~ Selected Chapters in Combinatorics ~

Lecture 7

* More on types
* Saturation

(* Omitting types)

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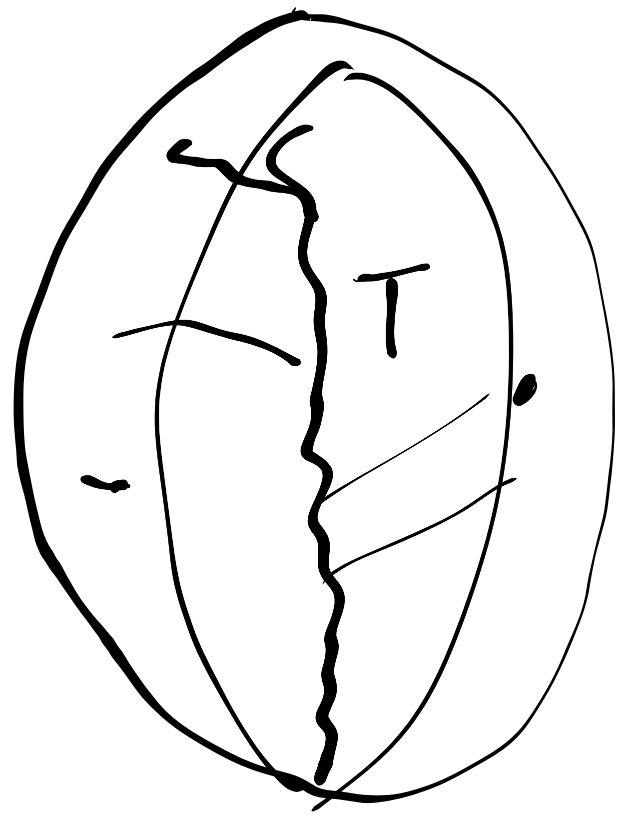
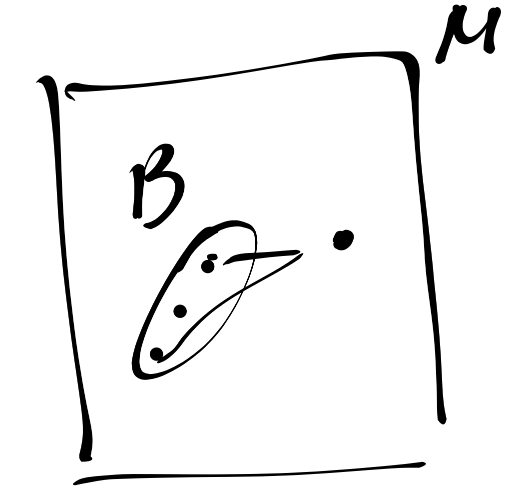
Types :

- Sets of formulas
- 

Lind. algebra of L

$L^* = L \cup \{x_1, \dots, x_n\}$

$\varphi(x_1 \dots x_n)$



- type of a tuple in a model $\bar{a} = (a_1, \dots, a_n)$

  $tp^{M}(\bar{a}) = \{\varphi(x_1 \dots x_n) : M \models \varphi(\bar{a})\}$ $\otimes$

- type of $\bar{a}$ over B = $tp(\bar{a})^{M_B}$

- Realised types (in $M \models T$)

  $p(x_1 \dots x_n)$ is realised in M

  if there is some $\bar{a} \in M^n$ s.t.

  for all $\varphi(x_1 \dots x_n) \in p$ $M \models \varphi(\bar{a})$

- Satisfiable type p: "types of possible elements"

  There exists a model that realises

  p

Lemma Σ: a set of formulas on $x_1 \cdots x_n$.
Σ is satisfiable iff each finite subset of Σ is satisfiable.

Pf: Compactness.

Lemma ($\ast$): Let A be an L-str. and Σ a set of L-fmlas on vars $x_1 \cdots x_n$. TFAE:

(1) Σ is an n-type of A.

(2) Every finite subset of Σ is realised in A

(3) A has an elem. ext B that realises Σ

Pf: (3) $\to$ (1) easy.

