~ Selected Chapters in Combinatorics ~

Lecture 4

* More on complete theories
  > Vaught's test
  > p-equivalence
* The Löwenheim-Skolem theorem
* Fraïssé amalgamation (Maybe)

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Recommended reading:

Henkin's construction:

- Marker (very detailed)
  pp. 35–38
- Hodges (different approach)
  pp. 124–126

Ultrafilters, ultraproducts:

- Thomas Jech: Set Theory Ch 7
- Hodges §8.5

Filters in topology:

- Boaz Tsaban
  u.cs.biu.ac.il/~tsaban/RT/Book/
  Chapter2.pdf
- Ryszard Engelking: General topology

More on complete theories

- Reminder:
  - Complete th.
  - Elementary embedding / elementary equivalence / elementary substructure

<u>Proposition</u>: A theory T is complete iff all models of T are elementarily equivalent

<u>Pf</u>: $\mathcal{M}, \mathcal{N} \models T$

if $\mathcal{M} \not\equiv \mathcal{N}$, then there is $\varphi(x)$

$\mathcal{M} \models \exists x\, \varphi(x)$ $\quad \mathcal{N} \models \neg \exists x\, \varphi(x)$

$T \cup \{\exists x\, \varphi(x)\}$, $\quad T \cup \{\neg \exists x\, \varphi(x)\}$

consistent proper extensions of T

contradicts completeness.

Supp. all models elem. eq.

φ sentence, $\varphi \notin T$

$\mathcal{M} \models T$, $\quad \mathcal{M} \models \neg\varphi$

$Th(\mathcal{M}) = T \longrightarrow \neg\varphi \in T$. $\square$

Thm (Łoś - Vaught test):

Let T be a theory without finite models. If T is κ-cat. for some κ, then T is complete.

Pf: Supp for a contradiction T is not complete.

There is sentence φ

$$T \not\models \varphi, \quad T \not\models \neg\varphi$$

$$T_0 = T \cup \{\varphi\} \quad T_1 = T \cup \{\neg\varphi\}$$

both satisfiable

T_0, T_1 have inf. models

(*) T_0, T_1 have models of card κ, non isomorphic

$\lightning$.

<u>Prop</u>: T a theory with inf. models
κ : a cardinal $\kappa \geq |L|$.
Then there is a model of T of card. $\geq \kappa$.

<u>Proof</u>: Extend the language by a set of constants ρ
$|\rho| = \kappa$
Consider
$$T' = T \cup \{ c_\alpha \neq c_\beta \mid \alpha \neq \beta \in \rho \}$$
By compactness, T' is sat.
$M \models T'$
$|M| \geq \kappa$
$M \models T$ $\square$

Thm (Tarski - Vaught test):

Let $\mathcal{N}$ be an L-str and $M \subseteq N$

$\mathcal{M}$ is an elem. substr of $\mathcal{N}$

iff for every fmla $\psi(\bar{x}, y)$ and tuples $\bar{a}$ from $\mathcal{M}$, if

$$\mathcal{N} \models \exists y\, \psi(\bar{a}, y)$$

then

$$\mathcal{N} \models \psi(\bar{a}, d)$$

for some $d \in \mathcal{M}$.

Pf: $i: \mathcal{M} \to \mathcal{N}$ inclusion

$\Rightarrow$/ if

$\mathcal{N} \models \exists y (\psi(\bar{a}, y))$

$\mathcal{M} \models \exists y\, \psi(\bar{a}, y)$

there exists $d \in M$ with $\psi(\bar{a}, d)$

$\mathcal{N} \models \psi(\bar{a}, d)$

$\Leftarrow$/ Induction on fmlas.

- Atomic fmlas: trivial
- φ is $\neg\psi$ and true for ψ

$\mathcal{M} \models \varphi(\bar{a})$ iff $\mathcal{M} \models \neg\psi(\bar{a})$

iff $\mathcal{M} \not\models \psi(\bar{a})$

(ind) iff $\mathcal{N} \not\models \psi(\bar{a})$

$\mathcal{N} \models \neg\psi(\bar{a})$

$\mathcal{N} \models \varphi(\bar{a})$

$\wedge, \vee$: same pattern

' φ is $\exists x\, \psi(\bar{y}, x)$

$\mathcal{M} \models \varphi(\bar{a}) \iff \mathcal{M} \models \exists x\, \psi(\bar{a}, x)$

$\iff \mathcal{M} \models \psi(\bar{a}, d)$ for some $d \in M$

(ind) $\iff \mathcal{N} \models \psi(\bar{a}, d)$

$\Rightarrow \mathcal{N} \models \exists x\, \psi(\bar{a}, x)$

$\mathcal{N} \models \varphi(\bar{a}, e)$ some e.

