~ Selected Chapters in Combinatorics ~

Lecture 5

* Fraïssé amalgamation

* Types

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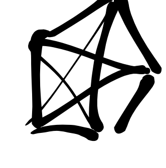
Def (homogeneous structure):

L-str M is <u>homogeneous</u> if every isomorphism between <u>finite</u> (finitely generated) substructures is restriction of some automorphism of M.

Def (age):

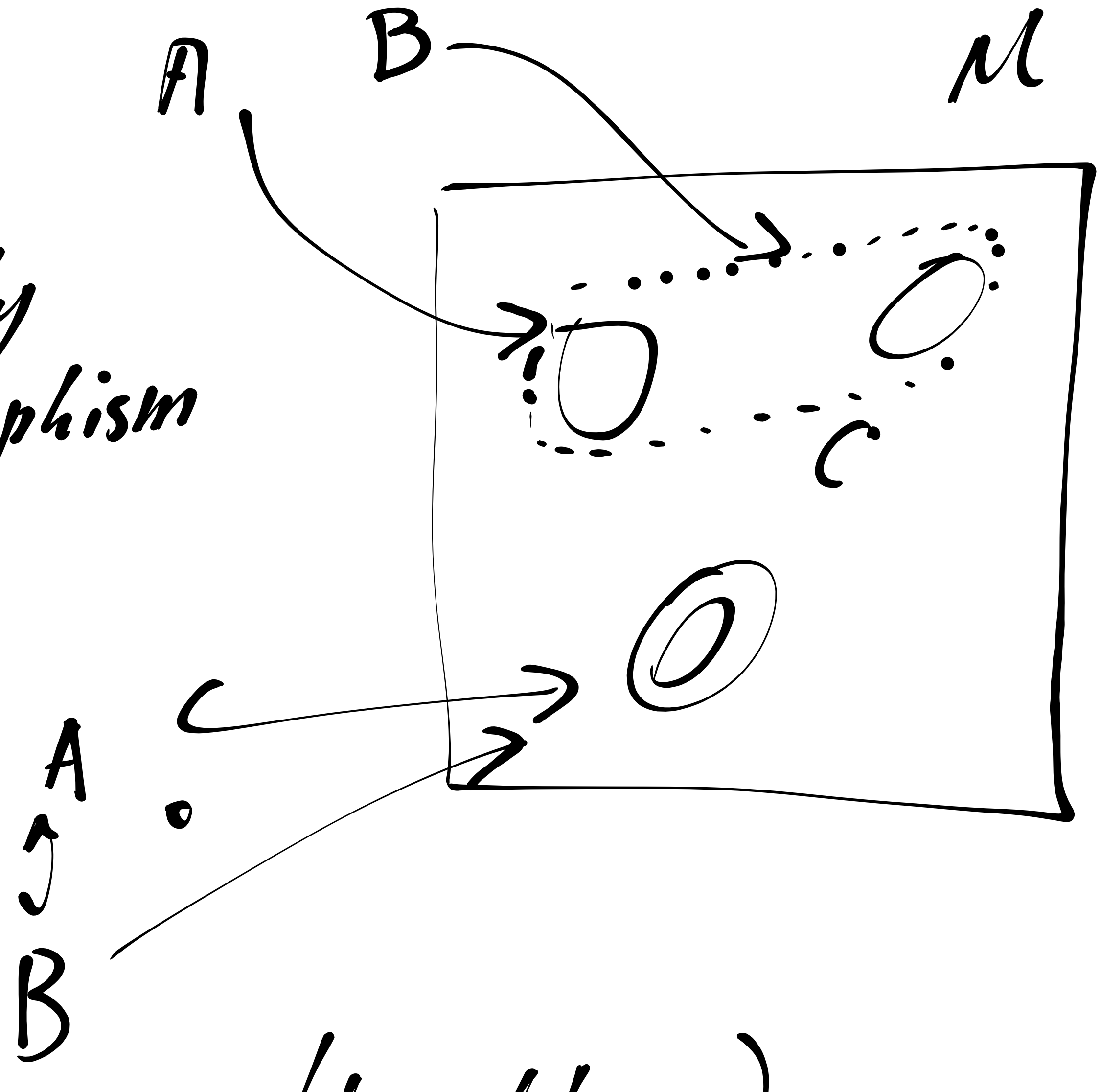
L-str. M.
$Age(M)$ is the class of all finite L-strs that embed into M.

Examples:

- $(\mathbb{Q}, <)$ $\quad Age((\mathbb{Q}, <))$ is all finite lin. orders.
- $(\omega, =)$ $\quad$ all finite sets.
- $\coprod_{n \in \omega} K_n$ $\quad$ / $\quad \Delta$ 

Examples (hom. str)

Properties of the age of a ctble rel. str.

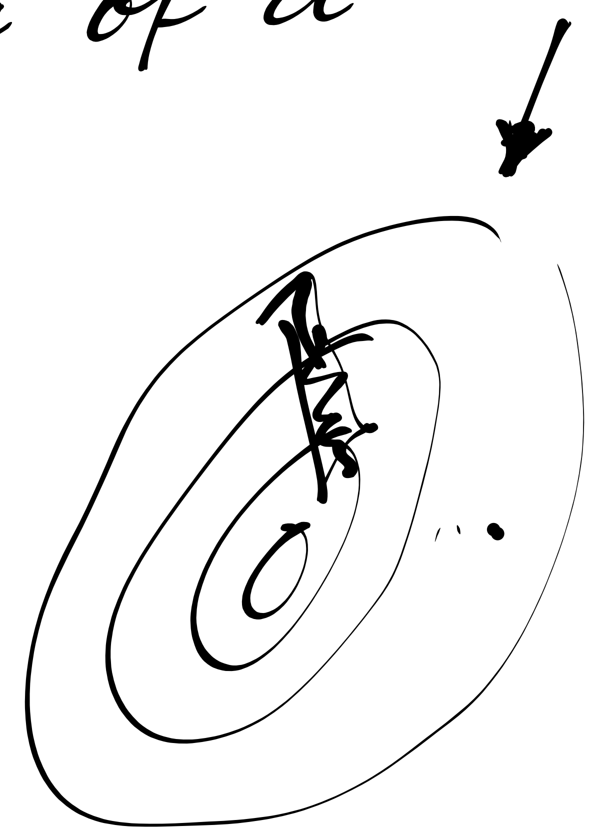


(i) only countably many isomorphism types

(ii) closed under isomorphisms, substructures —— (hereditary)

(iii) (JEP) for all $A, B \in Age(M)$ there exist $C \in Age(M)$ into which A, B embed.

