~ Selected Chapters in Combinatorics ~

* Homomorphism-homogeneity

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Reminders

- homogeneity "local" = finite domain. Countable str.

- Fraïssé thm.

<u>Example (Classification)</u>

<u>Lachlan - Woodraw thm</u>: If G is ctbly infinite homogeneous graph, then G or its complement is iso to

- $I_\omega[K_n]$ $n \geq 2$
- $I_n[K_\omega]$ $n \geq 2$
- R
- H_n $(n \geq 3)$
- K_ω

Homogeneity

homomorphism –

M is homogeneous if every local isomorphism is restriction of an automorphism of M

homomorphism

endomorphism of M

Later: more variations

Thm (Fraïssé)

Notion of hom

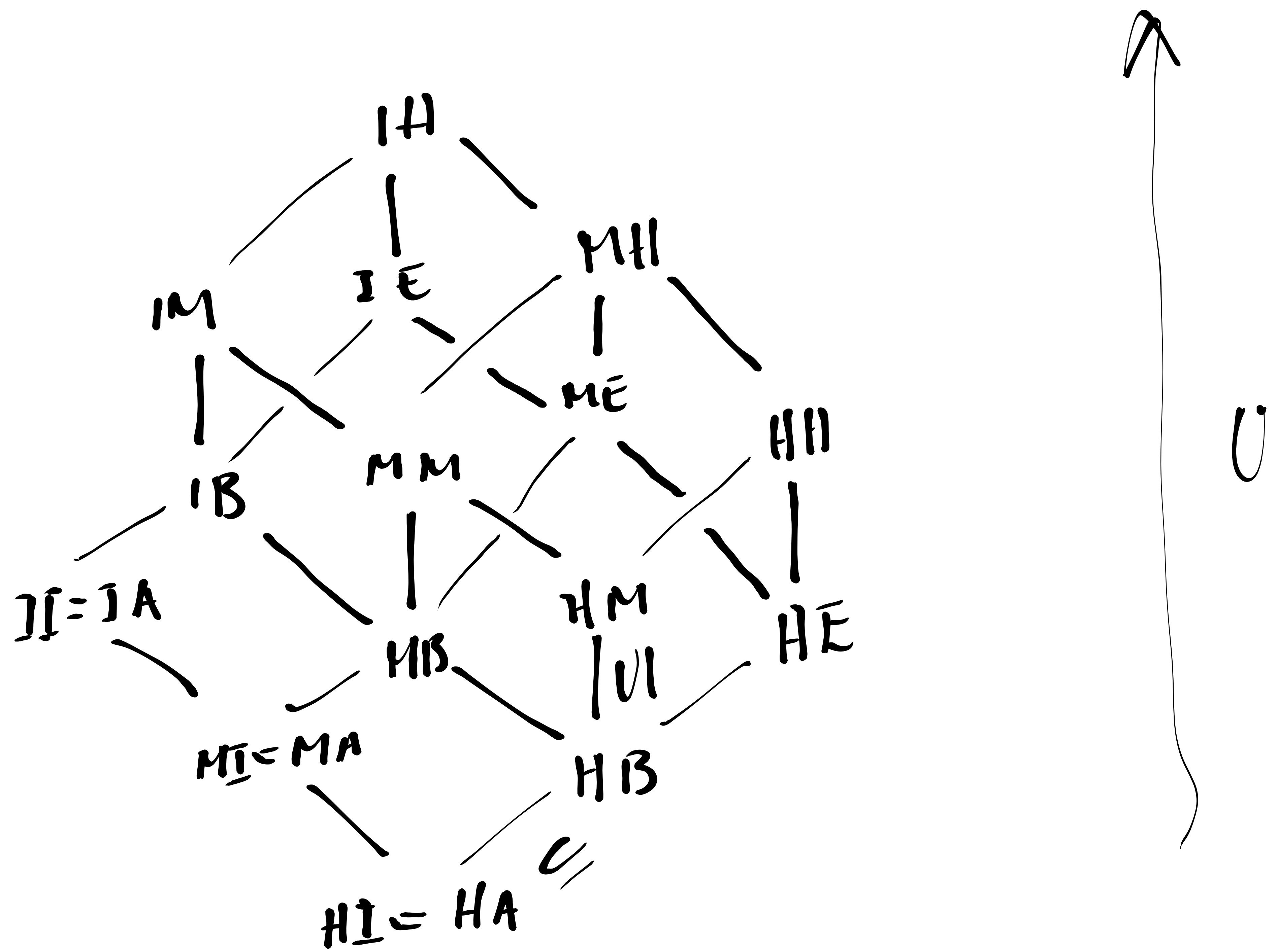
If M is a $\boxed{\text{homogeneous str}}$, then $Age(M)$ is a ctble (up to iso) hereditary class with JEP and $\boxed{AP.}$ Amalg. prop