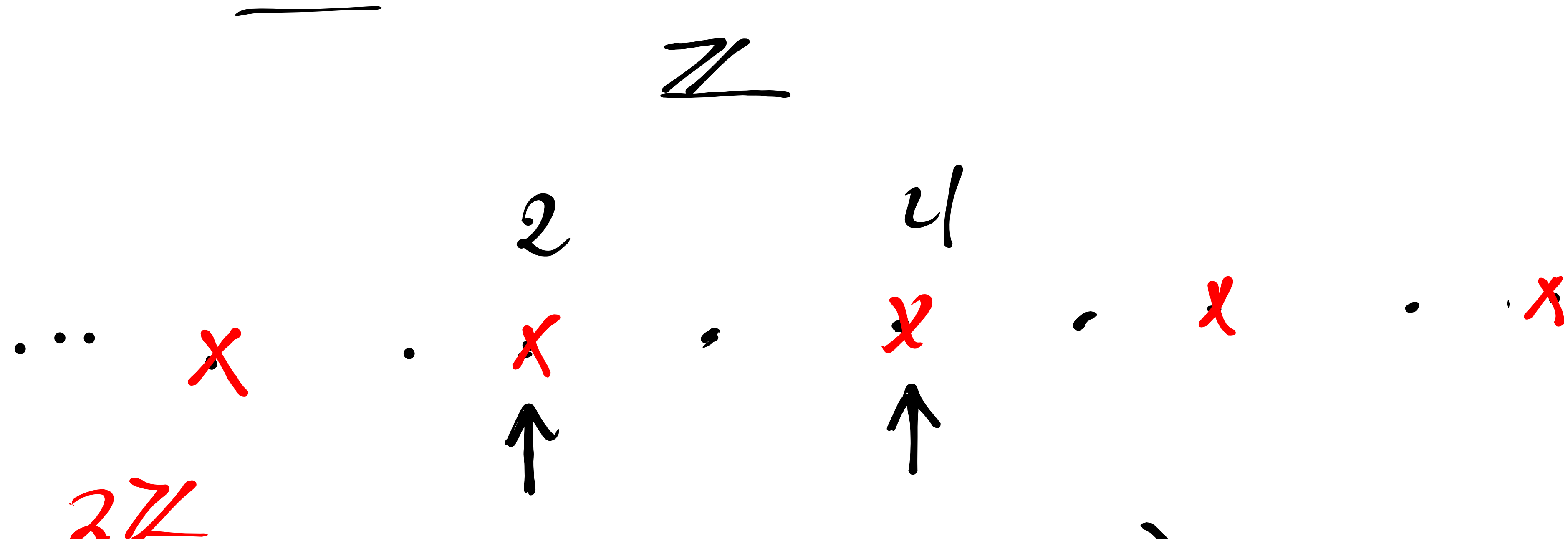
$\square$

Example from last week, revisited.

- $(2\mathbb{Z}, <)$ as substructure of $(\mathbb{Z}, <)$

  > is $(2\mathbb{Z}, <)$ an elem. substructure?

  <u>NO</u>



$$\exists x \, (2 < x < 4)$$

$$\psi(2,4)$$

$$\mathbb{Z} \models \psi(2,4)$$

$$2\mathbb{Z} \not\models \psi(2,4)$$

Another example from last week:

- Theory of discrete linear orders without endpoints.

> Is it complete?

<u>Obs</u>: $T_{dis.l.o.}$ is not κ-cat for any κ.

ω:

$(\mathbb{Z}, <)$

L: a countable lin. ord.

$\mathbb{Z} \times L$, with lexicographic ordering.

Local isomorphisms

R, R' two relations.

A <u>local isomorphism</u> $R \to R'$ is an isomorphism between finite restrictions of R, R'

<u>Define</u> $S_0(R, R')$ as the set of all local isomorphisms $R \to R'$.

$\phi \in S_0(R, R')$

Suppose $S_0(R, R'), \dots, S_p(R, R')$ have been defined.

A local iso s belongs to $S_{p+1}(R, R')$ if

<u>(FORTH)</u>: for all a in $\mathrm{dom}(R)$ there is $t \supseteq s$ defined on a and $t \in S_p$.