Prop: Let $\mathcal{C}$ be a class of finite L-strs. If $\mathcal{C}$ satisfies (i)–(iii) then $\mathcal{C}$ is the age of a countable L-str.



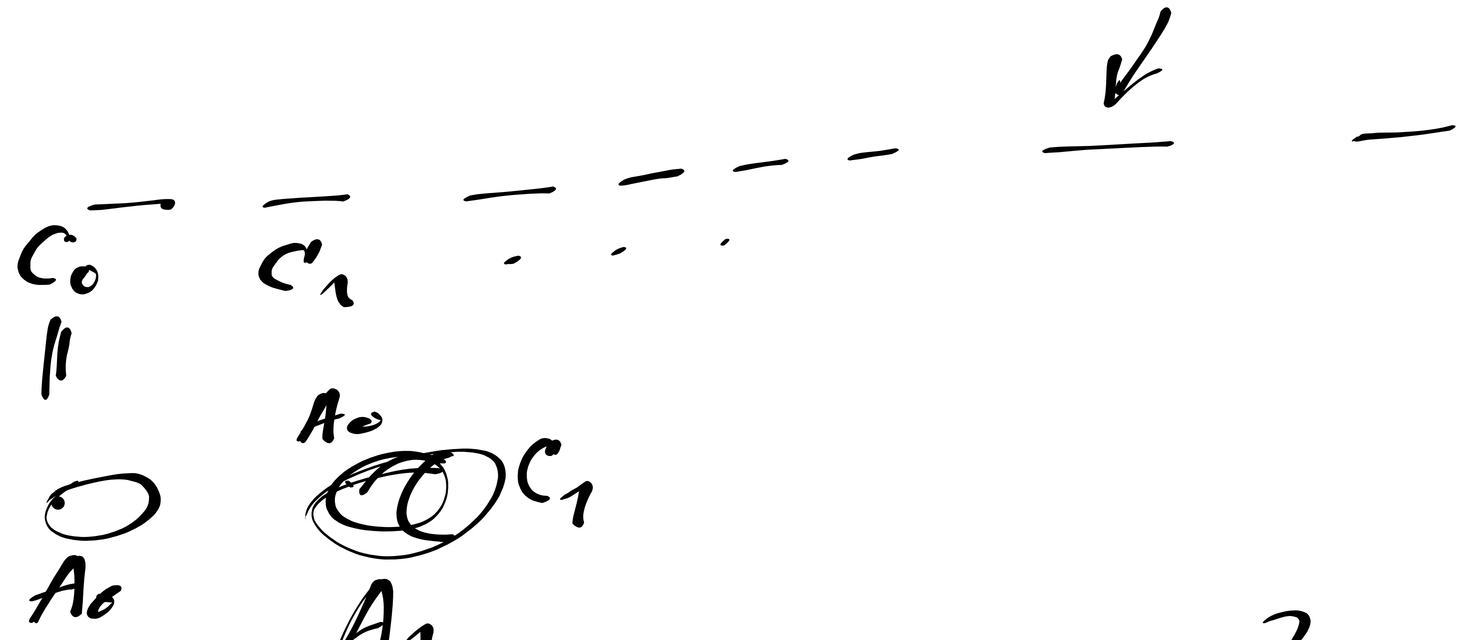
Pf: Let $C_0, C_1, \dots$ be representatives of iso. classes in $\mathcal{C}$

(i)

Construct a chain $A_0 \subseteq A_1 \subseteq \dots$

$A_0 := C_0$

If A_i has been chosen, then find $B \in \mathcal{C}$ s.t. $A_i \hookrightarrow B$ and $C_{i+1} \hookrightarrow B$



Let $A = \bigcup\{A_n : n \in \omega\}$

Verify $\mathrm{age}(A) \underset{\substack{\supseteq \\ \subseteq}}{=} \mathcal{C}$ $\square$

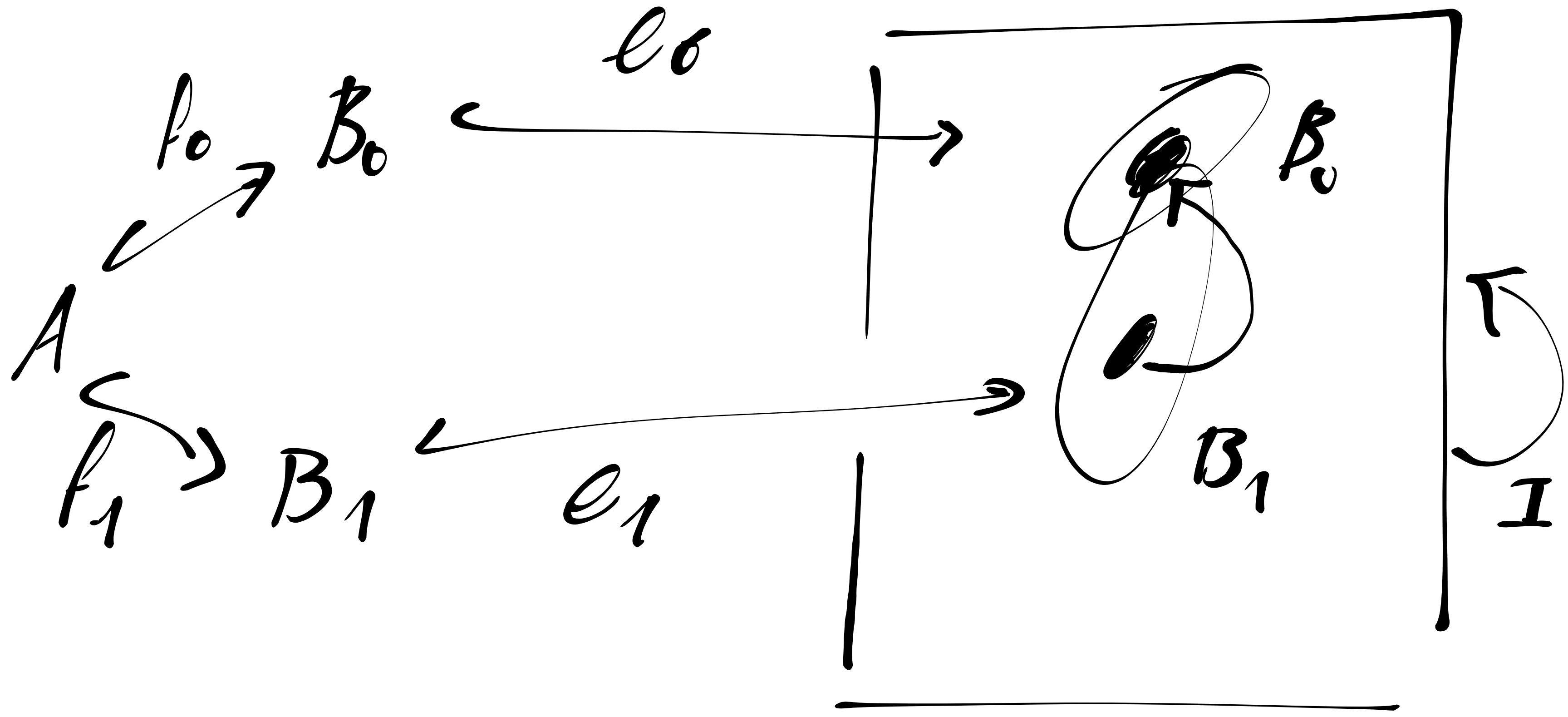
Prop: Let M be a homogeneous L-structure. then