(1) $\to$ (2): Supp. Σ is an n-type of A ($Th(A) \cup \Sigma$ consistent) So there is $B \models Th(A)$ s.t. for some $b \in B^n$ $B \models \Sigma(b)$,

If Φ is fin sset of Σ $B \models \exists x_1 \cdots x_n \, \Phi$

Since $Th(B) = Th(A)$, $A \models \exists x_1 \cdots x_n \, \Phi$

(2) $\to$ (3) Suppose every finite $\Psi \subset \Sigma$ is realised in A. In particular every fin. sset of $\Sigma \cup Th(A_A)$ is satisfiable.

$\Sigma \cup Th(A_A)$ has a model (Compactness) B'

Let B be the reduct of B' to L.

$B \models \Sigma$.

$\square$

Complete type of T:

- p theory in the language extended by $x_1 \dots x_n$ with $T \subset p$ and complete
- for every fmla $\varphi(x_1 \dots x_n)$, either $\varphi \in p$ or $\neg\varphi \in p$.

Stone space:

$$S^n(T) := \{ p(x_1 \dots x_n) : p \text{ is a complete } n\text{-type of } T \}$$

basic open sets:

$$[\varphi] = \{ p \in S^n(T) : \varphi \in p \}$$

$$[\neg\varphi]$$

Lemma: Stone top. on $S^n(T)$ is compact and totally disconnected.

Pf: Let $C = \{[\varphi_i] : i \in I\}$ be a cover of $S^n(T)$.

We know: $B \models T$ and $a \in B^n$, then $B \models \varphi_i(a)$ some $i \in I$.

So $T \cup \{\neg\varphi_i : i \in I\}$ is NOT satisfiable

By compactness thm, there is $F \subset I$ finite s.t. $T \cup \{\neg\varphi_i : i \in F\}$ is not sat.

So $\{[\varphi_i] : i \in F\}$ cover $S^n(T)$. $S^n(T)$ is compact.

Totally disconnected:

$p \neq q$ in $S^n(T)$

so there is φ s.t. $\varphi \in p$ $\neg\varphi \in q$.

$p \in [\varphi]$ and $q \in [\neg\varphi]$. $\square$

Are there types? $\searrow$ $(\mathbb{Z}, <)_{\mathbb{N}}$

$\Sigma = \{x > n : n \in \mathbb{N}\}$

$\Psi = \{x > n_1, x > n_2, \ldots, x > n_k\}$

$N = \max\{n_1 \ldots n_k\}$ $N+1$ satisfies Ψ

$\Sigma = \{x > 0\} \cup \{x < n : n \in \mathbb{N}^+\}$ in $\underline{(\mathbb{Z}, <)_{\mathbb{N}}}$.

$\Psi = \{x > 0, x < 1\}$

$\Sigma = \{x > 0\} \cup \{x < 1/n : n \in \mathbb{N}^+\}$ is $(\mathbb{Q}, <)_{\mathbb{N}, \{1/n : n \in \mathbb{N}\}}$

$(\mathbb{Q}, +, \cdot, {}^{-1}, 1, 0)$

$\varepsilon \quad 1/\varepsilon \quad 1+\varepsilon \quad$ &c.

Proposition $\mathcal{M}, \mathcal{N}$ L-strs. $(, , ,)$

$f : \mathcal{M} \to \mathcal{N}$ isomorphism.

Then $\bar{a} \in \mathcal{M}^n$ has the same type in $\mathcal{M}$ as $f(\bar{a}) = (f(a_1), \dots, f(a_n))$ in $\mathcal{N}$

Pf: Back and forth.

