~ Selected Chapters in Combinatorics ~

- Some topology
- Types
- Saturation

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<u>Def</u>: X: a set

$\tau \subseteq 2^X$ is a topology if

1. $\phi, X \in \tau$
2. whenever $U_1, \dots, U_n \in \tau$, $\cap\{U_1 \dots U_n\} \in \tau$
3. $A \neq \phi$ if $U_a \in \tau$ for all $a \in A$. then

   $\cup\{U_a : a \in A\} \in \tau$

Elts. of τ are called <u>open sets</u>.

$Y \subseteq X$ is <u>closed</u> if $X \setminus Y$ is open

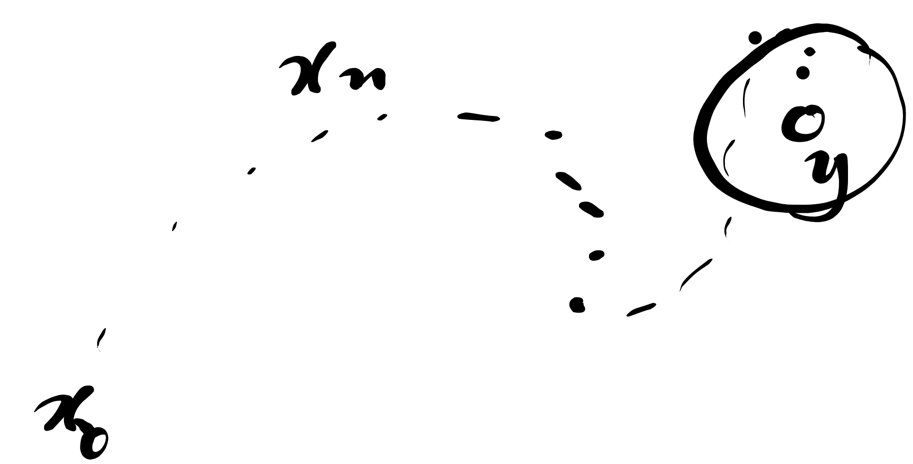
Examples

- $\{\phi, X\}$
- 2^X is a top. on X
- (X, d) a metric sp.

$$B_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$$

$\mathbb{R}$:

$$d(x,y) = |x - y|$$



Convergence, connectivity, continuity

T_0: $\forall x, y \in X$, $x \neq y$ there exists open $W \in \tau$ s.t.