<u>(BACK)</u>: for all b in $\mathrm{dom}(R')$ there is $t \supseteq s$ with $b \in \mathrm{im}(t)$, $t \in S_p$.

$S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots \supseteq$

- if $p < q$ and $s \in S_q(R, R')$ then $s \in S_p(R, R')$
- Every restriction of a p-iso is a p-iso.
- if $s : R \to R'$ is an iso, then any finite restriction of s is a p-iso (for all p)
- if $S_{p+1}(R, R') = S_p(R, R')$ then $S_q(R, R') = S_p(R, R')$ for all $q > p$.

ω-isomorphism : any $f \in \bigcap \{ S_i : i \in \omega \}$

p - equivalent

$R \underset{p}{\sim} R'$ if $S_p(R, R') \neq \emptyset$

Example:

Supp. R is 2-equivalent to an antisymmetric binary relation. Then R is antisymm.

$S \underset{2}{\sim} R$

$\phi \in S_2(R, S)$

given $\alpha, \beta \in dom(S)$

$S(\alpha, \beta)$

Apply back-forth conditions

: there is a 2-iso with image (α, β)

. there is (a, b) with $R(a, b)$ $\quad a \mapsto \alpha$, $b \mapsto \beta$ 2-iso

$\neg R(b, a)$, so $R(\beta, \alpha)$ does not hold (i.e $\neg R(\beta, \alpha)$)

Example (continued)

A binary rel 3-eq. to dense lin. ord without endpts is a dense lin. ord without endpts