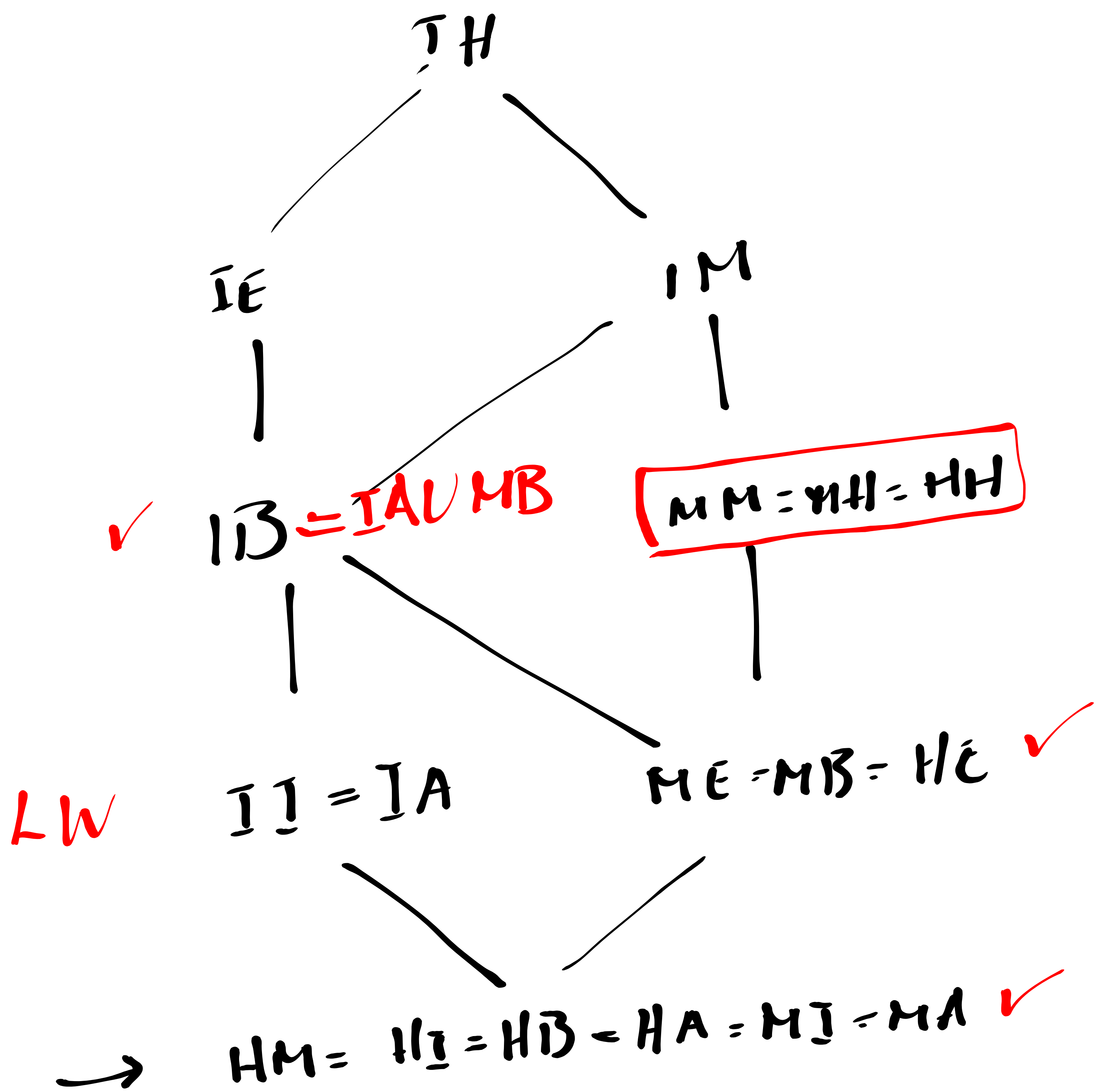
Conversely, if $\mathcal{C}$ is a ctble hereditary class of finite strs with JEP and AP, then there exists a ctble homogeneous $M_{\mathcal{C}}$ s.t. $Age(M_{\mathcal{C}}) = \mathcal{C}$. Correspondence Class - struct.