either $x \in W$ and $y \notin W$

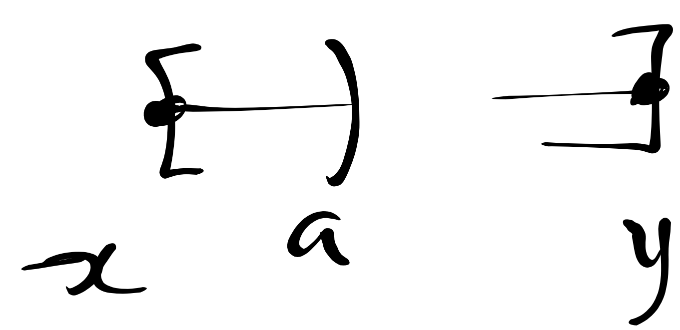
OR $y \in W$ and $x \notin W$

$\boxed{(X, \tau) \text{ top sp}}$

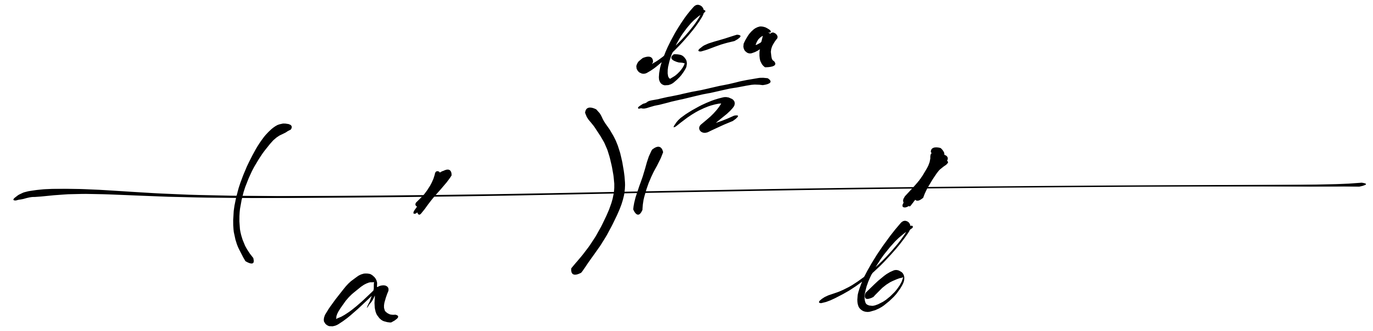
<u>Example</u>

$[1, 1] \subseteq \mathbb{R}$

Basic open: $[-1, b)$ (a, b) $(a, 1]$



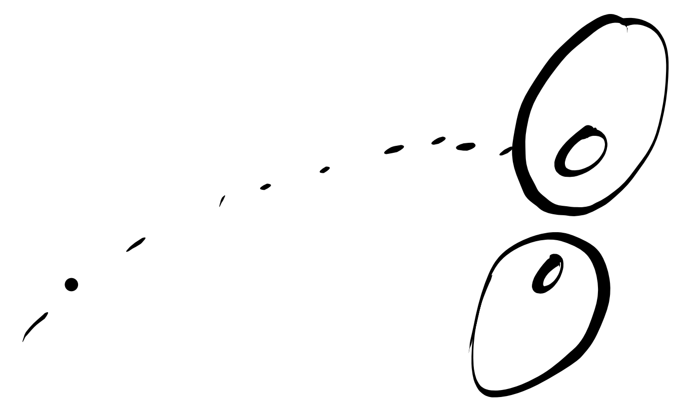
T_1: for all distinct $x, y \in X$ there exists $W \in \tau$ s.t. $x \in W$ $y \notin W$.

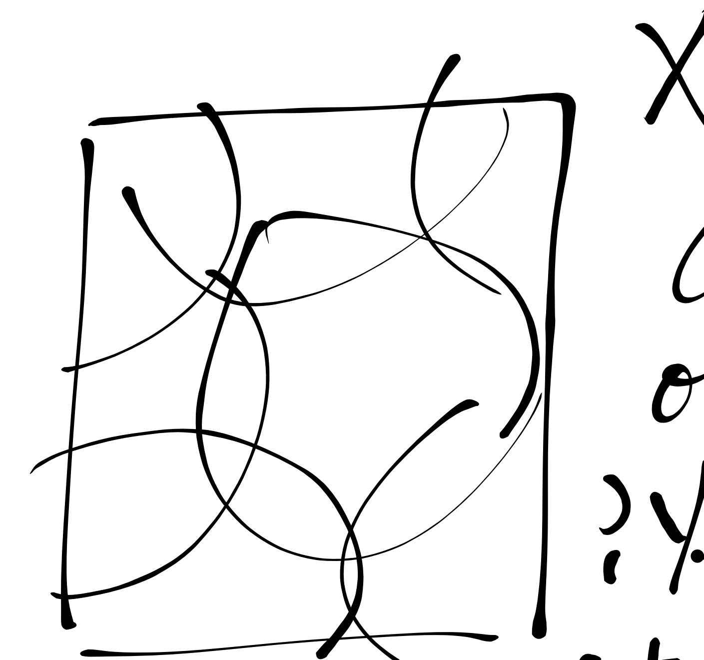


T_2 (Hausdorff spaces)

for all distinct $x, y \in X$ there exist disjoint open U, V s.t. $x \in U$ $y \in V$.

Example / prop : Sequences have at most one limit in Hausdorff spaces.