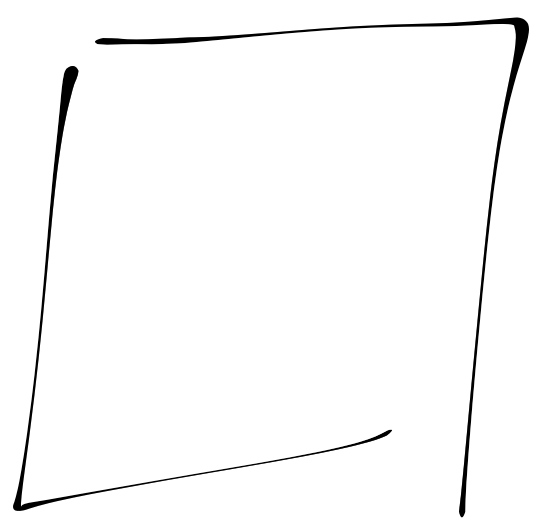
Cor: Tuples in the same orbit of the aut. group of a structure have the same type.

$$tp(\bar{a}) = \{\varphi(\bar{x}) : \mathcal{M} \models \varphi(\bar{a})\} = tp(f(a))$$

Having same type:

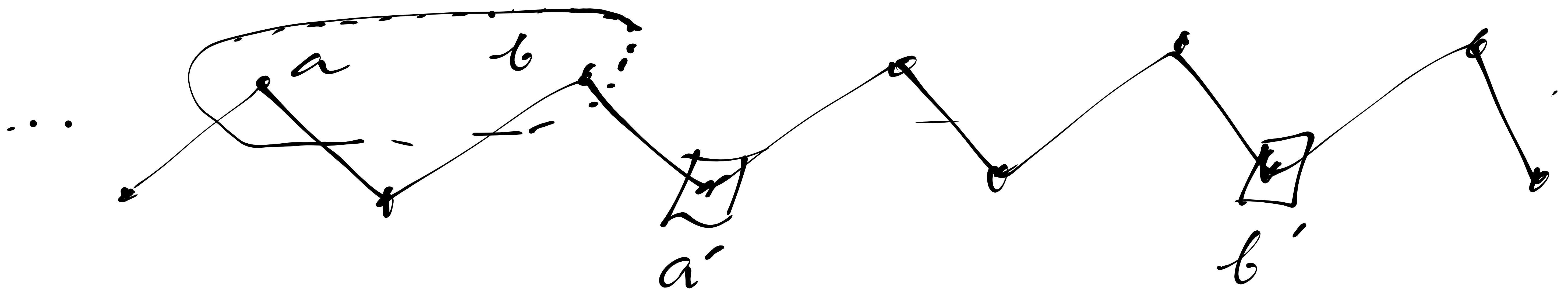
$$tp(a) = tp(f(a))$$

Same theories



M

Elementary equivalence



$tp(a,b) \ni$ "$\exists y : y \sim a \wedge y \sim b$" $\varphi(x,y)$

$tp(a',b') \not\ni \varphi(x,y)$

Saturation

Definition: for an infinite cardinal κ, $\mathcal{M}$ is κ-saturated if all 1-types over A are realised in $\mathcal{M}$, for all $A \subseteq M$ with $|A| < \kappa$.

$\mathcal{M}$ is saturated if it is $|M|$-saturated

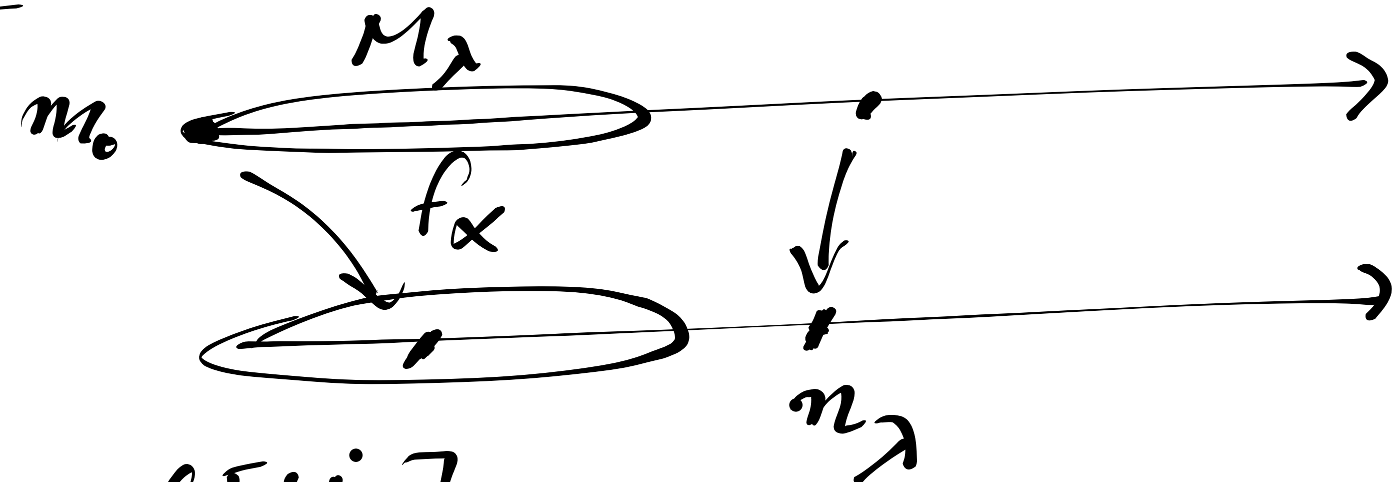
Thm: If $\mathcal{M}, \mathcal{N} \models T$ are saturated and $|\mathcal{M}| = |\mathcal{N}|$, then $\mathcal{M} \cong \mathcal{N}$.