Moreover, $M_{\mathcal{C}}$ is $\boxed{\text{unique up to iso}}$.

Uniqueness criterion.

Countable structures





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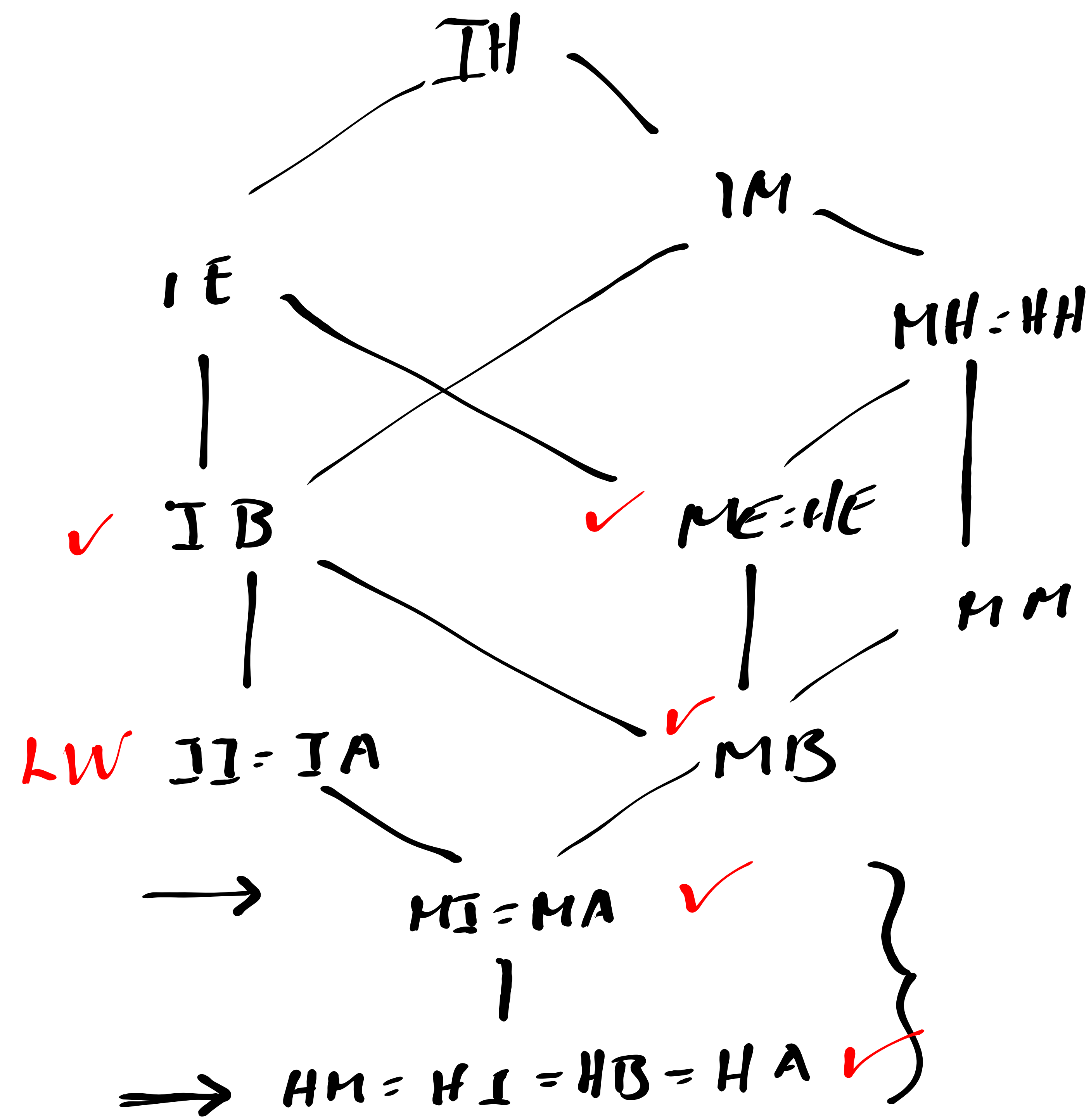


General

Ingredients for notion of homogeneity

- local homomorphism
  * homomorphism (H)
  * monomorphism (M)
  * isomorphism (I)

- Global extension
  * Endomorphism (H)
  * Monomorphism (M)
  * Epimorphism (E)
  * Self-embedding (I)
  * Automorphism (A)
  * Bimorphism (B)

Proposition: For countable strs, $HI = HA$, $MI = MA$, $II = IA$.