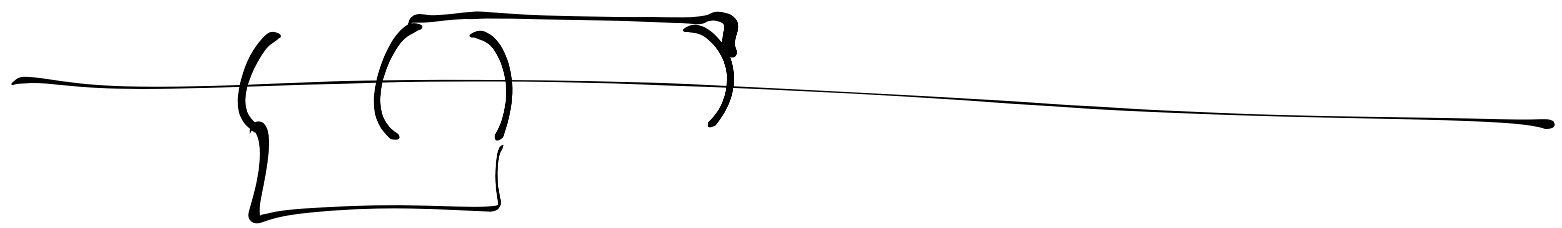


Any coll of sets $\{Y_i : i \in I\}$ s.t $\bigcup Y_i = X$

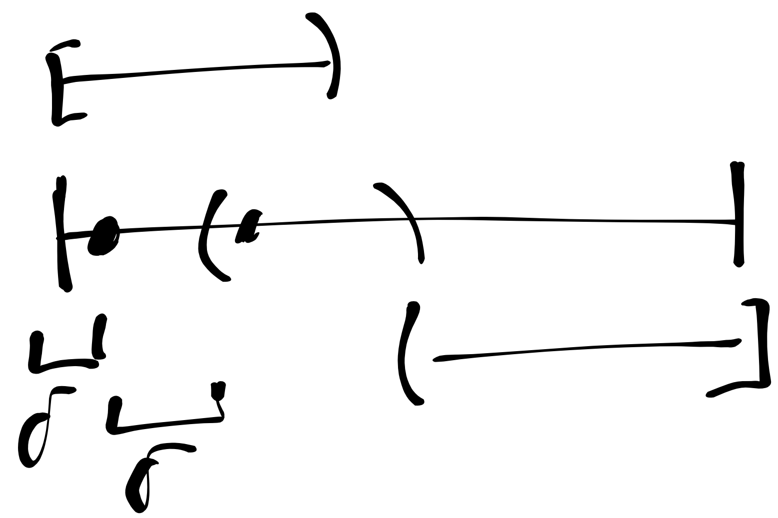
<u>Compactness</u> :

A top. sp. (X, τ) is compact if every cover of X by open sets contains a <u>finite</u> subcover.

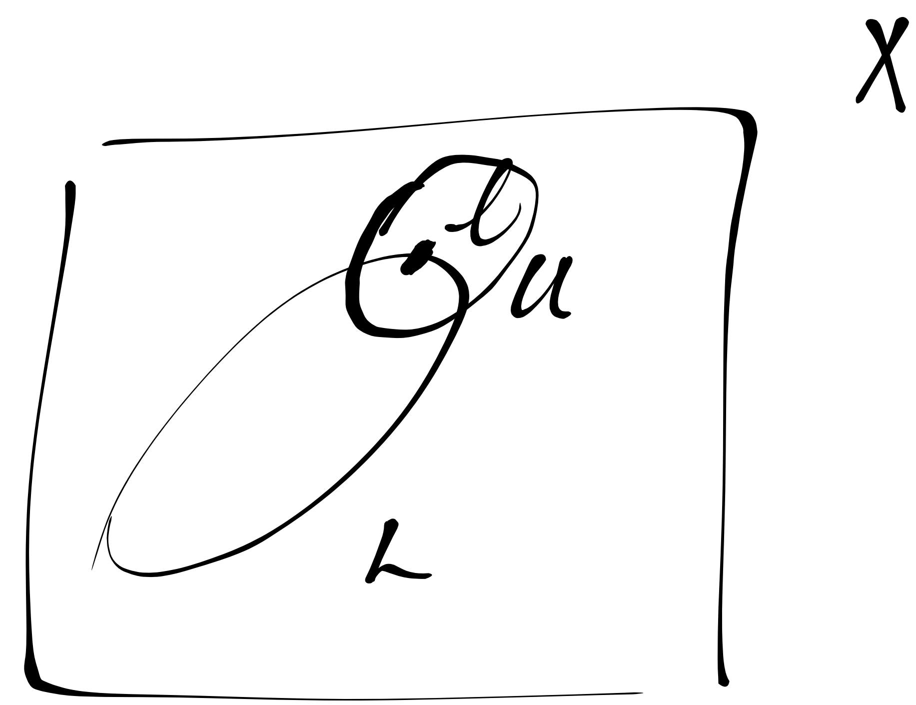
$\mathbb{R}$



$[0, 1]$



Prop Every infinite subset L of a compact space contains a limit point of L.



√ Every sequence in a closed and bdd subset of $\mathbb{R}^n$ contains a convergent subsequence

$\forall U \in \tau \, (l \in U \to \exists s \neq l, \, s \in L, \, s \in U)$

$Y \subseteq 2^X$ has the fin. int. prop if for any finite $F \subset Y$

$$\cap F \in Y$$

In compact spaces, families of closed sets with FIP have nonempty intersection.

$I \neq \emptyset$

X_i is a set for each $i \in I$

$$\prod_{i \in I} X_i = \left\{ f : I \to \bigcup \{X_i : i \in I\} \;\middle|\; \begin{array}{c} f(i) \in X_i \\ (\text{all } i \in I) \end{array} \right\}$$

Now supp (X_i, τ_i) is a top. sp.