(AP) (iv) for all $A, B_0, B_1 \in Age(M)$ and embeddings $f_i : A \to B_i$ there exist $C \in Age(M)$ and embeddings $g_i : B_i \longrightarrow C$ s.t. $g_0 \circ f_0 = g_1 \circ f_1$

$$\begin{tikzcd}
 & B_0 \arrow[dr, hook, "g_0"] & \\
A \arrow[ur, hook, "f_0"] \arrow[dr, hook, "f_1"'] & & C \\
 & B_1 \arrow[ur, hook, "g_1"'] &
\end{tikzcd}$$

Amalgamation Property.

Pf: Take A, B_0, B_1 as in the statement

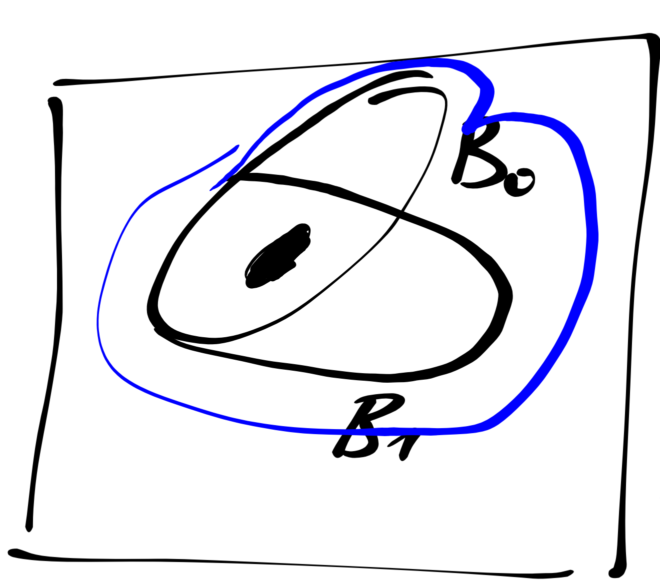


$$\left.\begin{array}{l} A' = e_1 \circ f_1 [A] \\ A'' = e_0 \circ f_0 [A] \end{array}\right\} \text{isomorphic}$$

there exists $i : A' \longrightarrow A''$ isomorphism

By homogeneity, i is restriction of an automorphism I

$$\begin{tikzcd} & B_0 \arrow[dr] & \\ A \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & C \\ & B_1 \arrow[ur] & \end{tikzcd}$$



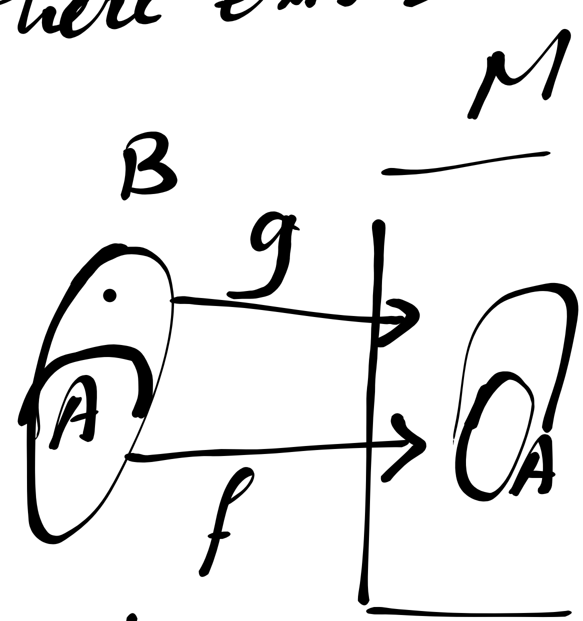
finite iso to $C \in Age(M)$

II

Extension property

M satisfies EP if for all
$e : A \hookrightarrow B \in Age(M)$ (embedding)
for all $f : A \to M$ there exists
$g : B \to M$