$\kappa = |\mathcal{M}| = |\mathcal{N}|$

Proof Back-and-forth

$\mathcal{M} = \{m_i : i < \kappa\}$

$\mathcal{N} = \{n_i : i < \kappa\}$



$f[M_\lambda]$

$f_{\alpha+1}$

At limit stages, $f_\beta = \bigcup_{\alpha<\beta} f_\alpha$

Ordinals

↳ Well-ordered by $\in$

$\{\}$ $\{\emptyset\}$ $\{\emptyset, \{\emptyset\}\}$

0 1 2 $n+1 = n \cup \{n\}$

→ ω well ordered by $\in$

first limit ordinal.

$\boxed{\text{Thomas Jech: Set theory}}$

Lemma Every L-str M has an elementary extension N
(**) that realises all 1-types over M.

(Example: $(\mathbb{Q}, <)$
$\{x < 1/n : n \in \mathbb{N}\}$)

Completion of orders, posets &c.

Proof: - Enumerate all 1-types $/M$
$(p_\alpha)_{\alpha \in \lambda}$

- Idea: Construct elem. chain
$M_\mu = M_0 \preccurlyeq M_1 \preccurlyeq M_2 \preccurlyeq \cdots \preccurlyeq M_\beta \preccurlyeq \cdots$
$(\beta < \lambda)$

Obs: We already know this for a single type
Lemma ✪

If we have $(M_\alpha)_{\alpha<\beta}$ already constructed

$M_{\alpha+1}$: Apply Lemma ✪ to M_α to obtain an elem. ext of M_α that realises $p_{\alpha+1}$

M_η (η limit): apply Tarski's elem. chain thm

$$M_\eta = \bigcup_{\beta<\eta} M_\beta$$

Take $N = \bigcup_{\beta<\lambda} M_\beta$.

$\square$

<u>Thm</u> M: an L-str.

κ: an infinite cardinal.

Then M has a κ-saturated elem. str. N

<u>Pf</u>: Chain $(M_\alpha)_{\alpha < \kappa^+}$

Start from $M_0 = M_M$

Induction:

$M_{\alpha+1}$ obtained from M_α by applying Lemma ☆☆.

$M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ (Tarski)