$W \subset \prod_{i \in I} X_i$ is open if for every $f \in W$ and $j \in I$

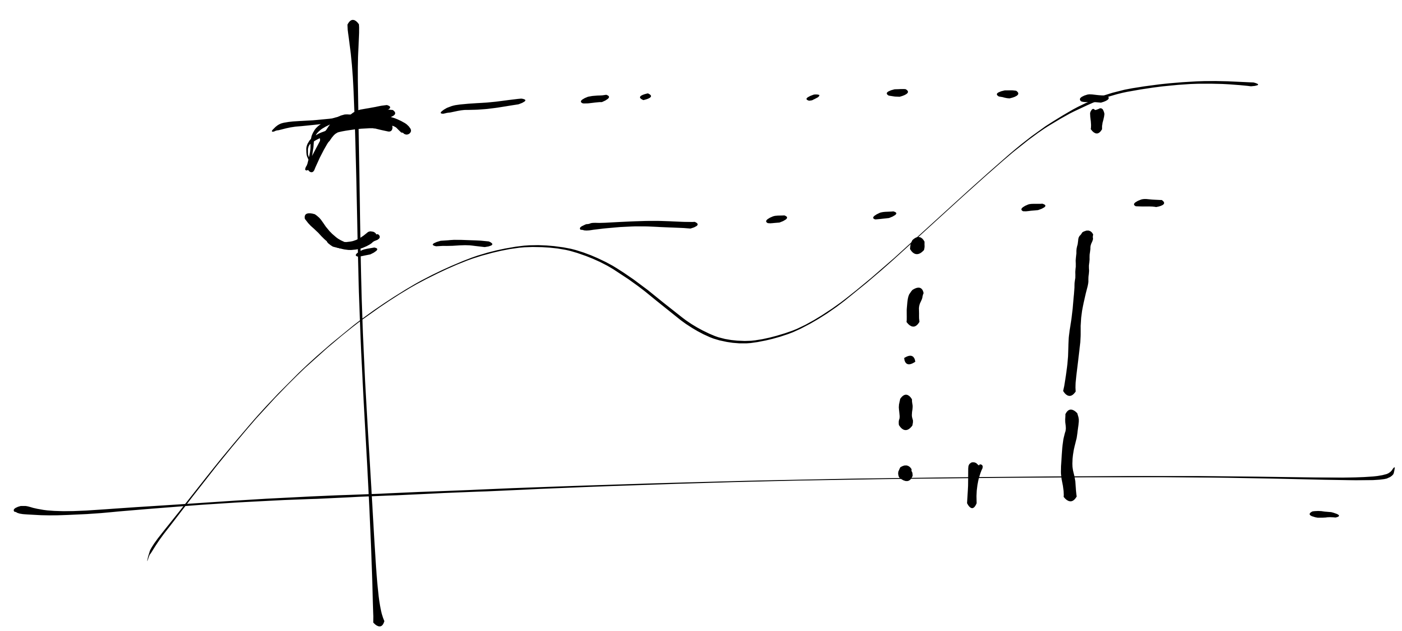
there exists $U_j \in \tau_j$ s.t.

- $f(j) \in U_j$
- $\prod U_j \subseteq W$
- $U_j = X_j$ for all but finitely many $j \in I$.

- All projections are <u>continuous</u>

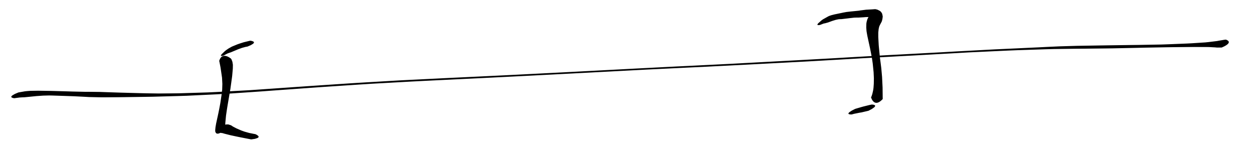
$$h : (X, \tau) \longrightarrow (Y, \eta)$$

$h^{-1}(U)$ is open in (X, τ) for all open subsets of $\mathrm{im}(h)$.



$$f : \mathbb{R} \to \mathbb{R}$$

∞?



Clopen sets with
$\cap, \cup, \neg$
is Bool. alg.