Pf: $IA \subseteq II$

Supp. M is II. Let $p: A \to B$ be a local iso

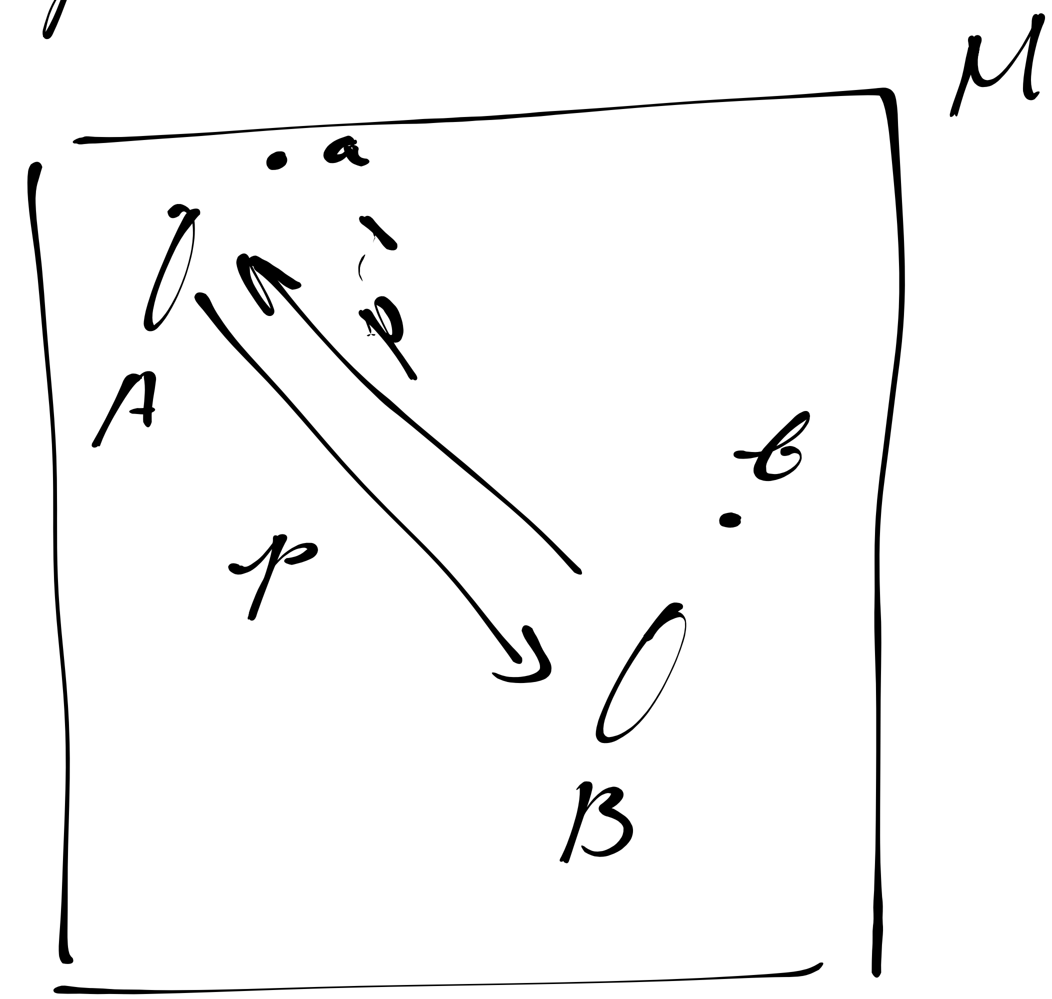
Consider $b \notin B$

p^{-1} is an iso.

Apply II to find a preimage for b

Repeat over an enumeration

Done. $IA = II$.



(Back only)

M is MI iff M is II and every local monomorphism is an isomorphism. $II = IA$, so $MI = MA$.

Similar argument: $HI = HA$. $\square$

Given my fav. class of relational structure, what is the poset of morphism-extension classes?

More detailed. What is the Lachlan-Woodrow analogue for each class?