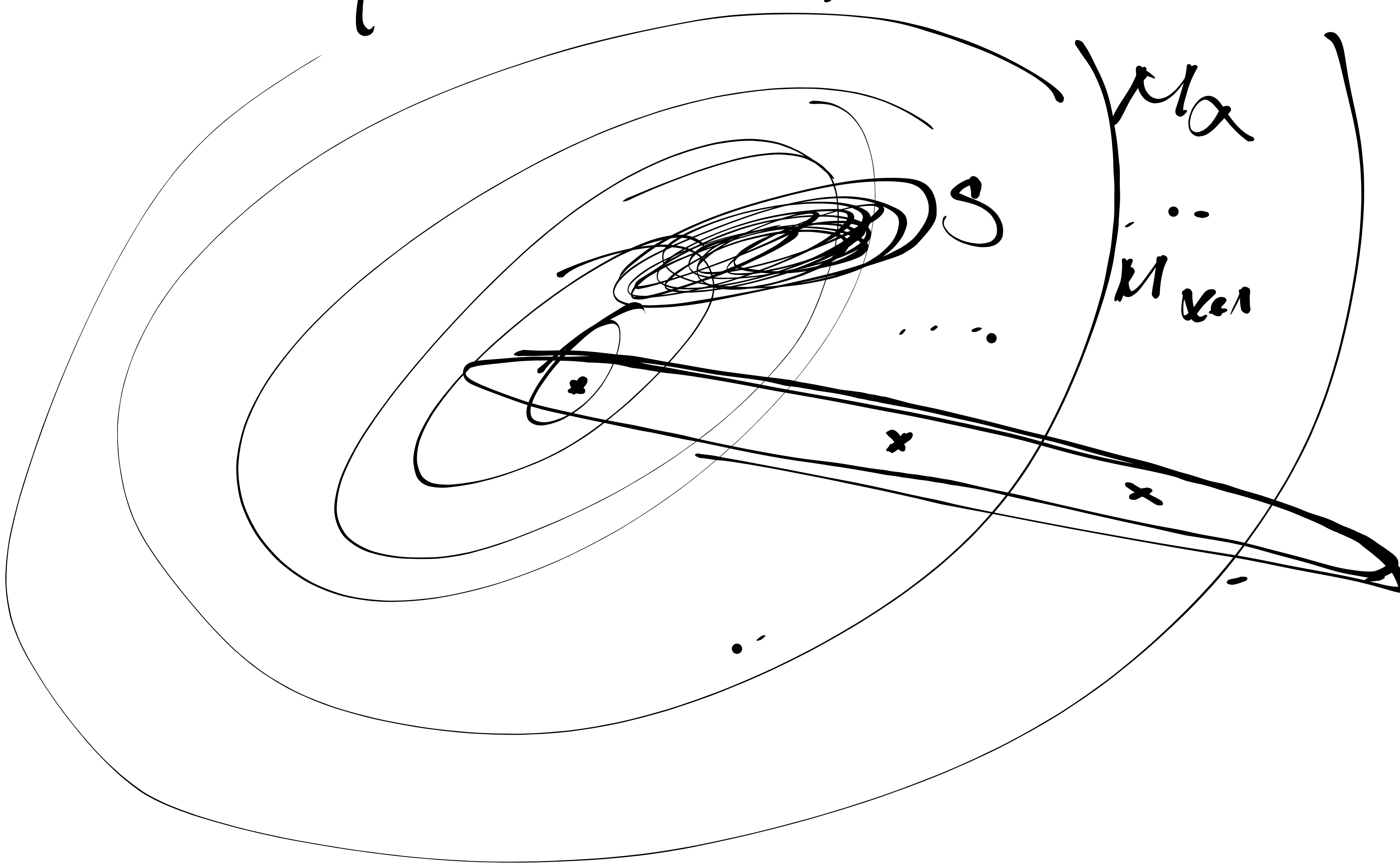
Let $N = \bigcup_{\alpha<\kappa^+} M_\alpha$

N is κ-saturated.

