Selected Chapters in Combinatorics

Lecture 3

- More compactness:
  - > Finish Henkin's proof
  - > Ultrafilter proof ←
- Elementary embeddings
- Löwenheim–Skolem theorem (maybe)

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From last week:

<u>Prop 1</u>: T a finitely satisfiable theory. Then T can be extended to finitely complete Henkin th. T^*.

<u>Lemma 2</u>: Every finitely complete Henkin th T^* with Henkin constants ρ has a model

<u>Theorem (compactness)</u>: A theory T is satisfiable iff every finite $F \subseteq T$ is satisfiable

<u>Proof</u>:

T fin. sat

By prop 1, we can extend T to fin. comp T^* (Henkin)

By Lemma 2, T^* has a model. $T \subseteq T^*$, so T has a model $\square$

Ultrafilters (yes!)

- **Def (filter):**

  X : a set. $\mathcal{F} \subseteq 2^X$ is a filter if:

  (1) $\emptyset \notin \mathcal{F}$.

  (2) $Y \in \mathcal{F}$ and $Z \supseteq Y$, then $Z \in \mathcal{F}$

  (3) if $Y_1, Y_2 \in \mathcal{F}$, then $Y_1 \cap Y_2 \in \mathcal{F}$

Note: if $A_0 \cdots A_{n-1} \in \mathcal{F}$ then $\bigcap\{A_i : i \in n\} \neq \emptyset$

(FIP) $\longleftarrow$

Lemma: If $S \subseteq 2^X$ has FIP, then there exists a $\subseteq$-smallest filter that contains S.

Pf: Start from S
- add all finite intersections
- add all supersets

Call the result S'.
- Verify S' is a filter, any other filter containing S contains S' (easy).

Examples

- Given $\emptyset \neq Y \subseteq X$,
  $$\{Z \subseteq X : Y \subseteq Z\}$$
  is a filter.

- (Fréchet filter)
  X: infinite set
  $$\{Y \subseteq X : X \setminus Y \text{ finite}\}$$

– Filters are partially ordered by $\subseteq$

Def: an ultrafilter is a $(\subseteq)$-maximal filter.

meaning: $\mathcal{F}$ ultrafilter and $\mathcal{G}$ filter, $\mathcal{G} \supseteq \mathcal{F}$ then $\mathcal{F} = \mathcal{G}$.

<u>Exercise</u>: Let $\mathcal{F}$ be a filter.
The following are equivalent

(1) $\mathcal{F}$ is an ultrafilter on X

(2) for all $A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$

(3) Whenever $A_1 \cup A_2 \cup \cdots \cup A_n \in \mathcal{F}$ then some $A_i \in \mathcal{F}$.

<u>Example</u>: (principal ultrafilter)

X a set

$a \in X$

$\{ Y \subseteq X : a \in Y \}$ is an ultrafilt.

Lemma: Every filter is contained in an ultrafilter.

Proof: $\mathcal{F}$ filter

$\mathcal{M}$: all filters on X that contain $\mathcal{F}$.

$(\mathcal{M}, \subseteq)$ poset.

Exercise: $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ sequence of filters

Then $\mathcal{F} = \bigcup \{ \mathcal{F}_i : i \in \omega \}$ is a filter

By (Exercise), we can apply Zorn's lemma to find maximal filter containin $\mathcal{F}$. $\square$

Lindenbaum algebra of L

$\mathbb{L}$: all sentences in the language.

Eq. rel $p \sim q$ if $\vdash p \to q$ & $\vdash q \to p$.

Usual operations $\wedge$ $\vee$

$$\exists x \varphi \sim \bigvee_{t \text{ term}} \varphi(t)$$

$$\forall \qquad \bigwedge$$

Theory : subset of $\mathbb{L}$ closed under deduction &c. (filter)

Complete theory : ultrafilter on Lind. algebra

Lindenbaum's lemma: Every consistent f.o. theory can be extended to a complete consistent theory.