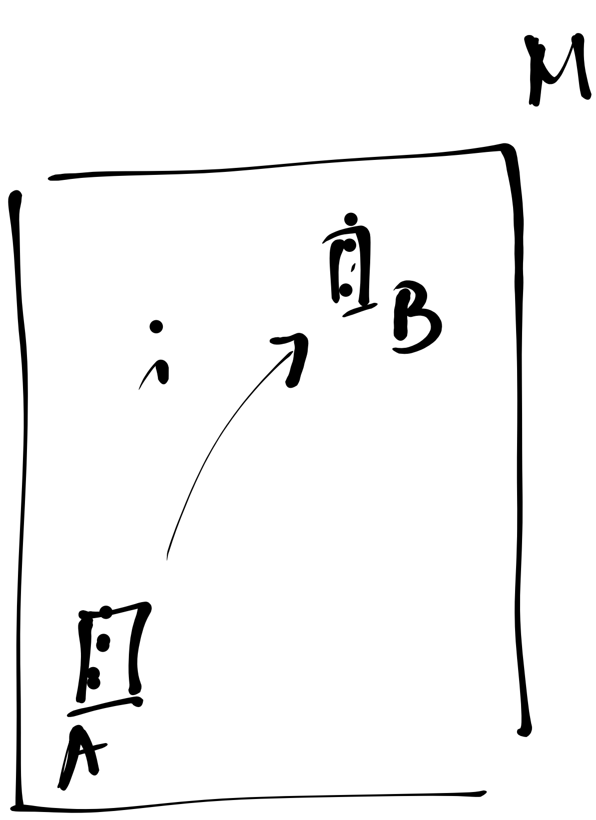
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e"'] & M \\ B \arrow[ur, dashed, "g"'] & \end{tikzcd}$$



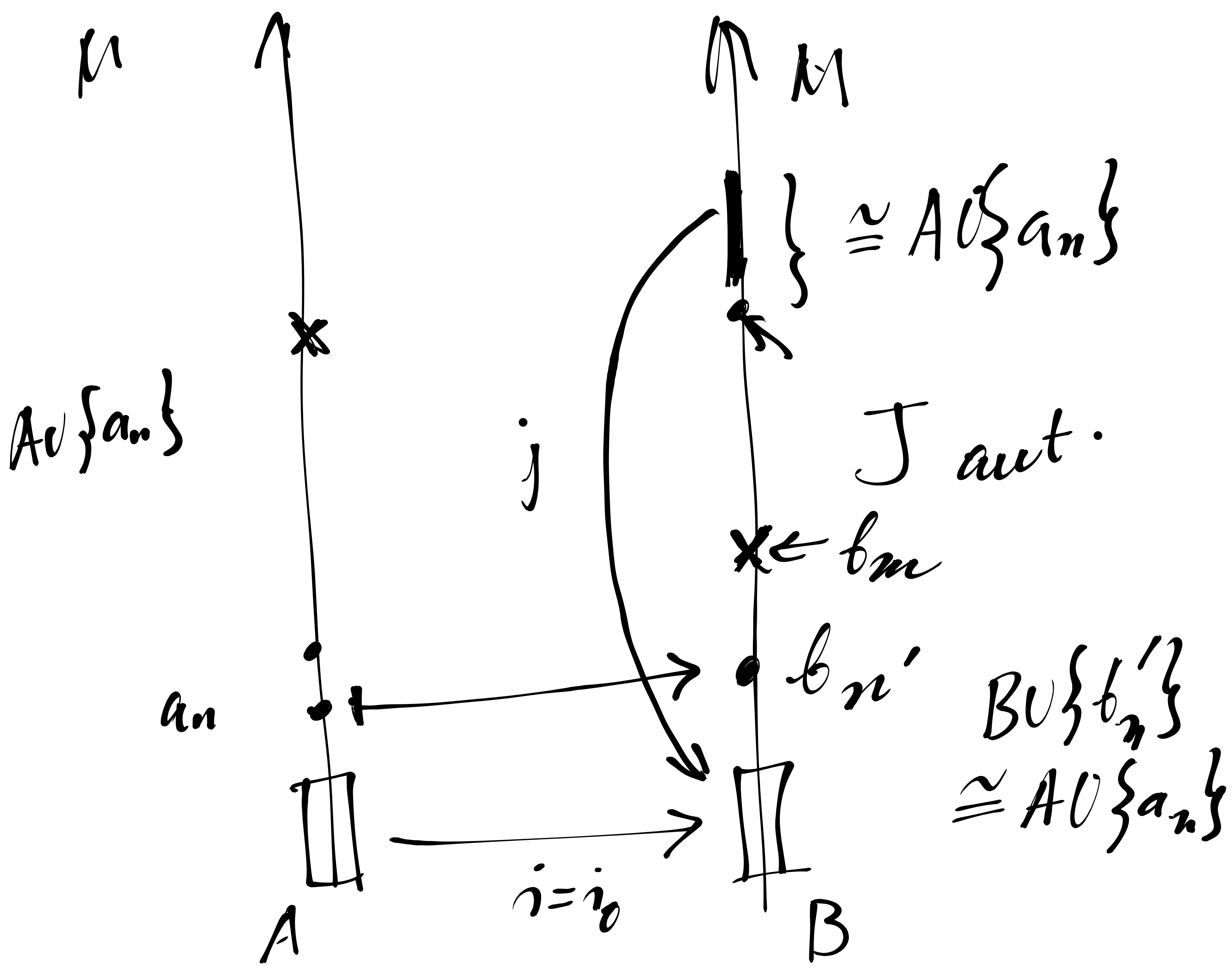
$g|_A = f$

Prop: The EP is equivalent to homogeneity for ctble strs.

1. Assume EP.



Enumerate M in two ways:
$\{a_i : i \in \omega\} =$
$= \{b_i : i \in \omega\} = M$
so that
$i : a_i \mapsto b_i$
for $i < n$



<u>forth</u>: Supp. i_n has been defined let c be the least nat. # s.t. i_n is not defined at a_c.

Apply the argument above to extend i_n to i_{n+1} which includes a_c in its domain.

<u>Back</u>

$$f := \bigcup \{ i_m : m \in \omega \} \quad \square$$

Theorem (Fraïssé): Let L be a ctble rel. language and $\mathcal{C}$ a class of finite L-strs with properties (i) – (iv). Then there exists a homogeneous L-str $M_{\mathcal{C}}$ such that $Age(M_{\mathcal{C}}) = \mathcal{C}$. For any ctble L-str N, if N is homogeneous and $Age(N) = \mathcal{C}$ then $N \cong M_{\mathcal{C}}$.

Pf:

Existence

Let P be a set of representatives of the iso. classes in $\mathcal{C}$

Enumerate all triples

(A, B, f)

where $f: A \to B$ is an embedding.