Take $S \subseteq N$, $\boxed{|S| < \kappa.}$

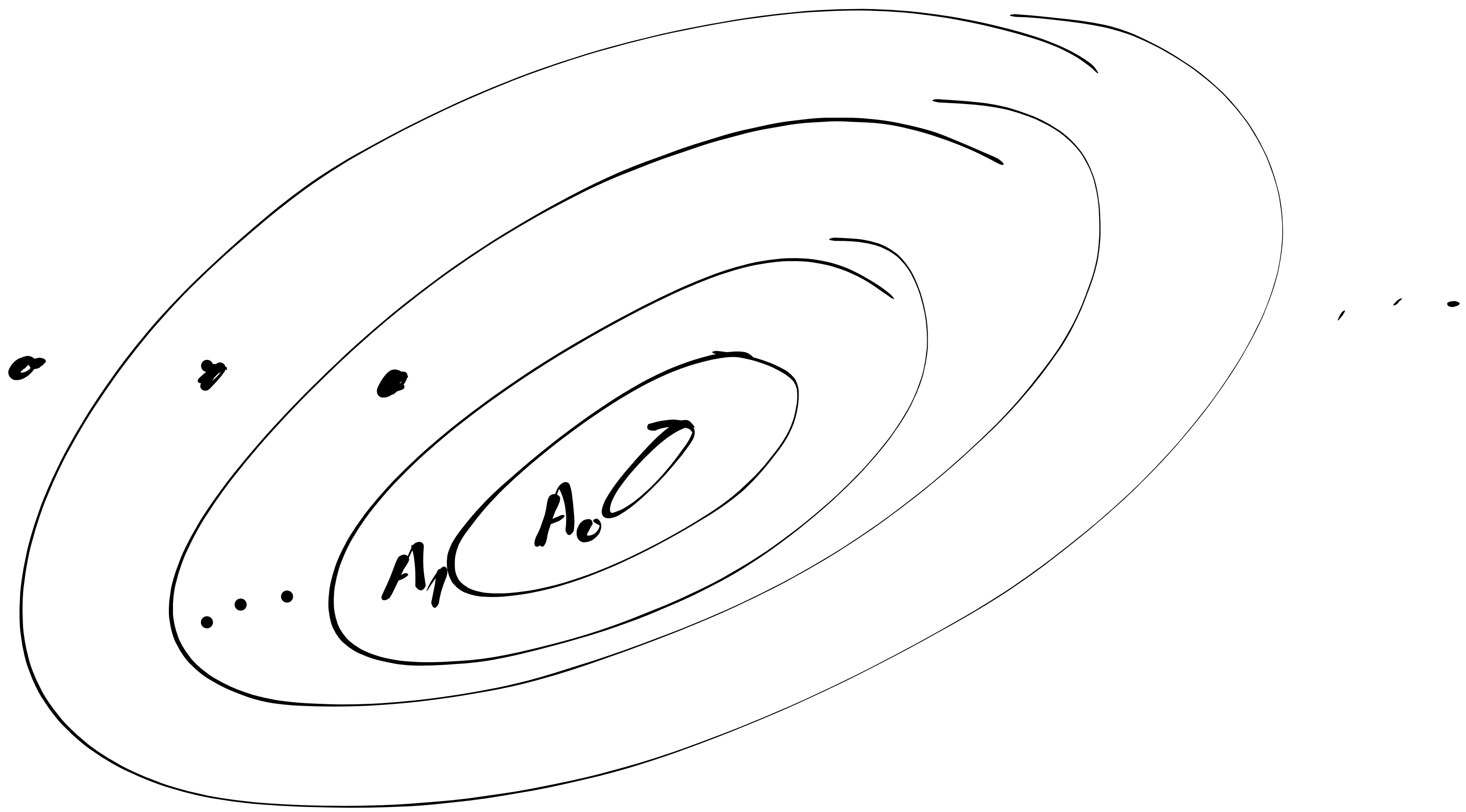
Then $S \subseteq M_\alpha$ for some $\alpha < \kappa^+$

(otherwise, there would be a cofinal sequence of κ^+ of length $< \kappa$)



$< \kappa^+$

Then at step $\alpha+1$, all types over S are realised.



"If X is a finite subset of $\bigcup A_i$, then there is a minimal N s.t. $X \subseteq A_N$"

Doing things carefully (accounting for cardinalities

Thm : L a language

$\kappa > |L|$ infinite.

Every L-str of size 2^κ has a κ^+-saturated ext of size 2^κ.

Assuming GCH, there are saturated models of size κ^+ for all κ.

Without CH,

$(\mathbb{Q}, <)_{\mathbb{Q}}$: $2^{\aleph_0}$ 1-types. If $2^{\aleph_0} > \aleph_1$, then there is no sat. model of size $\aleph_1$