What is the Fraïssé thm in each class?

Fraïssé $\longrightarrow$ uniqueness criterion = finest eq rel
for which it makes
sense to classify

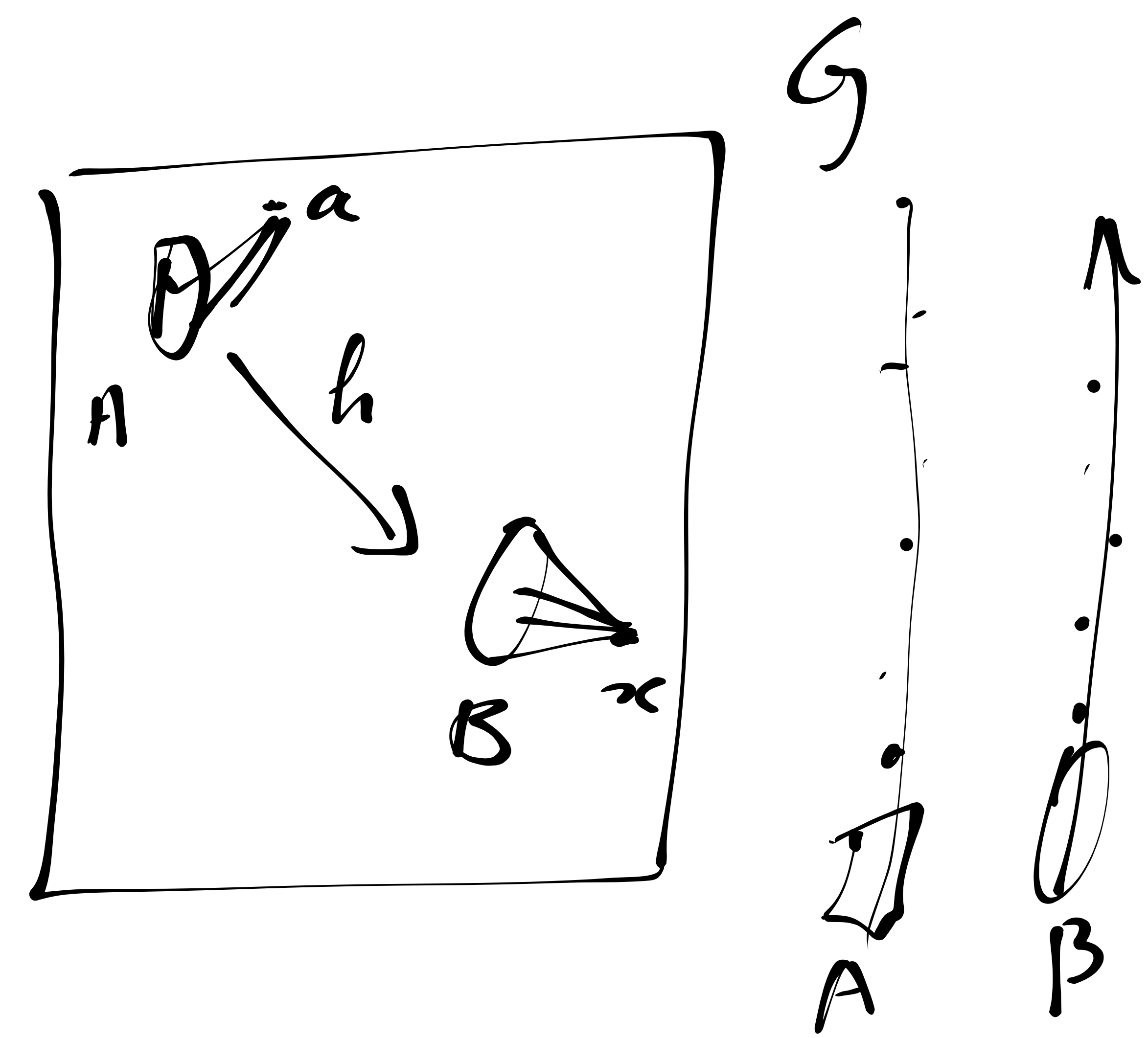
Case study: Graphs

Original classes: HH, MH

Obs: If G is a ctble graph and

(Δ) for every finite $F \subseteq G$ there exists x s.t. $F \subseteq N(x)$

then G is HH.