Stone - Čech compactification βX

If X is discrete, βX is the set of all ultrafilters on X

$\{ \{F : U \in F\} : U \subseteq X \}$

Stone's representation thm B Boolean algebra

$S(B)$ = all ultrafilters on B

Every Boolean algebra is iso. to the algebra of clopen sets of $S(B)$

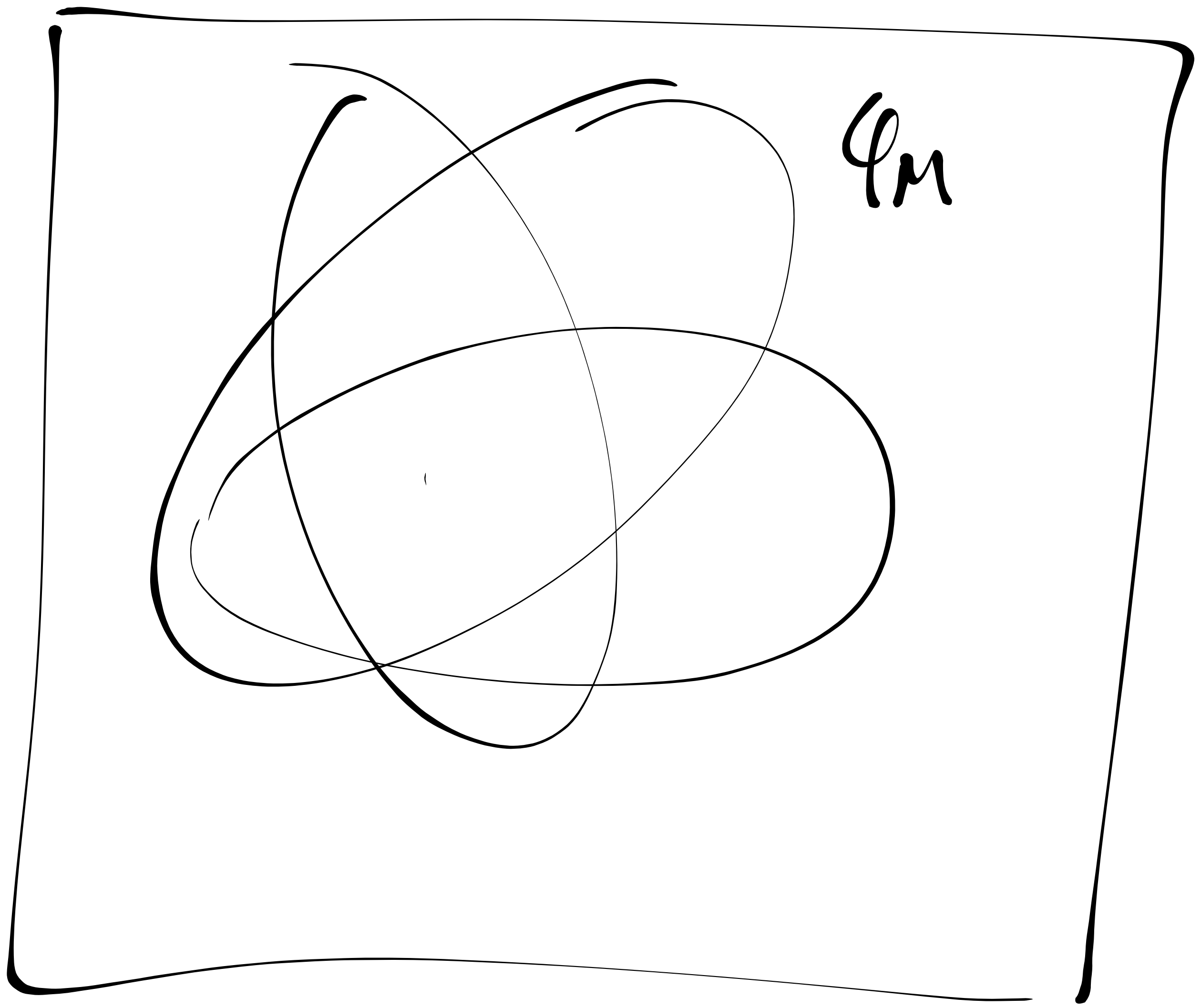
Types

types are sets of formulas that "correspond" to classes of elements in elementary extensions of $M \models T$

Σ a set of fmulas with free vars $x_1 \cdots x_n$

Σ is realised in $M \models T$ if there exist $a_1 \cdots a_n \in M$ s.t. $M \models \varphi(a_1 \cdots a_n)$ for each $\varphi(x_1 \cdots x_n) \in \Sigma$

$\Sigma = \{x = x\}$



M

$\Sigma(x)$

$\varphi(x) \in \Sigma$

$\varphi_M = \{ y \in M : M \models \varphi(y) \}$

Σ fmlas on $x_1 \cdots x_n$ is <u>satisfiable</u> if there exists a model $M \models T$ in which Σ is realised.

<u>Lemma</u> A set of fmlas Σ on $x_1 \cdots x_n$ is satisfiable iff each fin. subset of Σ is satisfiable

<u>Pf</u> - Extend the language with $x_1 \cdots x_n$ as constants
- Apply compactness thm.