$A, B \in P$

$\{(A, B. f) : \ldots\} \text{ "} \subseteq \text{" } \omega^3$

C_0 an element of $\mathcal{C}$

Assume C_k has been constructed

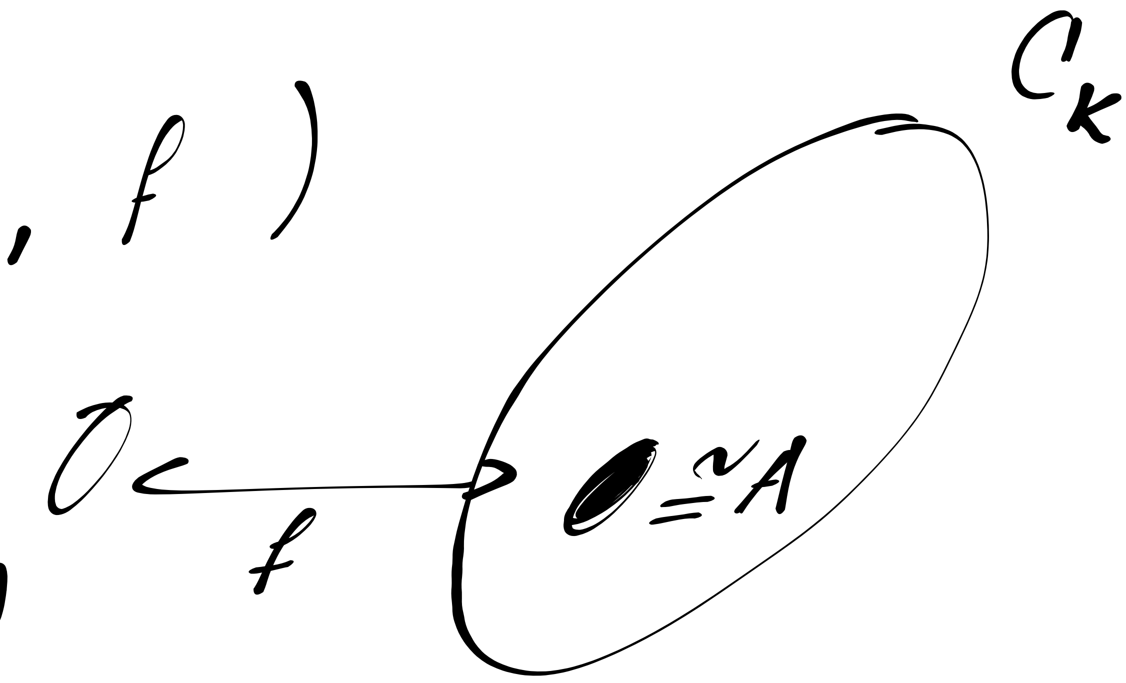
$\alpha : \omega^3 \to \omega$ bijective,

$\alpha(i,j,k) \geq i$

(A, B, f) enumerated by α

Consider all triples of the form

(A, C_k, f)



(A, B, g)

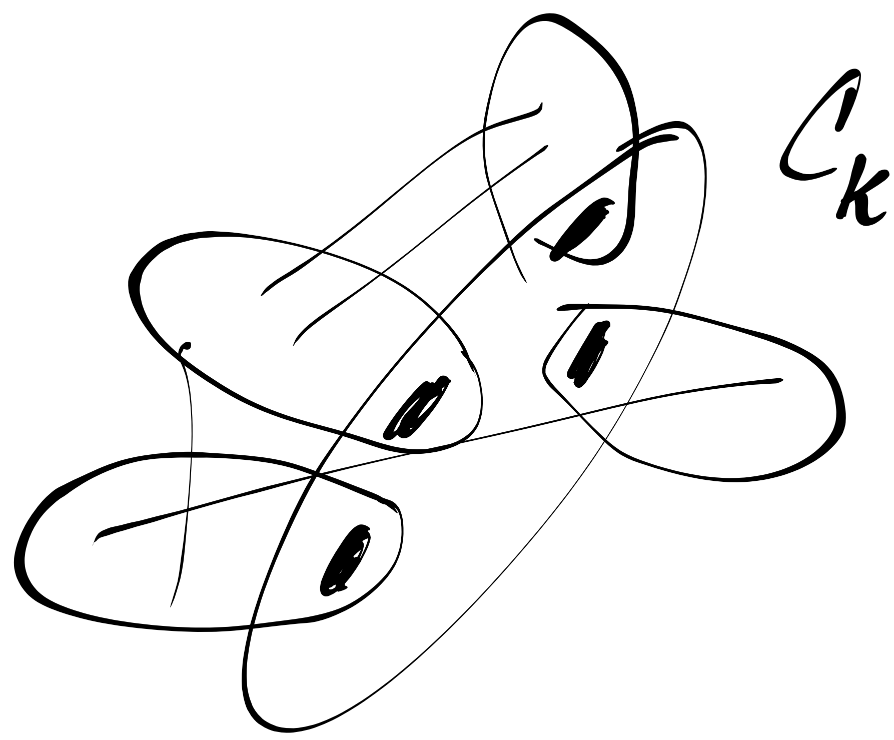
C_{k+1} as amalg. of B and C_k over A.

(A, C_k, f_n)

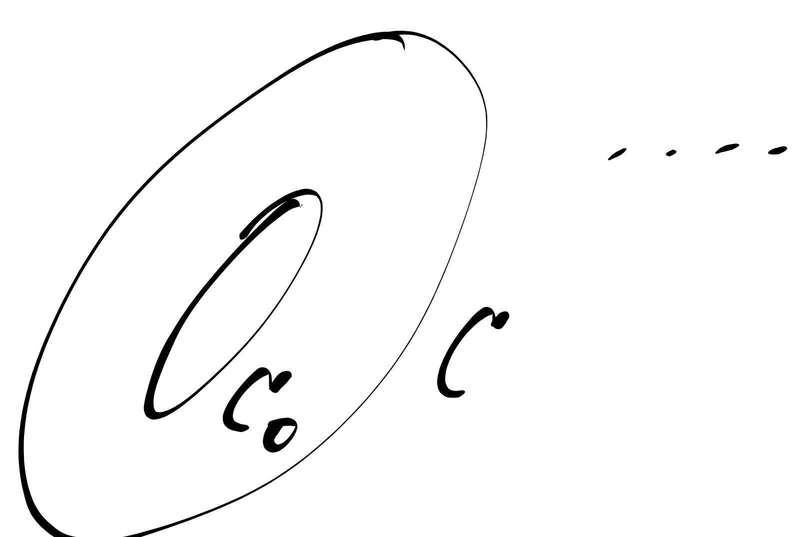
$\vdots$

(A, C_k, f_2)

