~ Selected Chapters in Combinatorics ~

Lecture 10

* More on ω-cat sets
* Interpretations
* ???

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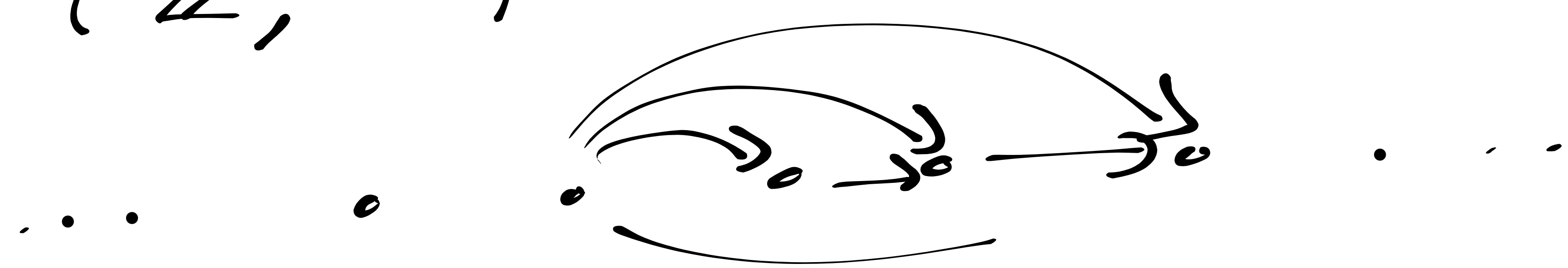
CSP

Def: $\mathbb{B}$: rel. str with language L

$CSP(\mathbb{B})$ decide, given finite L-str $\mathbb{A}$, whether there exists a homomorphism
$$\mathbb{A} \to \mathbb{B}.$$

$CSP(\mathbb{B})$: the class of all finite L-structs that map homomorphically to $\mathbb{B}$.

$CSP(\mathbb{Z}, <)$: all finite acyclic digraphs



CSP = Constraint Satisfaction Problem

Example (H-colorability)

H: a graph

CSP(H)

3-colorability $\qquad$ CSP(K_3)

n-colorability $\qquad$ CSP(K_n)

Feder–Vardi: CSP($\mathcal{B}$), where $\mathcal{B}$ is a finite relational structure is either in P or NP-complete.

Bulatov / Zhuk : Feder - Vardi conjecture is true.
Жук

<u>Lemma</u>: L: finite relational lang.
$\mathcal{C}$: class of finite L-structures. TFAE:
(1) $\mathcal{C} = CSP(B)$ for some structure B
(2) $\mathcal{C} = Forb^{hom}(\mathcal{F})$, where $\mathcal{F}$ is a class of finite connected L-strs.
(3) $\mathcal{C}$ is closed under under disjoint unions and inverse homomorphisms.
(4) $\mathcal{C} = CSP(B)$ for countably infinite B.

$\mathcal{F}$: a class of L-strs

$\mathrm{Forb}^{hom}(\mathcal{F})$: all L-strs A s.t.
for all $B \in \mathcal{F}$, there is no
<u>homomorphism</u> $B \to A$.

Disjoint union :

A, B L-structures

$A' = A \times \{0\}$

$B' = B \times \{1\}$

$\cong A$ $\cong B$

Copy <u>str.</u> of A on A',
B on B'

$$A \uplus B = A' \cup B'$$

Connectedness

L-str A is connected if it is not a disjoint union of nonempty structures.

Gaifman graph ($\underline{A}$)

Vertices: elts. of A

$a \sim b$ if a, b appear in some tuple that satisfies a relation in $\underline{A}$

Inverse homomorphisms

$\mathcal{C}$ is "closed under inverse hom" if whenever

$A \in \mathcal{C}$

and $h : B \longrightarrow A$ is a homomorphism

then $B \in \mathcal{C}$

(1) $\mathcal{C} = CSP(B)$ for some B

$\to$ (2) $\mathcal{C} = \dfrac{\mathrm{Forb}^{hom}(\mathcal{F})}{\text{connected finite}}$ for a class of L-strs.

Pf: Take $\mathcal{F}$ as the class of fin connected L-strs that do not map into B

(2) $\to$ (3): $\mathcal{C}$ closed under disj. unions and inverse homomorphisms.

Suppose we had a homomorphism from $C \in \mathcal{F}$ to $A_1 \uplus A_2$; then we would have homomorphism $C \to A_1$ or $C \to A_2$.

Closure under inv. homs: left to the enthusiastic student.

$\mathcal{C}$ closed under disj. unions and inv. homomorphisms
$\to$ $\mathcal{C} = CSP(\mathcal{B})$ for countably infinite $\mathcal{B}$.
(Requires finite relational language)
Breaking $\mathcal{C}$ into isomorphism classes, we obtain at most ctbly many classes
Let $\mathcal{C}' \subseteq \mathcal{C}$ contain only 1 representative from each iso. class of $\mathcal{C}$
$\mathcal{B}$: disjoint union of all elts of $\mathcal{C}'$
$CSP(\mathcal{B}) = \mathcal{C}$

A, B

$$\left.\begin{array}{l} h : A \longrightarrow B \\ g : B \longrightarrow A \end{array}\right\} \text{homomorphisms.}$$

$X \in CSP(B)$: $\exists\, u : X \longrightarrow B$

$g : B \longrightarrow A$

$gu : X \longrightarrow A$

<u>Obs</u>: If A and B are homomorphically equivalent then $CSP(A) = CSP(B)$.