Def: a set of fmlas Σ in vars $x_1 \cdots x_n$ is an n-type of T if $\Sigma \cup T$ is satisfiable

an n-type of a structure M is an n-type of $Th(M)$

Lemma Let M be an L-str and Σ a set of fmlas free vars $x_1 \cdots x_n$

TFAE

(1) Σ is an n-type of M

(2) Every finite subset of Σ is realised in M

(3) M has an elementary ext that realises Σ

Pf (1) $\to$ (2)

There is some $N \models Th(M)$ and $c \in N^n$

$N \models \Sigma(c)$. Let $\psi \subset \Sigma$ finite

$N \models \exists x_1 \cdots x_n (\bigwedge \psi)$

$M \models \exists x_1 \cdots x_n (\bigwedge \psi) \leftarrow$

(2) $\to$ (3): Supp every finite $\Psi \subset \Sigma$ is realised in M.

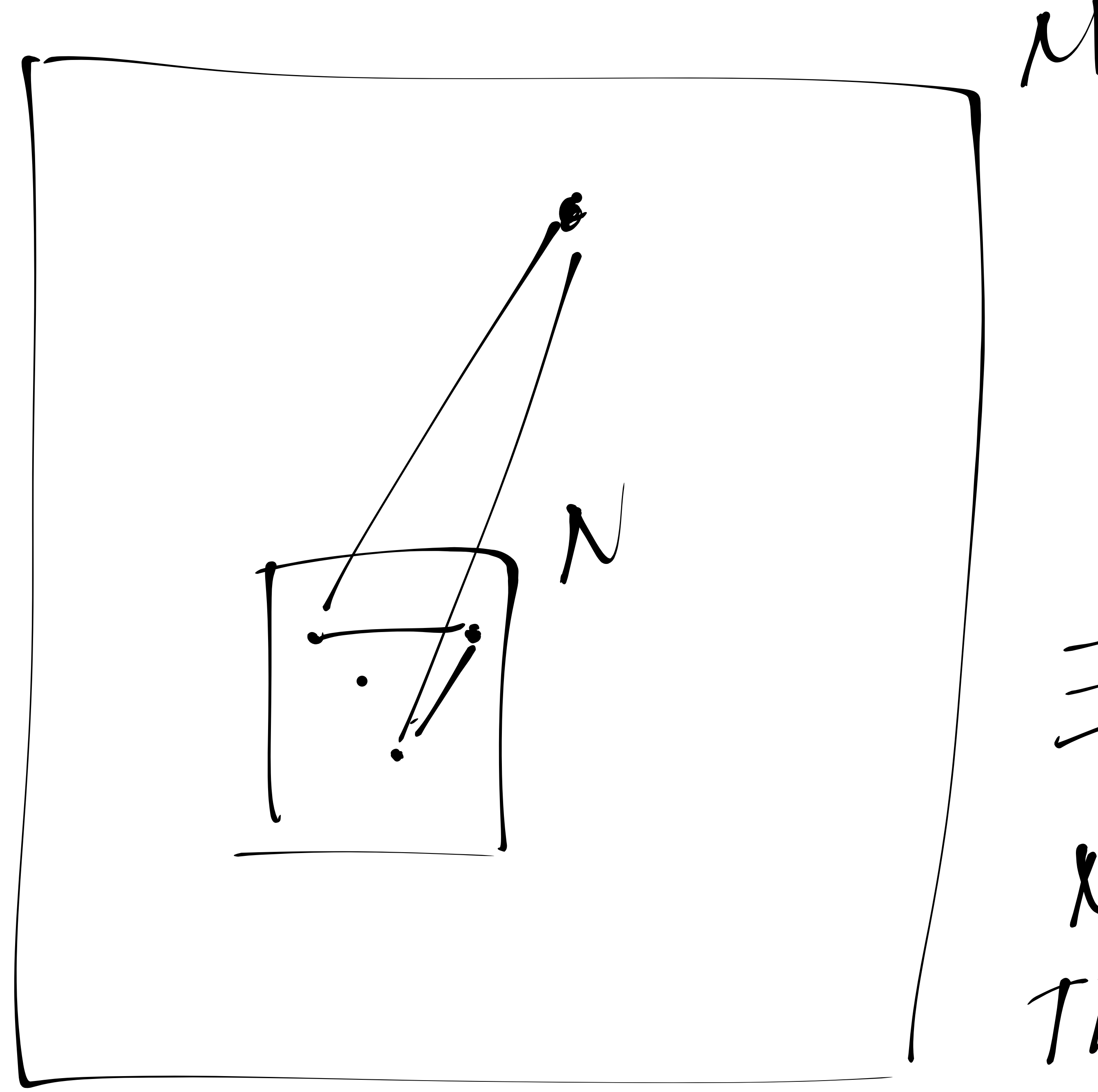
Every finite subset of $\Sigma \cup T(M_M)$ satisfiable.