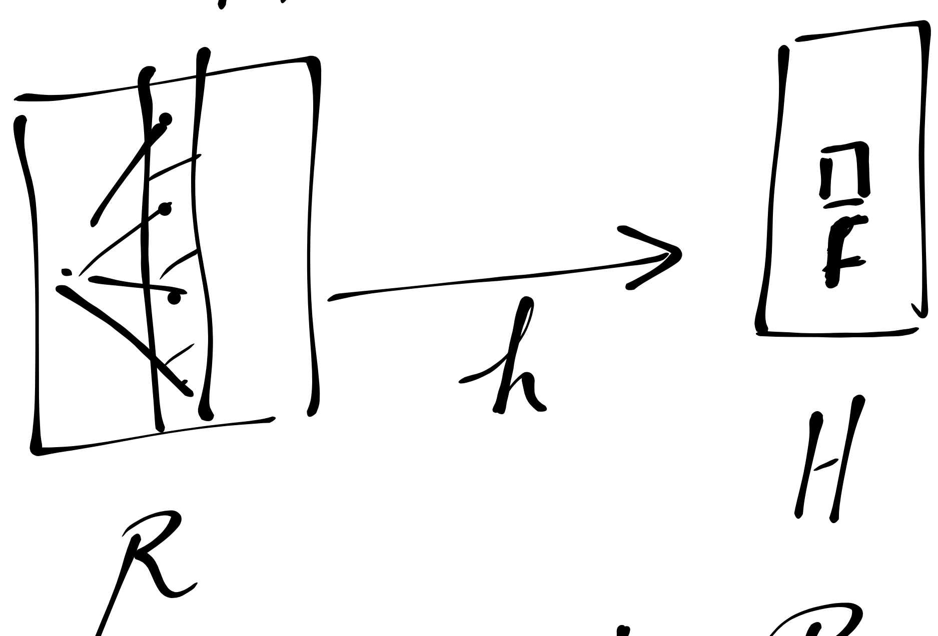
> Case 1: no edges $a - A$
map a to any vertex

> Case 2: Apply (Δ) over B
map a to x.

<u>Obs</u> Prop. (Δ) is equivalent to being a hom. image of R.

<u>Pf</u>: R clearly satisfies (Δ). Homomorphisms preserve edges, so images of R satisfy (Δ)

Now suppose $h: R \to H$ (surjective)



Consider $h^{-1}[F]$

$\ker h$ has fin. many classes

Take a transversal. t

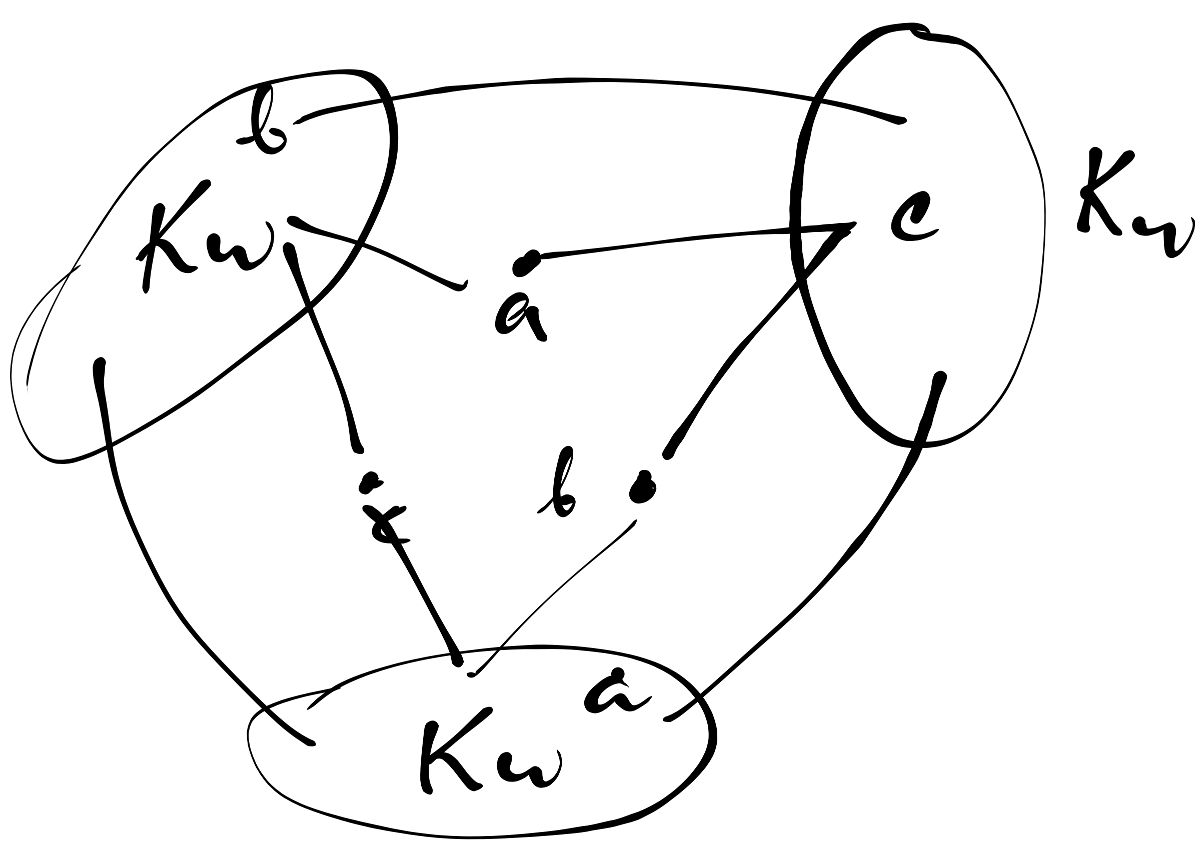
By axioms of R, there is $x \in R$ s.t. $N(x) \supseteq t$. x is mapped by h to a witness for (Δ).

