.~ Selected Chapters in Combinatorics ~

Lecture 8

* Omitting types thm ←
* ω-categoricity

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Compactness thm

Elem. substr / embeddings

Complete th.

Löwenheim-Skolem

Types, saturation

Omitting types

ω-categoricity

Fraïssé amalgation

Techniques

- Henkin's construction
- Ultraproducts
- Elementary chains
- Back and forth

Omitting Types

Def : p : an n-type over T
ϕ isolates p if for all $\psi \in p$
$$T \models \forall x(\phi(x) \to \psi(x))$$

$$[\phi] = \{p\}$$

Obs : if T is complete, then every principal (= isolated) type p is realised in every model of T.

Pf : There is $\underline{M \models \exists x\, \varphi(x)}$, where φ isolates p. Every other model of M realizes p.

Thm (omitting types thm)

L - a countable language

T - a consistent L-theory

p - a nonprincipal n-type.

Then T has a ctble model M that omits p.

Proof: Idea: Use Henkin's construction

witness property: if $\varphi(x)$ is a fmla, then

there exists $c \in L$ constant s.t

$$T \models (\exists x\, \varphi(x)) \to \varphi(c$$

How to do it:

over ctbly many steps. of 3 types.

Pf (cont):

$\mathcal{C} = \{c_i : i \in \omega\}$ new const. symbols

$L' = L \cup \mathcal{C}$.

Goal: Construct seq. $\{\vartheta_i : i \in \omega\}$ of L'-sentences

s.t. $\models \vartheta_t \to \vartheta_s$ for $t > s$

Use these:

$T^* = T \cup \{\vartheta_i : i \in \omega\}$. ← complete

To ensure completeness,

$\models \vartheta_{3i+1} \to \varphi_i$ or $\vartheta_{3i+1} \to \neg\varphi_i$

To ensure witness prop, if φ_i $\exists v \phi v$ and

$\models \vartheta_{3i+1} \to \varphi_i$, then $\vartheta_{3i+2} \to \phi(c)$ some $c \in \mathcal{C}$

$\{d_i : i \in \omega\}$ enumeration of n-tuples of constants in $\mathcal{C}$.

We will choose ϑ_{3i+3} to ensure that d_i^M does not realise p.

<u>Start</u>:

Step 0: $\vartheta_0 = \forall x\,(x = x)$

<u>Step $s+1 = 3i+1$</u>: (witness prop)

Supp. φ_i is $\exists v\, \phi(v)$ and $T \models \vartheta_s \to \varphi_i$

Take a constant $c \in \mathcal{C}$ not in $T \cup \{\vartheta_0 \dots \vartheta_s\}$

[If $\mathcal{N} \models T \cup \{\vartheta_s\}$ then for some $a \in \mathcal{N}$ $\mathcal{N} \models \phi(a)$, interpret c as a ~~a~~ to get $\mathcal{N} \models \vartheta_{s+1}$] $\vartheta_{s+1} = \vartheta_s \wedge \phi(c)$

If φ_i is not of the form $\exists v\, \phi(v)$, then

$\theta_{s+1} = \theta_s$.

<u>Step $s+1 = 3i+2$</u> (completeness)

If $T \cup \{\theta_s, \varphi_i\}$ is satisfiable,

$$\theta_{s+1} = \theta_s \wedge \varphi_i$$

Otherwise

$$\theta_{s+1} = \theta_s \wedge \neg\varphi_i$$

Either way $T \cup \{\theta_{s+1}\}$ satisfiable

Step $s+1 = 3i+3$: (omitting p)

Let $d_i = (c'_1 \cdots c'_n)$

Let $\psi(v_1 \cdots v_n)$ be the fmla obtained by replacing every occurence of c'_i by v_i and every other $c \in C \setminus \{c'_1 \cdots c'_n\}$ that appears in ϑ_s by by a variable v_c

add $\exists v_c$ in front of the fmla for each v_c.

Since p is not isolated, there is $\varphi(\bar{v}) \in p$ $\otimes$

s.t $T \not\vdash \forall v(\psi(v) \to \varphi(v))$

Let ϑ_{s+1} be $\vartheta_s \wedge \neg \varphi(\bar{d}_i)$

Why is $T \cup \{\vartheta_{s+1}\}$ satisfiable?

By $\otimes$, there is $N \models T$ and $\bar{a} \in N$

s.t. $N \models \psi(\bar{a}) \wedge \neg \varphi(\bar{a})$

We can turn N into a model of

ϑ_{s+1} by interpreting c_i' as a_i

$T^* = T \cup \{\vartheta_s : s \in \omega\}$

> T^* is complete.
Take any φ. So by const. $\varphi = \varphi_i$ for some i. At stage $3i+2$ we ensured $T^* \models \varphi$ or $T^* \models \neg\varphi$

> T^* has the witness prop. We did this at stage $3i+1$.

> Construct a model $M \models T^*$ from constants (as in lemma preceding Compactness thm)
Take $a \in M^n$. Since every elt. of M is interpretation of constant symbol, $a = d_i^M$
At stage $3i+3$ we ensured $M \models \neg\varphi_i(d)$ for some $\varphi_i \in p$.

$\square$

Additional bookkeeping: Same result, but omitting a countable collection of non-isolated types. $\oplus$

Look:
if a theory has only ctbly many n-types for all n, then $\oplus$ says we can omit all non-iso types, so there are models in which the ~~model~~ ~~a~~ type of any $\bar{a} \in M^n$ is isolated (atomic models).

Prop: If A, B are countable atomic models with $Th(A) = Th(B)$.

then $A \cong B$.

Pf: Back and forth.

Special case: $A = B$: if a, b have the same type, then they are in the same orbit of $Aut(A)$.