(A, C_k, f_1)

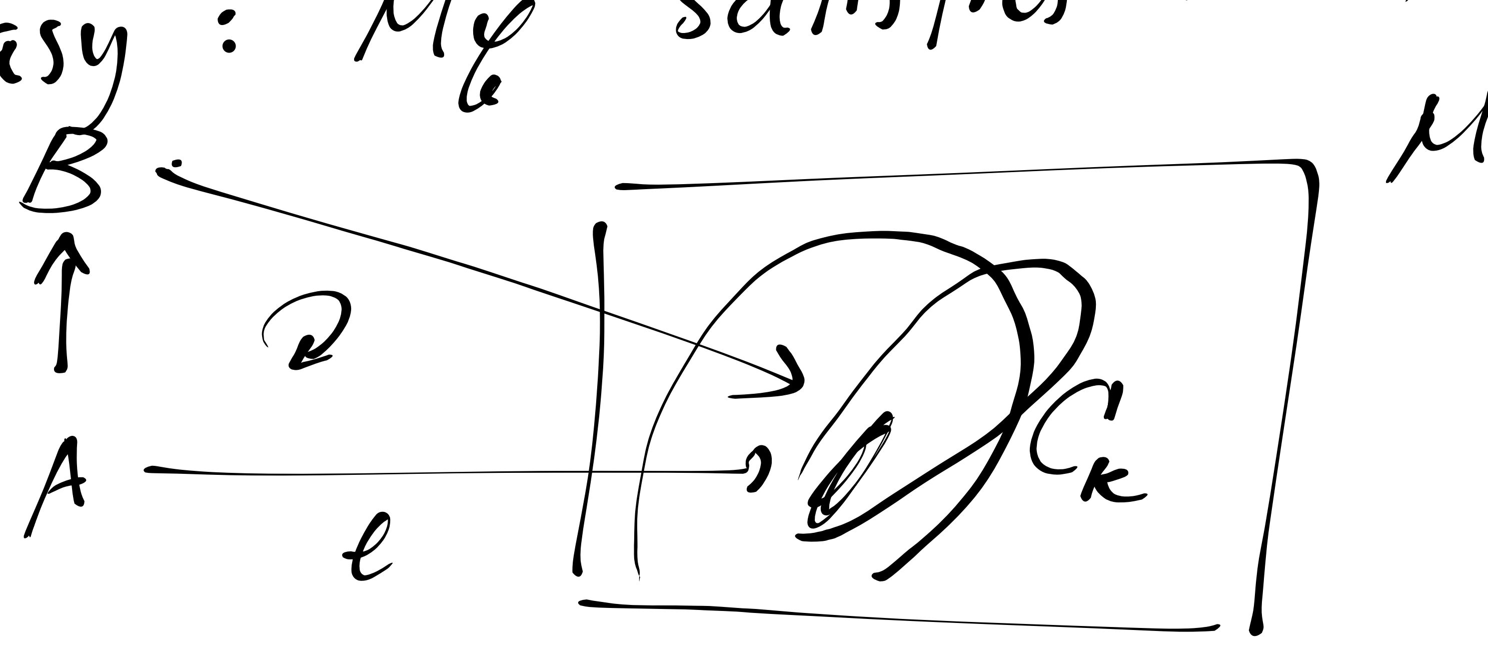


(A_i, B_j, e) $\quad \delta(i,j) = k$



$M_{\mathcal{C}} = \bigcup \{ C_i : i \in \omega \}$

Easy: $M_{\mathcal{C}}$ satisfies EP $\to M_{\mathcal{C}}$ hom

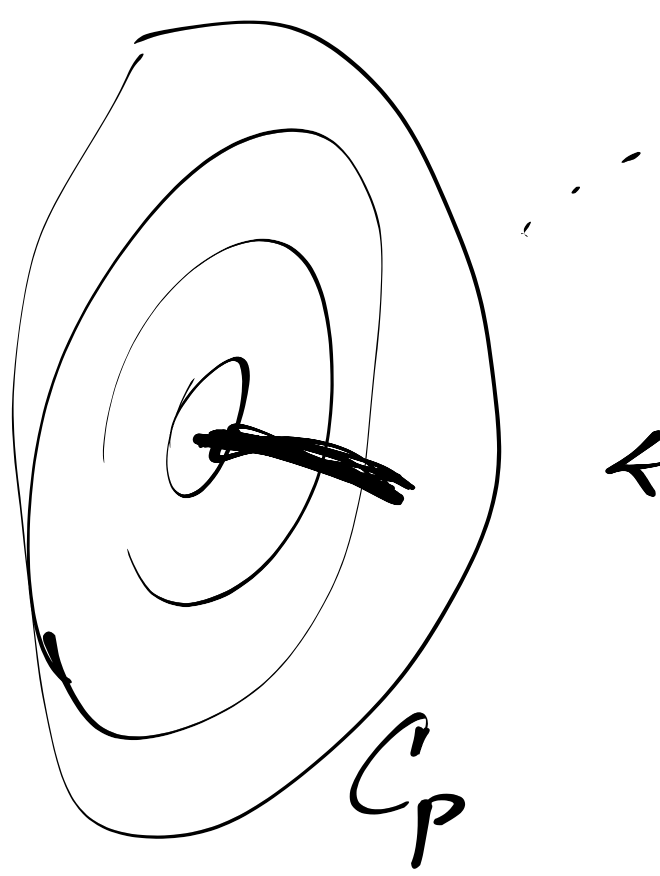


$\mathrm{age}(M_{\mathcal{C}}) = \mathcal{C}$

Given $A \in \mathcal{C}$, A embeds into some $C_k \subseteq_{fin} M_{\mathcal{C}}$

So A embeds into $M_{\mathcal{C}}$ and $\mathcal{C} \subseteq \mathrm{Age}(M_{\mathcal{C}})$.