3-eq to lin. od is lin. ord

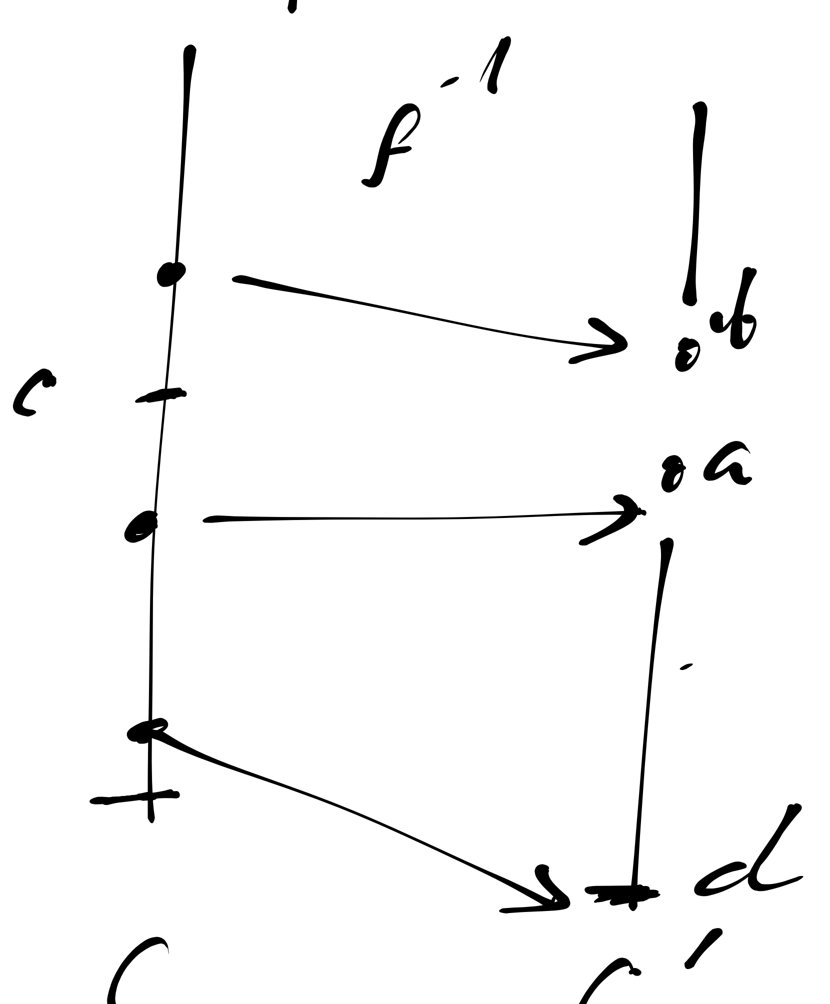
Density

C dlo, no endpts

$C' \underset{3}{\simeq} C$

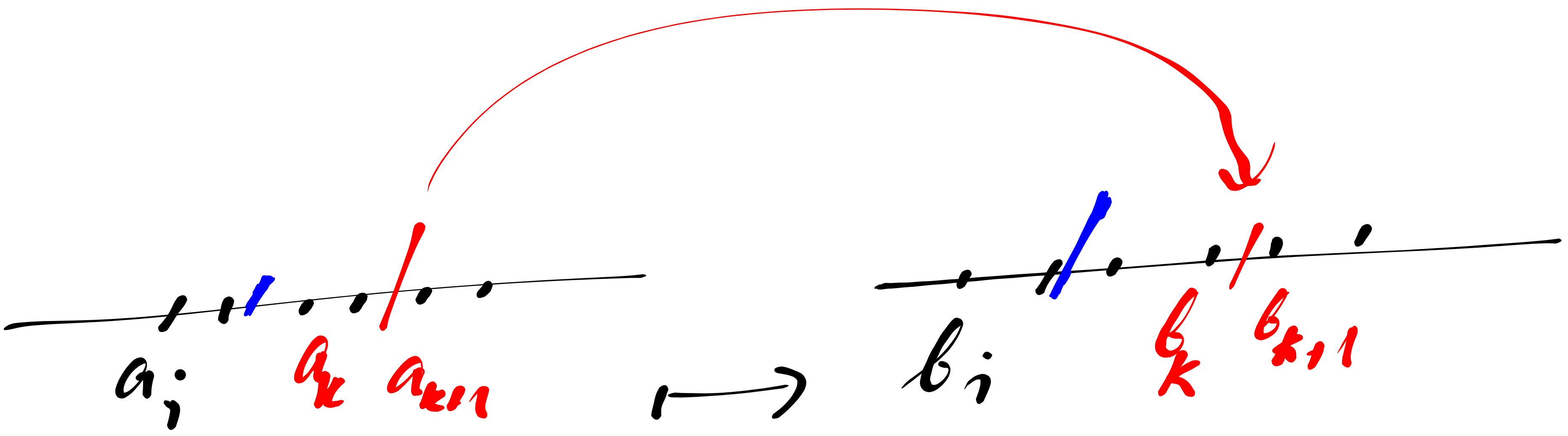
Supp C' not dense; so there are 2 consecutive elts $a < b \in C'$