Def: B is a core if all endomorphisms of B are embeddings.

B is a core of A if B is a core and B is homomorphically eq. to A.

Prop: All finite structures A have a core; all cores of A are isomorphic.

(orbits of n-tuples under $\mathrm{Aut}(A)$ are pp-definable)

pp: primitive-positive   pp-def: definable by pp-fmla

pp-fmla: $\exists x_{n+1} \dots \exists x_m (\psi_1 \wedge \dots \wedge \psi_n)$

ψ_i atomic $R(y_1 \dots y_k)$ $y_j \in \{x_1 \dots x_m\}$

Preservation thms

φ, ψ are equivalent modulo T (T theory)
if $T \models \forall \bar{x} \, (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$

Next lecture proof

Homomorphism preservation thm: T: a f.o. theory

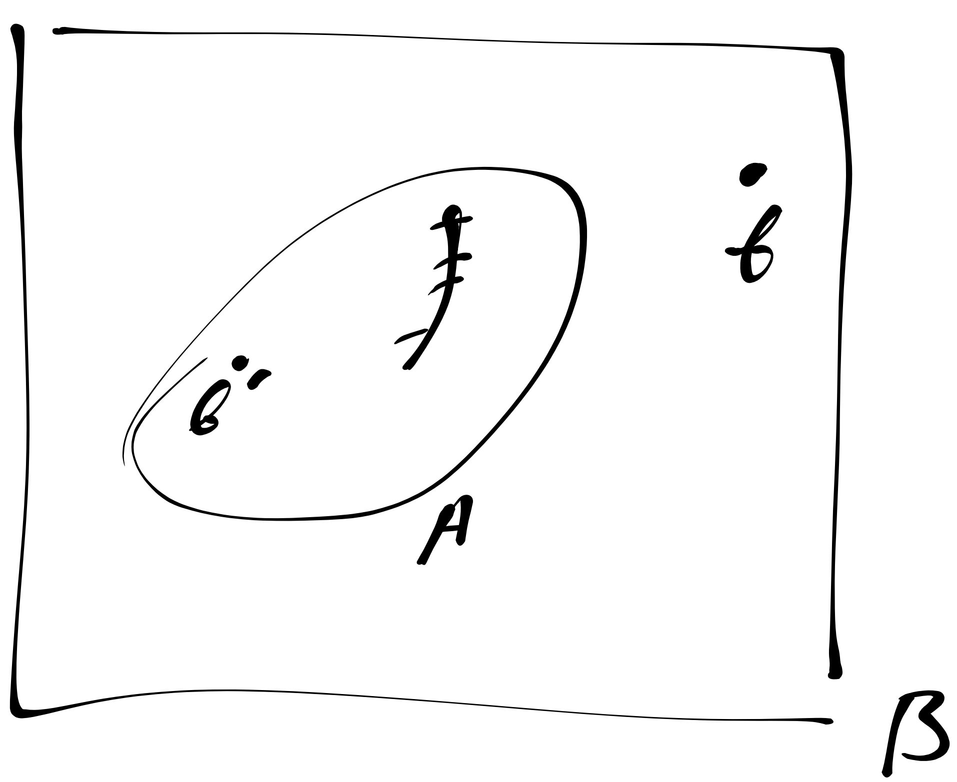
A f.o. fmla φ is equivalent modulo T to an existential positive fmla iff φ is preserved by all homomorphisms between models of T.

Corollary (Łoś–Tarski) φ is equivalent modulo T to an existential fmla iff φ is preserved by all embeddings btn models of T.

φ is preserved by $h : A \to B$

$A \models \varphi(\bar{a}) \longrightarrow B \models \varphi(h(\bar{a}))$

<u>Definition</u>: T is model-complete if every embedding between models of T is elementary.



$\sim$ some level of quantifier elimination

Thm: T: a theory. TFAE:

(1) T is model-complete

(2) Every f.o. fmla is equivalent mod. T to an existential fmla

(3) for every embedding $e: A \to B$ and $\bar{a} \in A^n$ and every existential fmla φ, if $B \models \varphi(e(\bar{a}))$, then $A \models \varphi(\bar{a})$

(4) Every $\exists$ fmla is equivalent to $\forall$ fmla

(5) Every f.o. fmla is equivalent to a universal fmla.

$\boxed{\text{Arrows: next time}}$

(1) $\to$ (2) : by Łoś-Tarski thm.

(2): Every fmla is eq to $\exists$ fmla

$\to$ (3) : $B \models \psi(e(\bar{a})) \implies A \models \psi(\bar{a})$

(ψ existential, e embedding, $\bar{a}$ tuple from A)

$\neg\psi$ is equivalent to an existential fmla

e preserves $\neg\psi$

$B \models \psi(e(\bar{a}))$: Since e preserves

$\neg\psi$, we must have $\psi(\bar{a})$