Q: Are there any HH graphs that do not satisfy (Δ)? YES

Q: MH = HH for ctble graphs? YES

(Cameron–Nešetřil: Homomorphism-homogeneous structures)

Example: HH, $\neg(\Delta)$:



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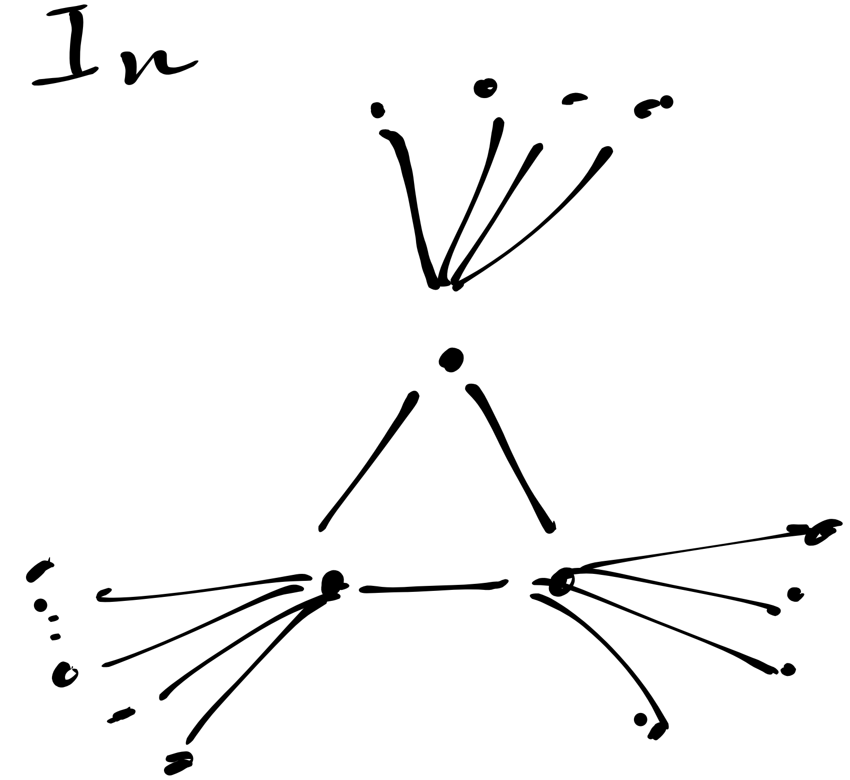