So $\Sigma \cup Th(M_M)$ has a model N (compactness)

Let N' be the L-reduct of N.

N' realises Σ.

(3) $\to$ (1) Nothing to do here. $\square$



$i : \mathcal{N} \to \mathcal{M}$

$\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$

$\exists x\, \varphi(x, \bar{a}) \quad \bar{a} \in \mathcal{N}^n$

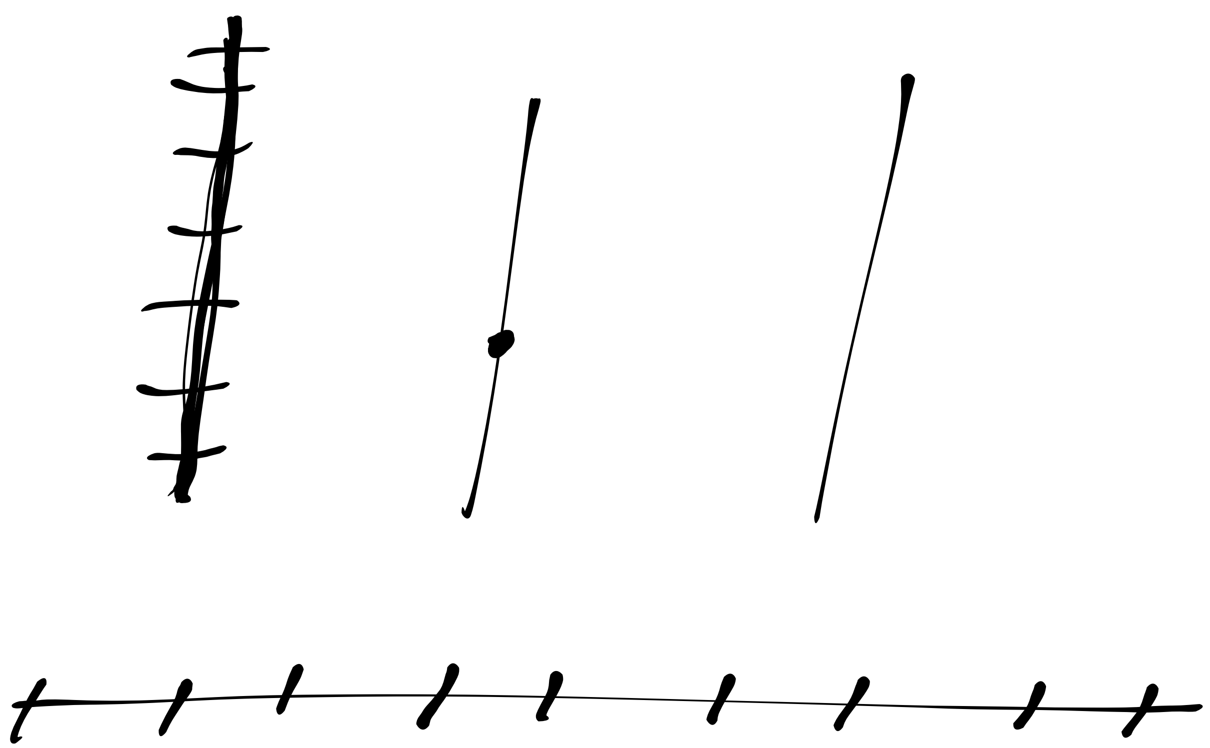
$\mathcal{M} \models \exists x\, \varphi(x, \bar{a})$