Pf: Cons. theory = filter
Complete theory = ultrafilter
$\square$

Although only defined filter as subset of a powerset, the definition can be adapted to any partial order (simply change $\subseteq$ to $\leq$)

$\longrightarrow$

$L \supseteq \{x_i : i \in a\}$

Ultraproducts

L: a language

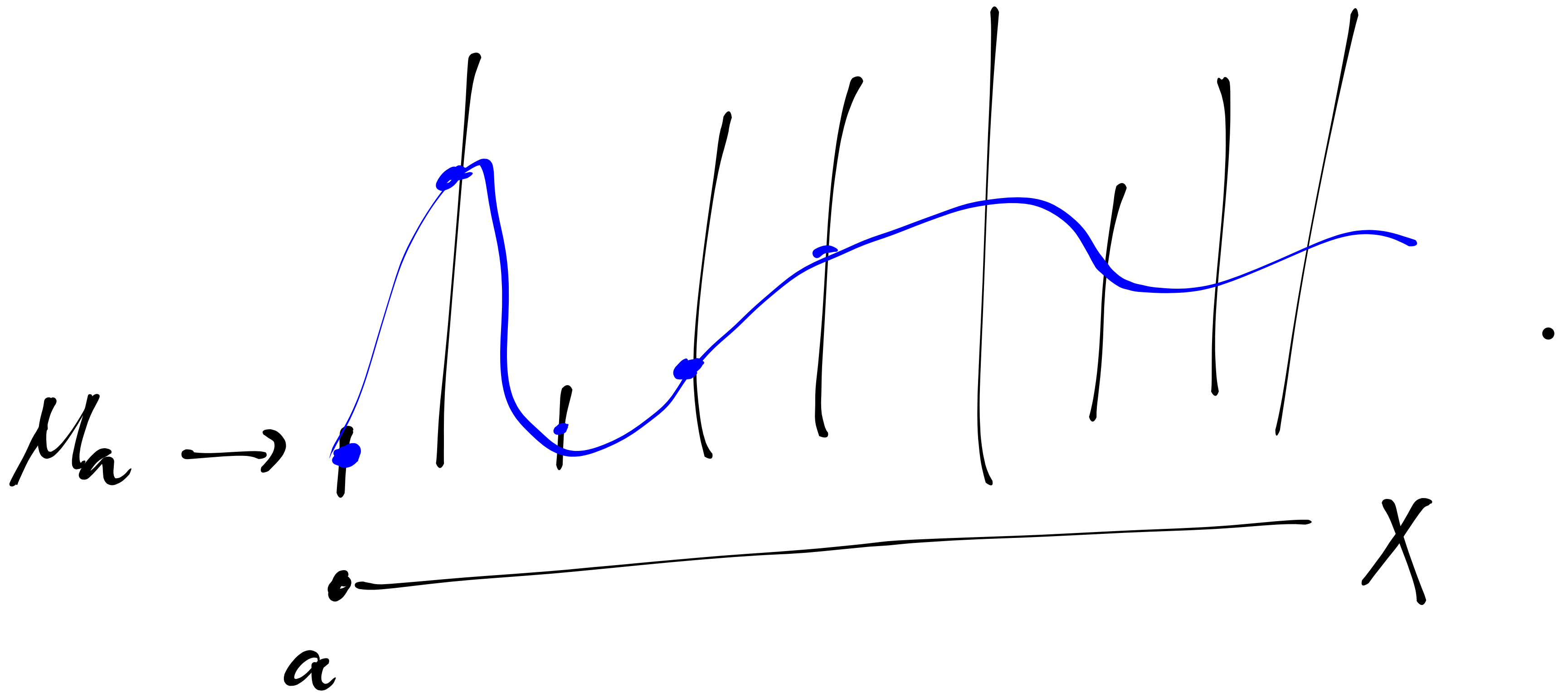
$\mathcal{U}$: an ultrafilter on X

M_a : an L-str for each $a \in X$

Def: $\prod_{a \in X} M_a / \mathcal{U}$ is the L-str

M with domain

$$\prod_{a \in X} M_a = \{ g : X \to \bigcup_{a \in X} M_a \mid \forall a \in X \; g(a) \in M_a \}$$