$f : C' \to C$ 1-iso def. on a and b



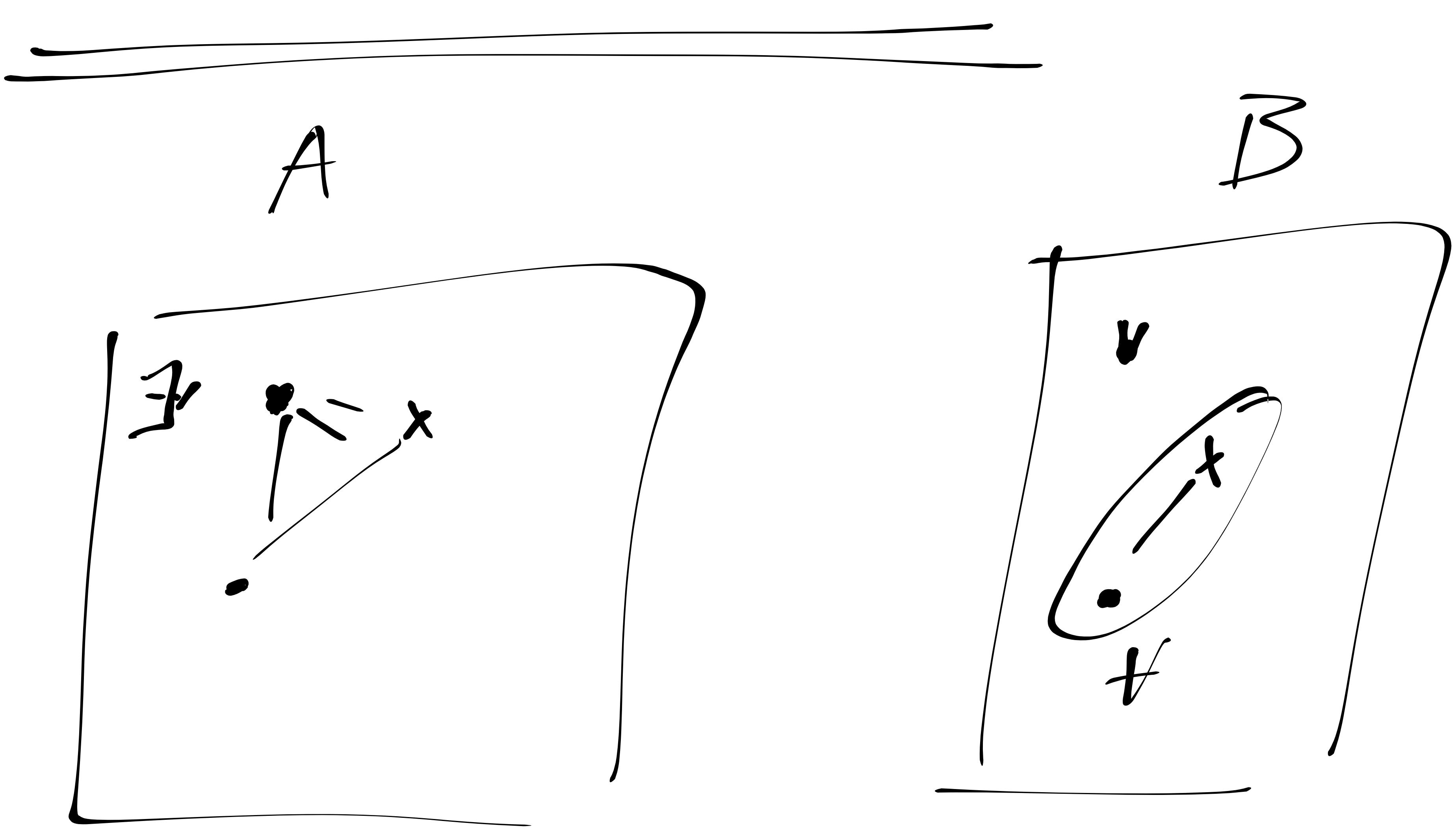
there is no way to extend f^{-1} to c ↯

Any two dense linear orders without endpts are ω-equiv.



$$S_0(C, C') = S_1(C, C')$$

$$S_0(C, C') = S_k(C, C') \text{ all } k.$$



Ehrenfeucht-Fraïssé games.

Exercise:

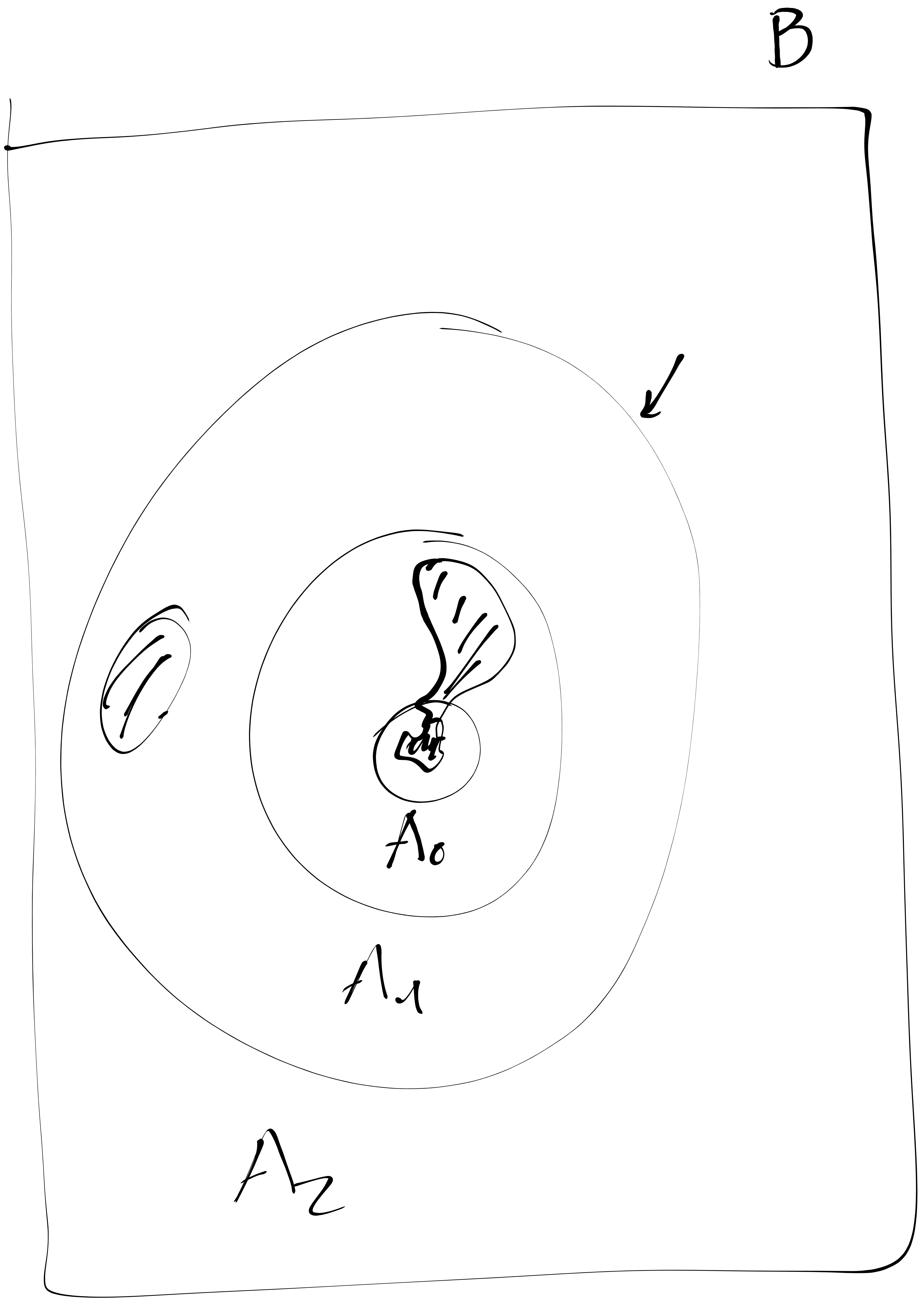
Any discrete linear order without endpoints is ω-eq to $(\mathbb{Z}, <)$