$B \in \mathrm{Age}(M_{\mathcal{C}})$



Each C_p is product of an amalgamation therefore (by AP of $\mathcal{C}$) iso. to an elt. of $\mathcal{C}$

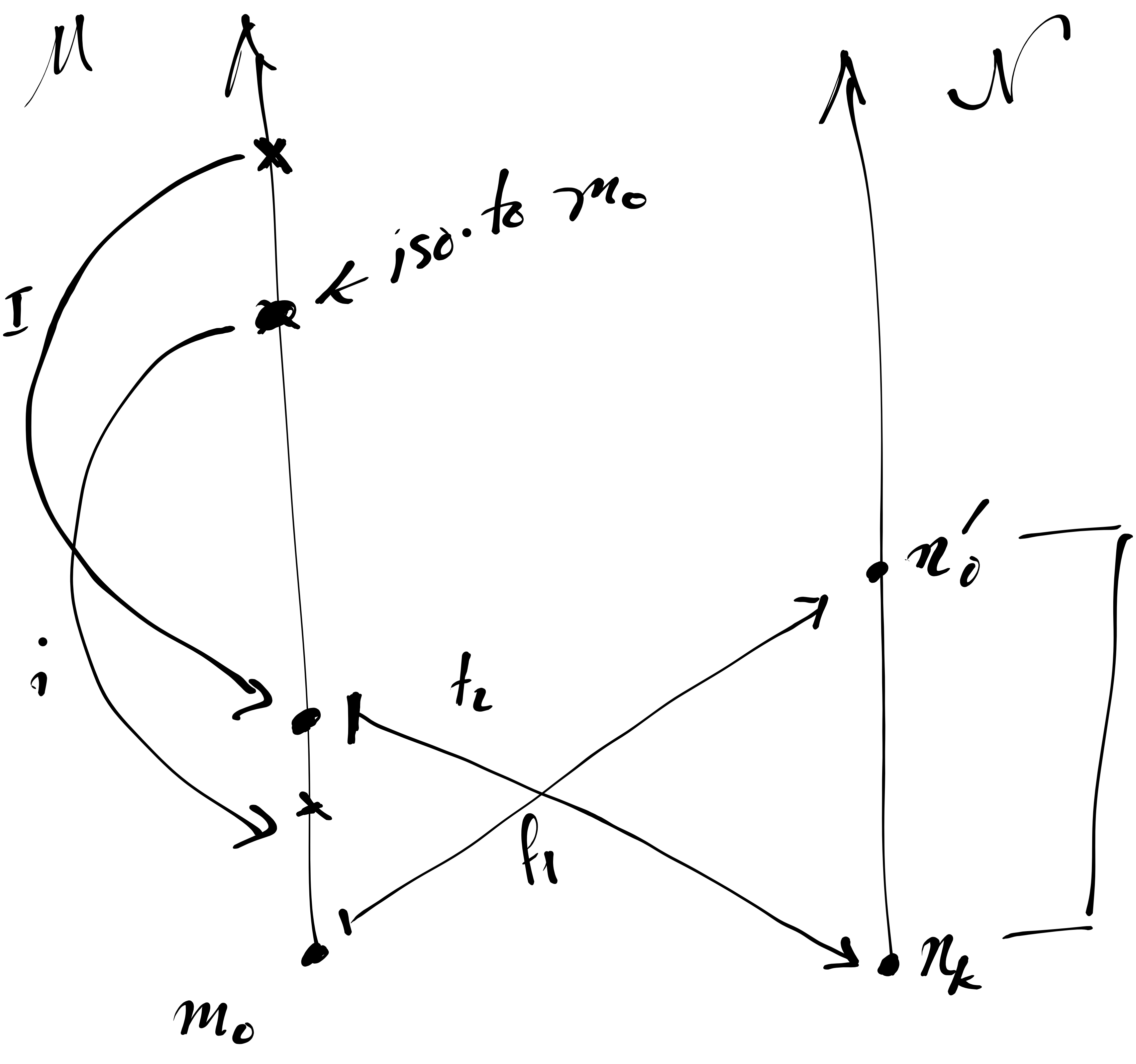
Since $\mathcal{C}$ is closed under iso, $B \in \mathcal{C}$.