Eq. rel $\sim$

$g \sim g'$ iff

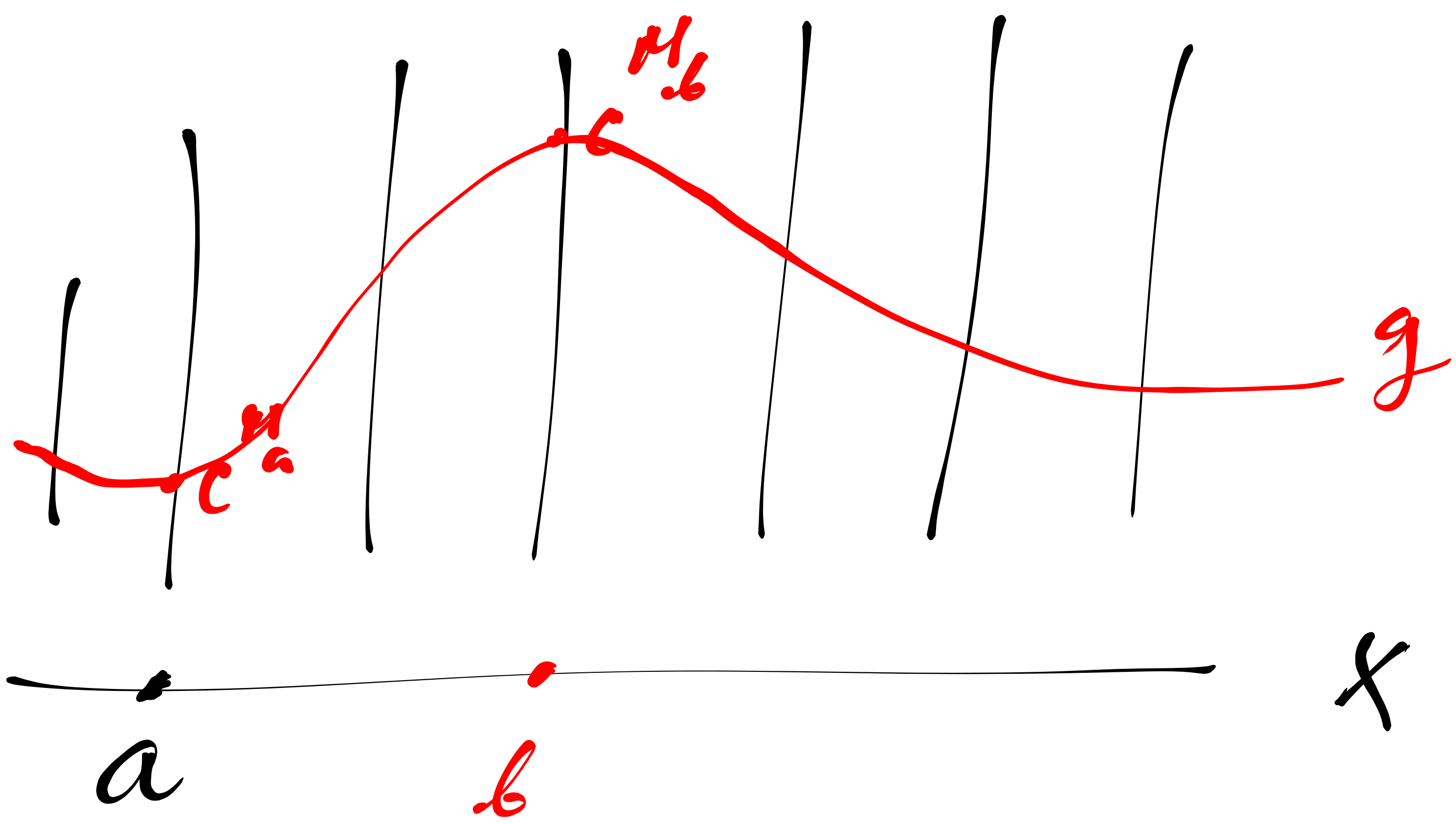
$$\{a \in X : g(a) = g'(a)\} \in \mathcal{U}$$

$$\prod_{a \in X} M_a / \mathcal{U}$$

Relations, fns, constants:

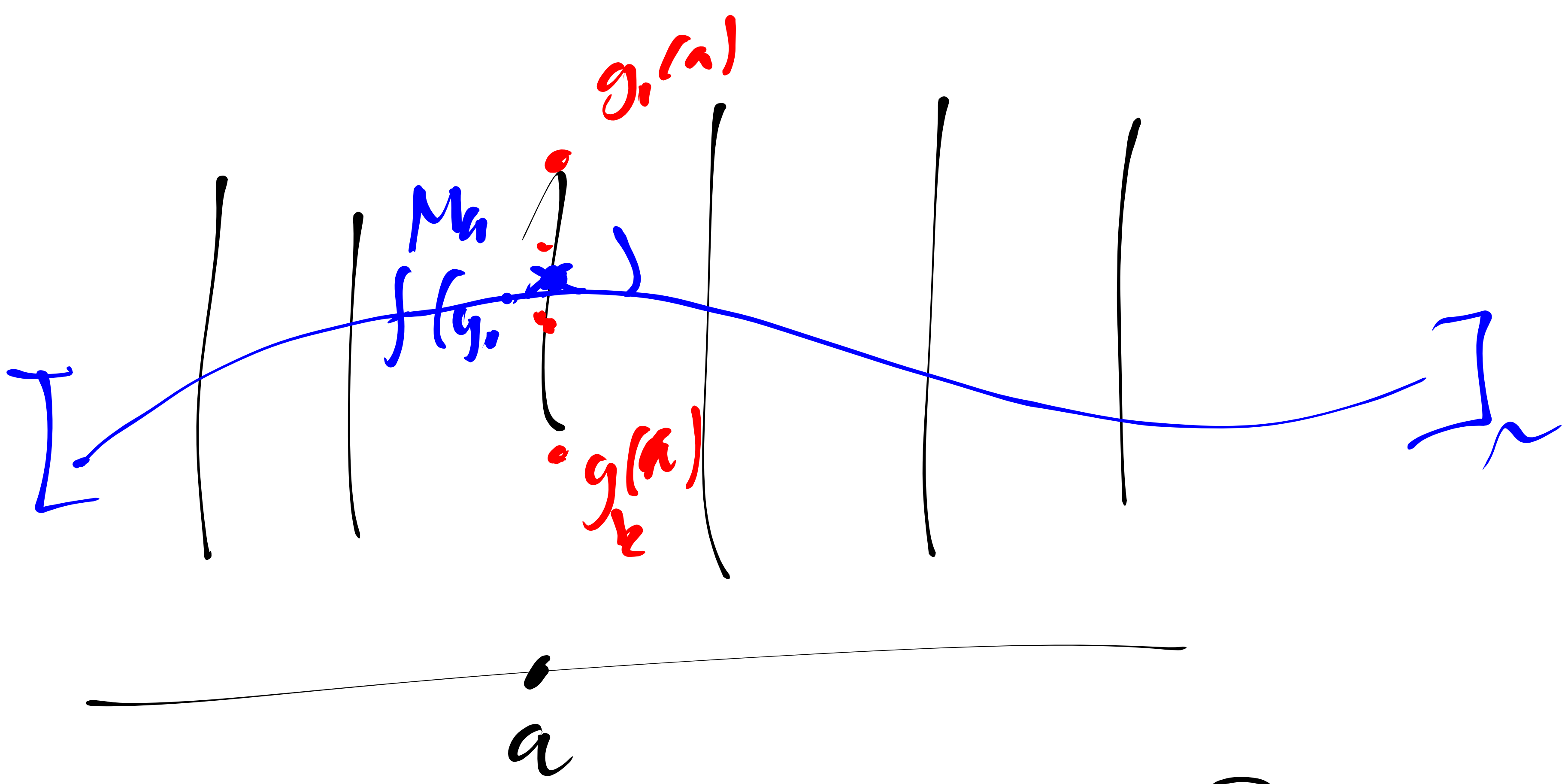
- c constant symbol

$$c^M = [a \mapsto c^{M_a}]_\sim$$



- fn. symbol f k-ary

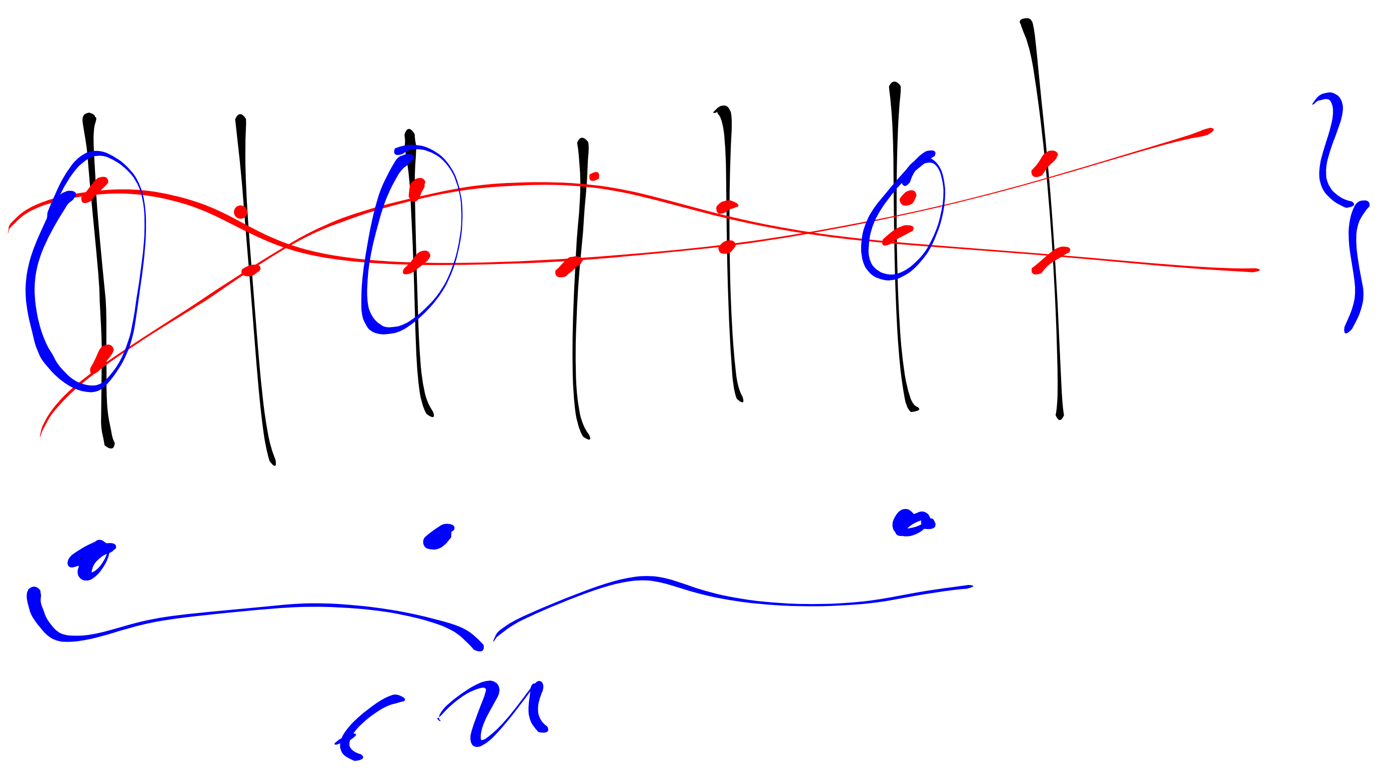
$$f^{\mathcal{M}}([g_1], [g_2], \dots, [g_k]) = [a \mapsto f^{\mathcal{M}_a}(g_1(a), \dots, g_k(a))]_{\sim}$$



- k-ary rel symbol R

$R^{\mathcal{M}}([g_1] \dots [g_k])$ iff

$$\{a \in X : (g_1(a) \dots g_k(a)) \in R^{\mathcal{M}_a}\} \in \mathcal{U}$$



Thm (Łoś): Let $\phi(x)$ be a first-order fmla and $\overline{[g]}$ a tuple of elements of $M = \prod_{a \in X} M_a / \mathcal{U}$. Then

$$M \models \phi(\overline{[g]}) \text{ iff } \{a \in X : M_a \models \phi(\overline{g(a)})\} \in \mathcal{U}$$