Then $\mathcal{N} \models$

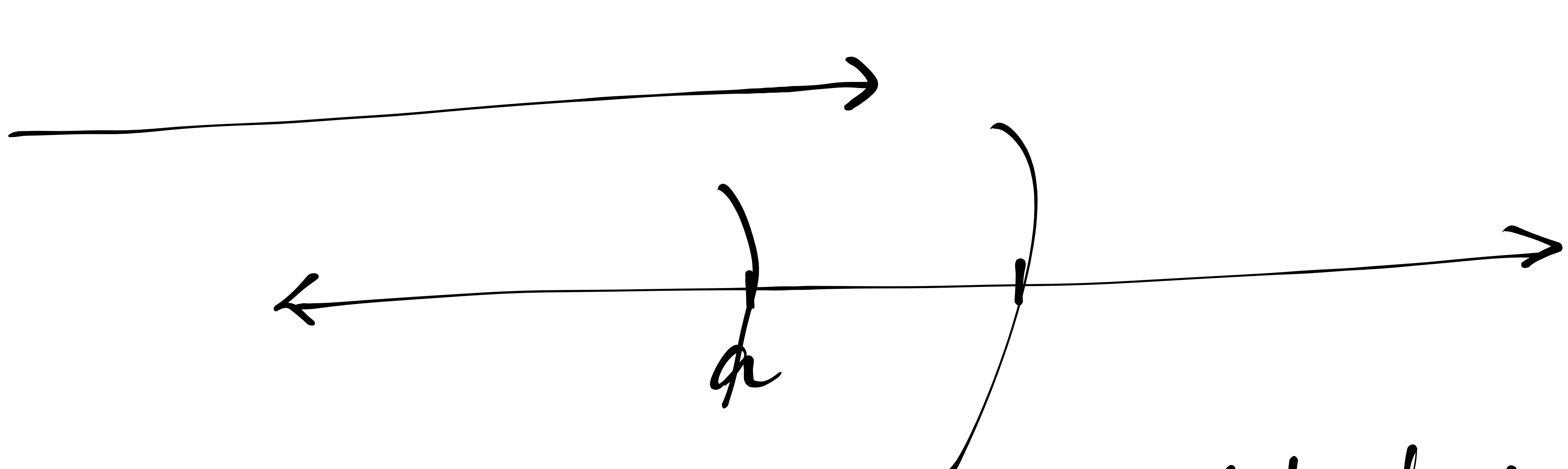
$\{x > n : n \in \mathbb{N}\} = p \qquad (\mathbb{N}, <)_{\mathbb{N}}$

There exist models of $Th(\mathbb{N}, <)_{\mathbb{N}}$ in which p is satisfied

Models of $(\mathbb{Z}, <)_{\mathbb{N}}$



$(\mathbb{Q}, <)_{\mathbb{Q}}$



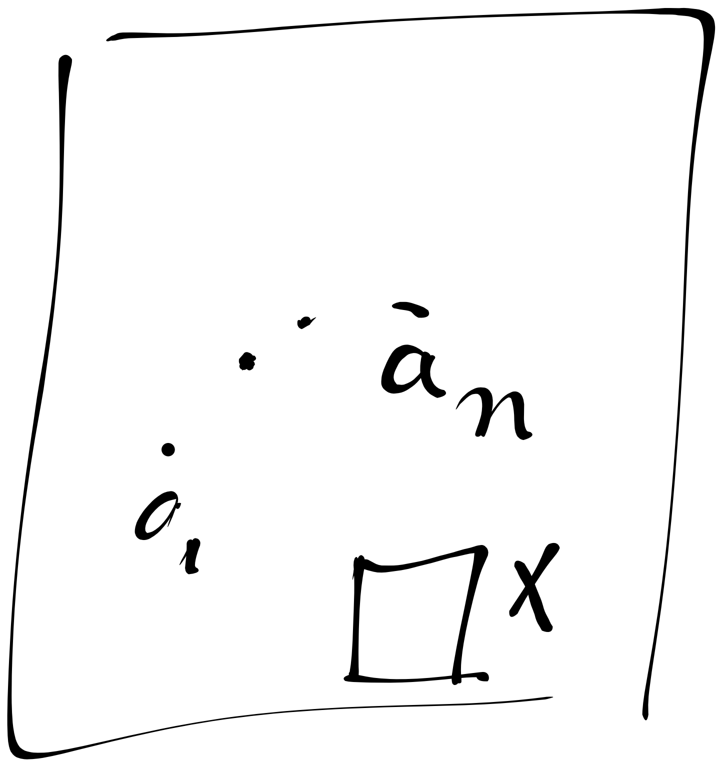
D : Dedekind cut

$\{x > a\}$

$\{\text{"}x \in D\text{"}\}$

$\rightsquigarrow$ Completions

$q \in D$ $\{x > q : q \in D\}$



$M \models T$

$\{\varphi(x_1 \dots x_n) : M \models \varphi(a_1 \dots a_n)\}$

"the type of $a_1 \dots a_n$ in M"

"the type of $a_1 \dots a_n$ over X"

p is a Complete type if

- Set of fmlas p s.t. (0) $T \subset p$

→ $p \cup \{\varphi(x_1 \dots x_n)\}$ is $\underline{\text{not}}$ sat for any $\varphi \notin p$

- $\varphi \in p$ or $\neg\varphi \in p$

$\mathcal{L}$: all formulas on free vars $\underline{x_1 \dots x_n}$

$\wedge \quad \vee \quad \neg$

T