Corollary theory of disc. l. o. without endpoints is complete.

Application

- Dense linear orders without endpoints

- Discrete lin. ords without endpts.



The Löwenheim–Skolem Theorem

* Elementary chains

<u>Def</u>: $(A_i : i \in \omega)$ is an elem. chain of L-structures if $A_i \preccurlyeq A_{i+1}$ for all $i \in \omega$

The union of $(A_i : i \in \omega)$ is the L-str with domain $B = \bigcup \{A_i : i \in \omega\}$ and

- for each R relational k-ary $a \in B^k$ satisfies R if $a \in R^{A_i}$ for some i.

- for each f, $f^B(\bar{a}) = a'$ if $f^{A_i}(\bar{a}) = a'$ for some i.

Thm (Tarski's elem. chain theorem):

$(A_i)_{i \in I}$ elementary chain of L-structures. $B := \bigcup \{A_i : i \in I\}$

Then $A_i \preccurlyeq B$ for all $i \in I$.

Pf: Induction on fmlas.

<u>Proposition</u>: $\mathcal{B}$ an L-str., $S \subseteq B$.

Then $\mathcal{B}$ has an elementary substructure $\mathcal{A}$ with $S \subseteq A$ and $|A| \leq \max\{|S|, |L|\}$.

<u>Pf</u>: Start with $S_0 = S$

Suppose S_i has been defined

for each $\boxed{L \cup S_i}$ - fmla $\phi(x)$ satisfiable in $\mathcal{B}$, choose a a_ϕ (<u>witness</u>) $\phi(a_\phi)$ true

$S_{i+1} = S_i \cup \{a_\phi : \phi \dots\}$

By Tarski-Vaught test,

$A = \bigcup\{S_i : i \in \omega\}$ is domain of elem substr of $\mathcal{B}$

$|A| \leq \max\{|S|, |L|\}$

Thm (Löwenheim - Skolem): Let A be an infinite L-structure, $S \subseteq A$, and κ an infinite cardinal.

$\downarrow$: If $\max\{|S|, |L|\} \leq \kappa \leq |A|$, then A has an elem. substr of card κ containing S.

$\uparrow$: If $\max\{|A|, |L|\} \leq \kappa$, then A has an elementary extension of cardinality κ.

Pf: $\downarrow$: Choose $S' \supseteq S$ of card. κ apply prop from last slide

$\uparrow$: Construct elem. ext A' of A

$\hookrightarrow$ ρ set of new cts $|\rho| = \kappa$

$\rightarrow$ Add $\{c \neq d : c, d \in \rho, c \neq d\}$

$\rightarrow$ A'' model of extended thery $|A''| \geq \kappa$

$\rightarrow$ Let A' be reduct of A'' to orig. lang.

$\rightarrow$ Apply $\downarrow$ LS to get correct card.

ω-categoricity, Fraïssé amalgamation

- The age of a structure