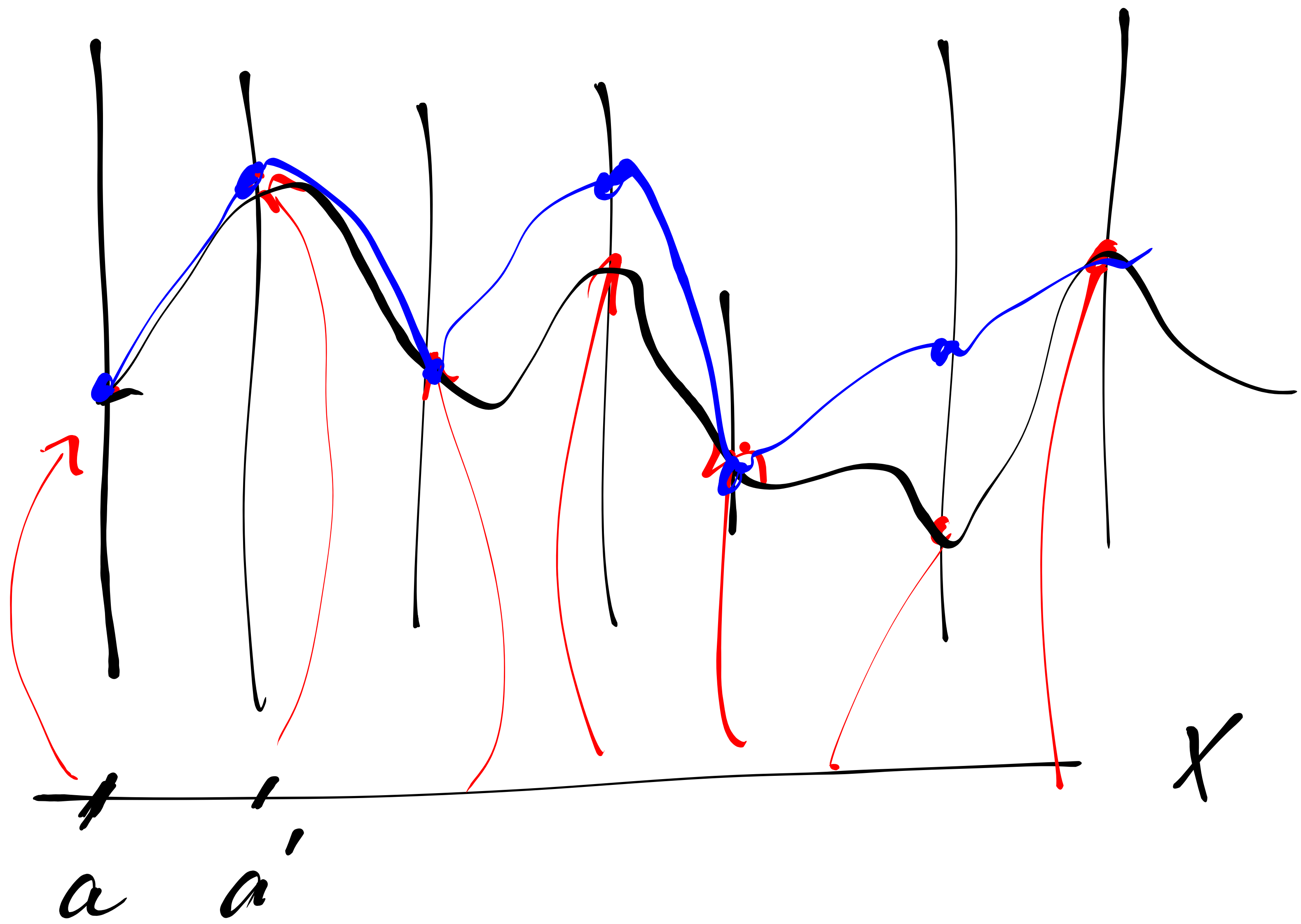
Proof: 97% routine
Left as an exercise

Idea: use induction on formulas.

Induction
- Prove for atomic
- $\neg$, $\wedge$, $\vee$
  (assume true for φ, prove true for $\neg\varphi$)
- Assume true for φ
  prove for $\exists x \varphi(x)$

Elements of $\prod M_a / \mathcal{U}$.

L-str M_a



$f : X \longrightarrow \bigcup \{M_a : a \in X\}$

$\{x \in X : f(x) = g(x)\}$ "large"

$f \sim g$ if $\uparrow$ is in $\mathcal{U}$

$[f]_\sim \leftarrow$ elts of the ultraprod.

Proof of compactness thm

wtp: if T is finitely satisfiable, then T is satisfiable.

X set of all fin. subsets of T

$Y \subseteq_{fin} T \quad M_Y \models Y$

Given φ a fmla,

$$X_\varphi = \{ S \in X : M_S \models \varphi \}$$

Claim: $\{ X_\varphi : \varphi \in T \} = Q$ has the FIP.

So there is an ultrafilter $\mathcal{U}$ with $Q \subseteq \mathcal{U}$.

$\mathcal{M} = \prod_{S \in X} M_S / \mathcal{U}$ is a model of T (apply Łoś's thm)

$\square$

Complete theories

Def Sets of L-sentences T s.t. given any L-sentence ψ, either $\psi \in T$ or $\neg\psi \in T$.

