q_1 \dots Q_{i-1} \xi$

- If $Q_i = \dots = Q_n = \forall$ then by (4)
  $\underbrace{\exists x_i \dots \exists x_n \neg\varphi}_{\text{universal } \varphi'}$ is equivalent to

  $Q_1 x_1 \dots Q_{i-1} x_{i-1} \neg\varphi$ is eq to φ
  with fewer alternations. Repeat.

Every fmla is equivalent to universal $\Rightarrow$ T is model comp.

φ a f.o. fmla.

By (5), $\neg\varphi$ is eq. to universal fmla,

so φ is eq. to $\exists$ fmla and

$\exists$ fmlas are preserved by embeddings. $\square$

Cores of ω-cat structures

For finite strs: use endomorphisms, choose smallest image.

Idea: choose "youngest" image ($\subseteq$-minimal age).

König's tree lemma: Every infinite tree contains either a vertex of infinite degree or an infinite path.

Proposition (A): Let Γ be an ω-cat L-structure.

Then there exists $c \in \mathrm{End}(\Gamma)$ s.t. for every $g \in \mathrm{End}(\Gamma)$ $\mathrm{Age}(c(\Gamma)) \subseteq \mathrm{Age}(g(\Gamma))$

Proof: Let $\mathcal{S}$ be the set of all finite L-strs S s.t. there exists $g \in \mathrm{End}(\Gamma)$ S does not embed into $g(\Gamma)$.

Idea construct c "avoiding" all $S \in \mathcal{S}$.

How to do it: define an infinite, fin. branching tree of partial endo.

Apply König's lemma.

Enumerate $\Gamma = \{a_i : i \in \omega\}$

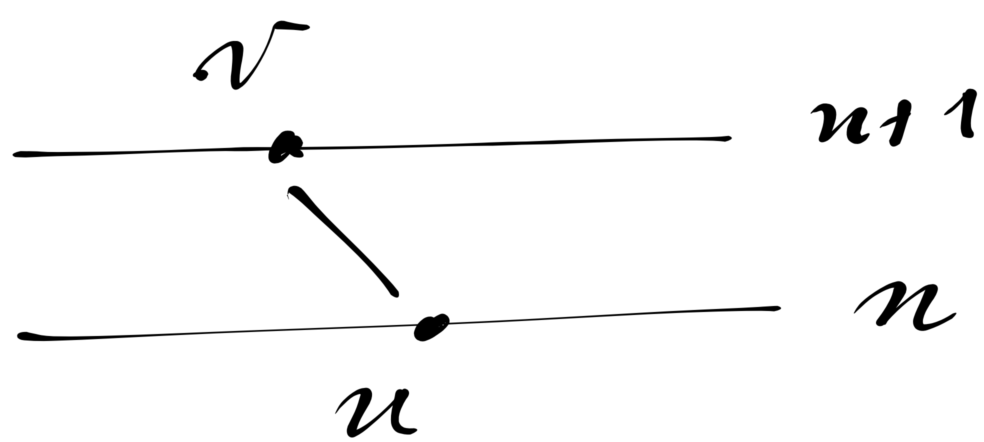
<u>Vertices of level n</u>: Equivalence classes of "good" homomorphisms $\{a_0 \dots a_{n-1}\} \to \Gamma$

Good: $h : \{a_0 \dots a_{n-1}\} \to \Gamma$ is good if it is δ-free

<u>Eq. rel:</u> $g \sim h$ if there is $\alpha \in \mathrm{Aut}(\Gamma)$ s.t. $g = h\alpha$.

σ : if h is good and $g \sim h$, then g is good.

EDGES:



$E(v, u)$ if some $h \in u$ is restriction of some $f \in v$.

T is ω-cat. By Ryll-Nardzewski, the tree is finitely branching. Need to prove existence of good homomorphisms $\{a_0 \dots a_{n-1}\} \to T$ for all n.

$\{a_0 \cdots a_{n-1}\} =: A_n$

Either A_n is $\mathcal{S}$-free and we use the id.

Or there is $e \in End(\Gamma)$ s.t. $e(\Gamma)$ has no copy of DES.

$\underbrace{e|_{A_n}}$ omits D. Repeating, eventually (after fin. many steps) we get a good homomorphism.

Apply König's lemma: there is an infinite branch, so an endomorphism c.

By construction $c(\Gamma)$ is $\mathcal{S}$-free. $\square$

e : distinguished endomorphism with youngest image.

Lemma ©: Γ ω-categorical and L contains predicates for all $\exists^+$ definable relations of Γ. Then every homomorphism $\Gamma_1 \to \Gamma_2$ with $Age(\Gamma_1) = Age(\Gamma_2) = Age(e(\Gamma))$ is an embedding.

{ finite
  ω-cat

Corollary: Every ω-cat structure has a core.

Core: endomorphisms are embeddings

$\to$ Expand the language with predicates for $\exists^+$-def relations. Apply © to find core, take L-reduct of that structure.

M, N are ω-cat.

$CSP(M) = CSP(N)$ iff M, N hom. equivalent.

Antoine

$1/2$

$$\bullet \to \bullet \to \bullet \to \cdots$$

$$\cdots \to \to \underbrace{\to \quad \to \quad \cdots}$$

$\exists x \exists y \ (x \neq y)$

Bodirsky:

Cores of countably categorical structures.