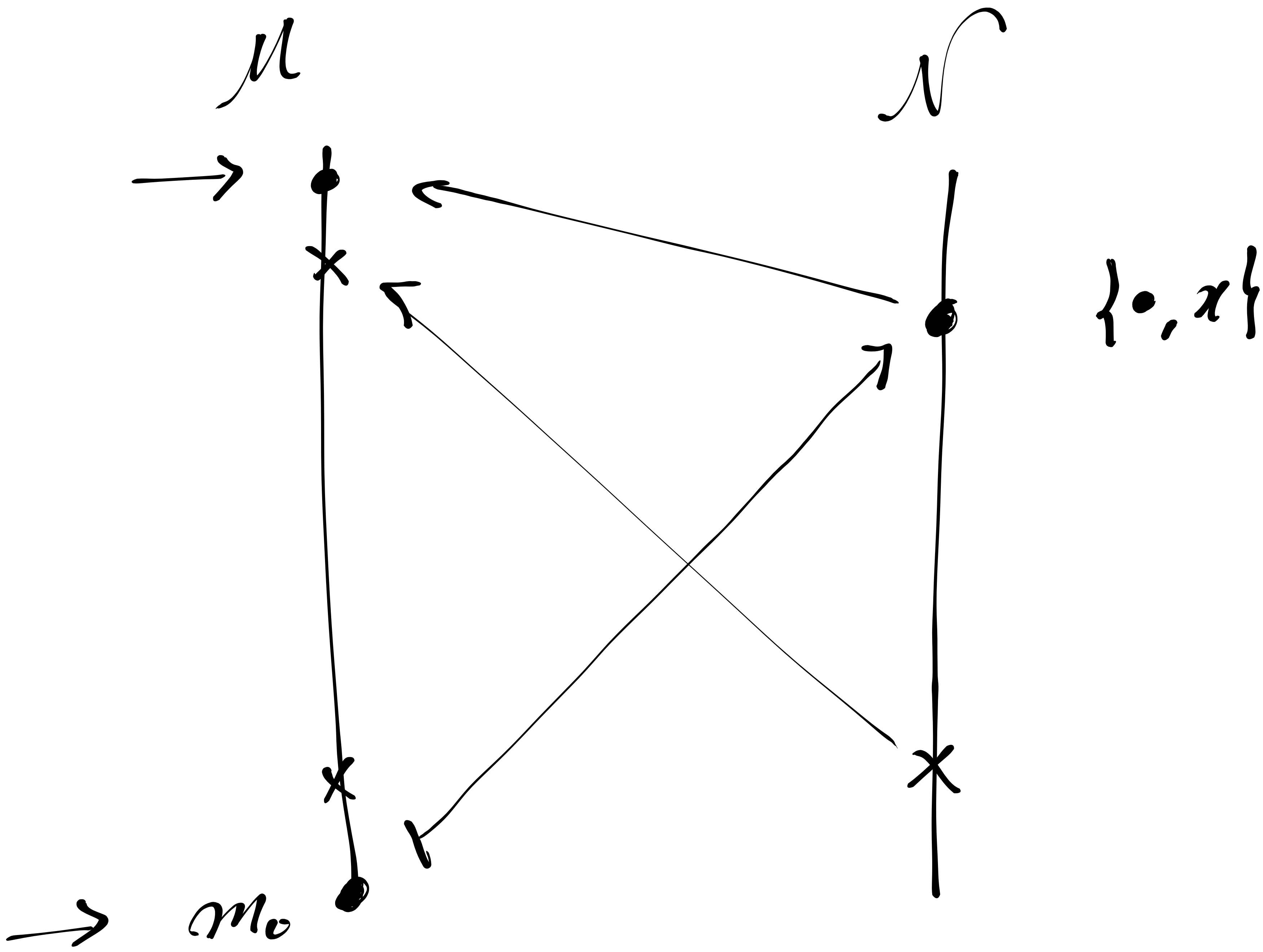
Uniqueness

Suppose $\mathcal{M}, \mathcal{N}$ are ctble, hom. and $Age(\mathcal{M}) = Age(\mathcal{N})$

Enumerate $\mathcal{M} = \{m_i : i \in \omega\}$

$\mathcal{N} = \{n_i : i \in \omega\}$





$\underline{\mathrm{Age}(M)} = \underline{\mathrm{Age}(N)}$

$\{m_0\} \subseteq \mathcal{M}$

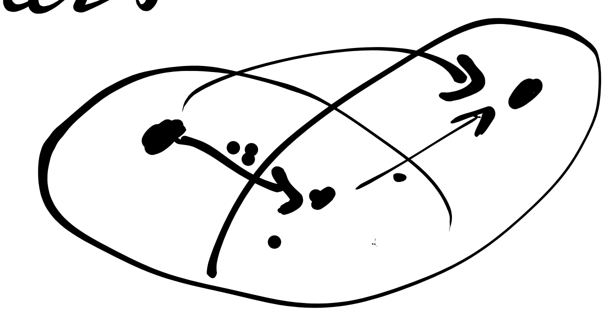
$A_0 \in \mathrm{Age}(M) = \mathrm{Age}(\mathcal{N})$

$\begin{array}{c} A_0 \\ \cong \\ \{m_0\} \end{array}$

<u>Fraïssé limits</u>

Examples

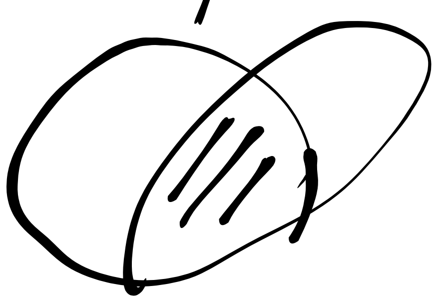
- Partial orders -



transitive closure

$a < b < c$
$a < c$

- Graphs

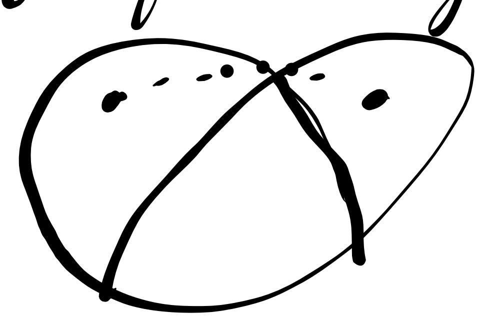


Ra(n)do(m)

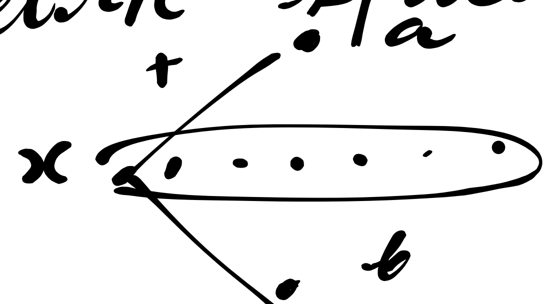
$\swarrow$

Universal hom. K_n-free graph

- K_n-free graphs



- Metric spaces



distances $= \mathbb{Q}$
Completion of Fr. limit is Urysohn space.

- Linear orders $\rightsquigarrow$ Fr. limit iso. to $(\mathbb{Q}, <)$

- Finite sets $\rightsquigarrow$ Fr. limit countable set

Types

> bla bla

variable

$$\{x > 0\} \cup \{x < \tfrac{1}{n} : n \in \mathbb{N}\}$$

Def (realised, satisfiable, fin. sat)

- An n-type is a set of fmlas on free vars $x_1 \dots x_n$
- p n-type realised in M if there exist $a_1 \dots a_n \in M$ s.t. $M \models \varphi(a_1, \dots a_n)$ for all $\varphi \in p$.
- p is satisfiable if realised in some elementary extension
- p finitely satisfiable if every fin. subset of p is realised.

Def (n-type)

Examples

Lemma Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\Sigma(x_1, \dots, x_n)$ be a set of fmlas. TFAE:

(1) Σ is an n-type of $\mathcal{M}$

(2) Every finite subset of Σ is realised in $\mathcal{M}$.

(3) $\mathcal{M}$ has an elementary extension that realises Σ

Def (maximal/complete type, $tp(a/X)$.)