Examples / non-examples

- Given any L-str. M $Th(M)$ (all sentences true in M) is complete.
- Theory of groups.

κ-categoricity

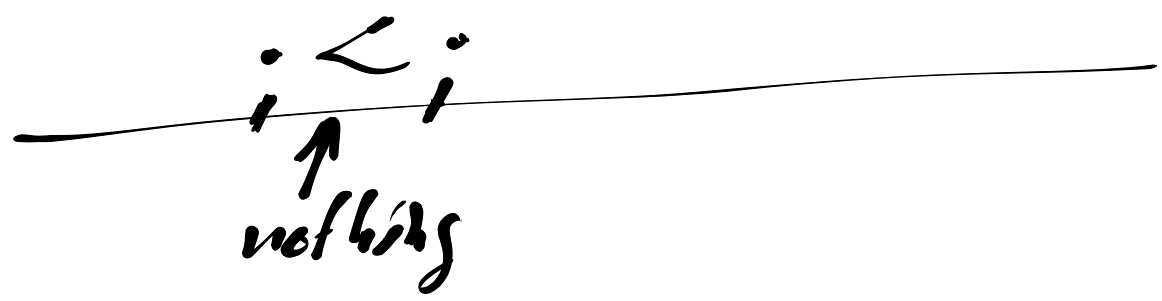
Def: κ a cardinal, T a theory. T is κ-categorical if all models of T of card κ are isomorphic.

- $\aleph_0$-categorical:

  Dense linear ordering without endpoints. $(\mathbb{Q}, <)$ is "the" ctble model.

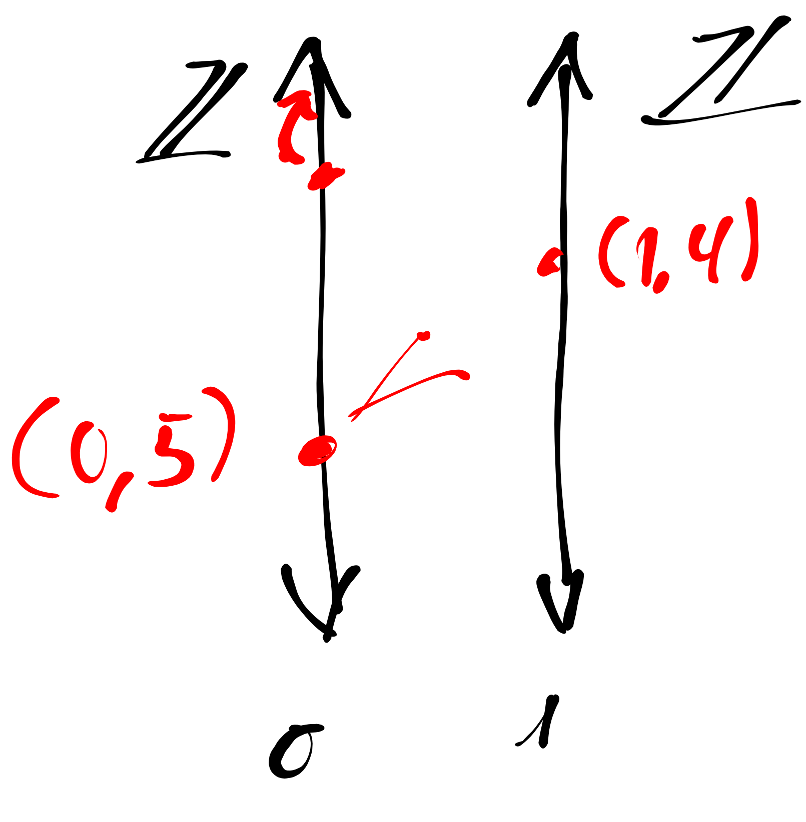
- Not $\aleph_0$-categorical:

  Discrete linear orderings without endpoints.