Comp of this

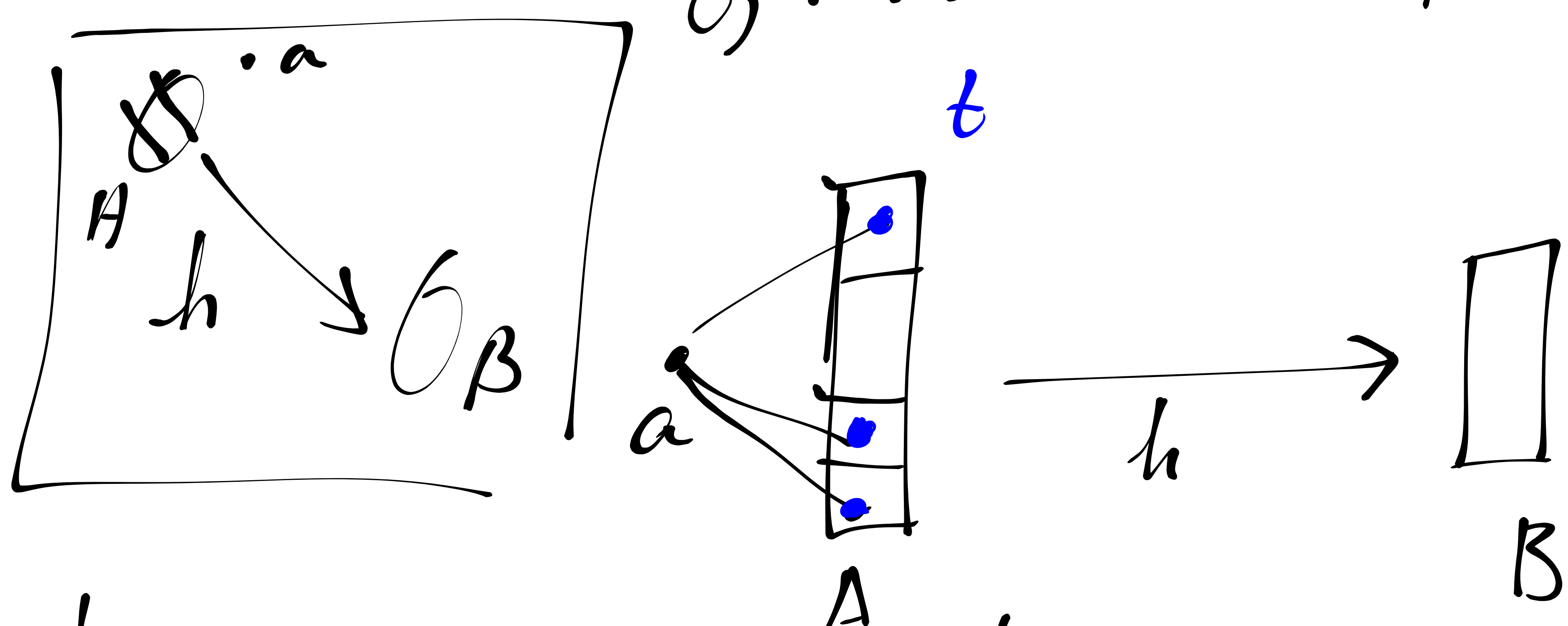
MH = HH / Rusinov – Schweitzer Homomorphism - hom. graphs

Thm A ctbly infinite graph G is HH iff it is MH.

Pf $MH \supseteq HH$ is clear.

Now suppose G is MH and $h : A \to B$ is local hom

G : MH $a \notin A$.



$h|_t$ is a mono.

Apply MH to find image for a.

Repeat over an enumeration. $\square$

But we have a large number of morphism-ext classes.

Indep nr of HH graphs

Thm: If G is a ctbly infinite connected HH graph with finite star number $\sigma(G) \geq 2$ then

$$\alpha(G) < 2\sigma(G) + \left\lceil \frac{\sigma(G)}{2} \right\rceil - 1$$

$$\sigma(G) = \sup \{ n : K_{1,n} \in \mathrm{Age}(G) \}$$
$$\alpha(G) = \sup \{ n : I_n \in \mathrm{Age}(G) \}$$

Corollary: An HH graph with infinite indep # is an image of R.

+ a little work + some results from others.

Thm: If G is a ctbly inf. MB-homogeneous graph, then G is bimorphism-equivalent to one of the following or its complement

$G \underset{B}{\approx} H$ if there exist bijective homo. $G \to H$ and $H \to G$.

1. K_ω
2. $I_\omega[K_\omega]$
3. R

- Fraïssé thms are important
- We don't know Fraïssé thms for all morph-ext classes.

  Open: IE, IH

  (I have ideas)

- "Partial quantifier elimination" in morph. ext classes.
- Ryll-Nardzewski analogues (Ch + M Pech)
- Polymorphism - homogeneity

  PH $\subseteq$ HH (each Cartesian power is HH)