ω-categoricity

Structure M is ω-categorical if $Th(M)$ is ω-cat.

If M is a ctble ω-cat str. then $Aut(M)$ is a subgroup of $Sym(M)$

Def: A permutation gp G on a ctbly inf set X is oligomorphic if there are only finitely many orbits of n-tuples under the action of G (all n)

Orbit of $(a_1, \dots, a_n) = \{(\sigma(a_1), \dots, \sigma(a_n)) : \sigma \in G\}$

Thm (Engeler / Ryll-Nardzewski / Svenonius):

If M is a ctbly infinite str. with ctbl language, tfae:

(1) M is ω-categorical

(2) all types of M are principal

(3) all models of $Th(M)$ are atomic

(4) for each fin. n, there are only finitely many non-equivalent formulas on variables $x_1 \dots x_n$

$$\varphi \underset{T}{\sim} \psi \quad \text{if} \quad T \models \forall x \, (\varphi(x) \Leftrightarrow \psi(x))$$

(5) M has finitely many n-types for all n

(6) all models of $Th(M)$ are ω-saturated

(7) Every relation preserved by $Aut(M)$ is f.o. def.

(8) $Aut(M)$ is oligomorphic.

Plan: $1 \to 2 \to 3 \to 1$

$2 \to 4 \to 5 \to 6 \to 1$

$2 \wedge 3 \wedge 4 \to 7 \to 8 \to 4$

(1) → (2): By omitting types thm, if $Th(M)$ has a nonprincipal type p, then there is a model N that omits p.

By L-S there is a ctble $\mathcal{O} \models Th(M)$ that realises p

This contradicts ω-categoricity.

(2) → (3): If all types are principal, then every model satisfies all types, so has to be atomic.

(3) → (4) : If we had infinitely many inequivalent $\varphi(x_1 \dots x_n)$, then

$S_n(Th(M))$ ← compact, infinite

So there is a non-iso type.

(4) → (5) : If for each n there are $\psi_1 \dots \psi_n$ s.t any $\varphi(x_1 \dots x_n)$ is equivalent to ψ_i for some i, then every n-type p is determined by the set of ψ_i's in p. (fin many).

(5) → (6): Let M be a model $a \in M^n$

p a 1-type of (M, a)

Suppose M has m $(n+1)$-types

$p_1 \dots p_m$

for each $i \neq j$ $i, j \in \{1, \dots, m\}$

let φ_{ij} be a fmla in $p_i \setminus p_j$.

p_i is isolated by $\varphi_{i,1} \wedge \varphi_{i,2} \wedge \dots \wedge \varphi_{i,m}$.

$\rightarrow$ p_i is realised in M.

(6) $\longrightarrow$ (1)

We know that κ-sat models of the same size of any given theory are isomorphic. In part. if all ctble models are ω-sat, then all ctble models are iso.

$(2) \wedge (3) \wedge (4) \longrightarrow 7$

Suppose R is an n-ary relation preserved by $Aut(M)$.

→ R is a union of orbits of n-tuples

(> by (4) (only fin. many ineq fmlas)
R has to be a <u>finite</u> union of orbits

> by (3), tuples of the same type are in the same orbit

(> by (2) n-types are principal, so use the isolating fmla.

(7) $\to$ (8)

Suppose that $\mathrm{Aut}(M)$ has infinitely many orbits on M^n for some n.

The union of any subset of orbits is preserved by $\mathrm{Aut}(M)$

$\to$ uncountable set of invariant n-ary relations

On the other hand, there are only ctbly many fmlas on $x_1 \dots x_n$

So there are invariant relations which are not definable.

(1) → (4)

If $\mathrm{Aut}(M)$ is oligomorphic, then there are only finitely many inequivalent fmlas $\varphi(x_1 \ldots x_n)$ for each n.

(Automorphisms preserve f.o. formulas)

$\square$

Quantifier elimination in ω-cat strs.

QE means that every $\mathcal{L}$-fmla $\varphi(x_1 \dots x_n)$ is equivalent modulo T to a quantifier-free $\psi(x_1 \dots x_n)$

$$T \models \forall x_1 \dots x_n \, (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

Trick: Extend language to eliminate quantifiers:
Simply add for each fmla $\varphi(x_1 \dots x_n)$ a rel. symbol R_φ and an axiom

$$\forall x_1 \dots x_n \, (R_\varphi(x_1 \dots x_n) \leftrightarrow \varphi(x_1 \dots x_n))$$

Prop: An ω-categorical str. M has qe iff it is homogeneous.

Pf

qe $\to$ hom: Supp $\bar{a} = (a_1 \dots a_n)$ and $\bar{b} = (b_1 \dots b_n)$ are such that $f : a_i \mapsto b_i$ is a local iso. By qe, $\bar{a}$ and $\bar{b}$ have the same type.

By RN, orbits are f.o. definable, so $\bar{a}, \bar{b}$ are on the same orbit, so f is restriction of an automorphism of M.

hom $\to$ qe :

$\varphi(x_1, \dots x_n)$: a f.o. fmla.

By RN, there are only finitely many orbits $O_1 \dots O_m$ of n-tuples that satisfy φ

Claim: each of $O_1, \dots, O_m$ is def. by qfree fmla.

- Consider the set of all qfree fmlas satisfied by $\bar{a} \in M^n$ $M \models \varphi(\bar{a})$

If $(b_1 \dots b_n) = \bar{b}$ satisfies the same qfree fmlas, in part $f: a_i \mapsto b_i$ is a local iso. Apply homogeneity. Since all fo fmlas are preserved under Auto, $M \models \varphi(\bar{b})$ $\square$