- $(\mathbb{Z}, <)$ a ctble model

- $(\mathbb{Z} \times 2, <)$



$\forall v \big((0,v) < (1,v) \big)$

Elementary embeddings

Def An elem. embedding is a homomorphism that preserves all f.o. formulas

$f : A \longrightarrow B$ and for all $\psi(\bar{x})$ and all tuples $\bar{a} \in A$

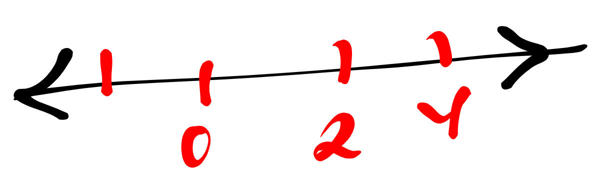
$$A \models \psi(\bar{a}) \quad \text{iff} \quad B \models \psi(f(\bar{a}))$$

a

$\exists x\,(x < 2 \wedge x > 0)$

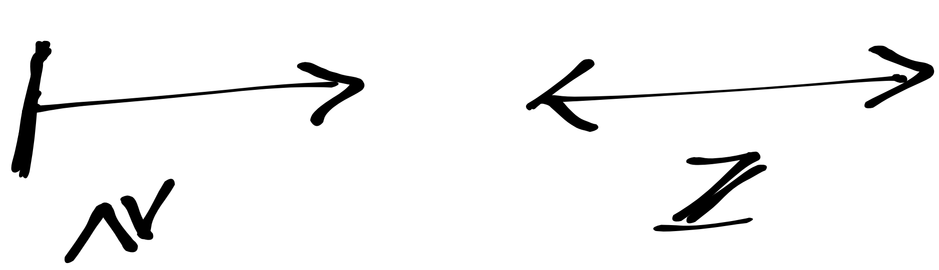
$\psi(0, 2)$

$\psi(x, y)$



Examples / non-examples

- $(\mathbb{Z}, \leq) \hookrightarrow (\mathbb{Q}, \leq)$
- $\mathbb{Q} \hookrightarrow \mathbb{R}$ as rings
- $(\mathbb{Z}, +, <) \xrightarrow{x \mapsto 2x} (2\mathbb{Z}, +, <)$
- Algebraic numbers $\hookrightarrow \mathbb{R}$ as fields
- $(\mathbb{N}, <) \hookrightarrow (\mathbb{N} + \mathbb{Z}, <)$
- Isomorphisms



$(\mathbb{Z}, <) \overset{i}{\hookrightarrow} (\mathbb{Q}, <)$

not elementary

$\psi(x, y)$:

$\forall z \, (z < y \wedge z \geq x \rightarrow z = x)$

in $\mathbb{Z}$,

$\psi(1, 2)$

but

$\psi(1,2)$ false in $\mathbb{Q}$

<u>Tarski-Vaught test</u>: A, B L-strs, $A \subseteq B$. $A \preccurlyeq B$ iff for every fmla $\varphi(x, y_0, \ldots, y_{n-1})$ and any n-tuple $\bar{a}$ from A, if

$$B \models \exists x \, \varphi(\bar{a}),$$

then there is $d \in A$ s.t.

$$A \models \varphi(d, \bar{a})$$

<u>Pf.</u>:

Prop: Let T be a theory with infinite models. If κ is an infinite cardinal and $\kappa \geq |L|$, then there is a model of T of cardinality κ.

Proof

<u>Thm</u> (Vaught's test): If T is a satisfiable theory with no finite models that is κ-cat. for some κ. Then T is complete.

The Löwenheim - Skolem thm.

<u>Thm</u> (L-S):

$\uparrow$: Let $\mathcal{M}$ be an infinite $\mathcal{L}$-str and κ an infinite cardinal $\kappa \geq |\mathcal{M}| + |\mathcal{L}|$. Then there is an $\mathcal{L}$-str $\mathcal{N}$ of cardinality κ and an elementary embedding

$$j : \mathcal{M} \to \mathcal{N}$$

$\downarrow$: Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $X \subseteq \mathcal{M}$. There exists an elementary submodel $\mathcal{N}$ of $\mathcal{M}$ s.t. $X \subseteq \mathcal{N}$ and $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$

Read all about it!

- Henkin's construction
  - Marker (very detailed, stronger compactness thm) pp 35 – 38.
  - Hodges (different approach) pp 124–126.
- Ultrafilters, ultraproducts
  - Thomas Jech: Set Theory Ch 7
  - Hodges §8.5
  - filters in topology:
    - Boaz Tsaban u.cs.biu.ac.il/~tsaban/RT/Book/Chapter2.pdf
    - Ryszard Engelking: General topology