Fraïssé theorems for IB- and IM-homogeneous structures (DRAFT)

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Abstract

After Coleman's work in [1], Fraïssé theorems are known for 12 of the 18 morphism-extension classes introduced by Lockett and Truss in [2]. We prove Fraïssé theorems for IM and IB, two of the remaining classes.

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1. Introduction

A structure M is ultrahomogeneous if every isomorphism between finitely generated substructures is restriction of an automorphism of M. Ultrahomogeneous structures have been heavily studied from several perspectives. They are interesting to model theorists (they are ω-categorical, their theory eliminates quantifiers, and the method by which they are constructed is amenable to modifications that produce some exotic structures), to group theorists (because of their large automorphism groups and their connection via Ramsey theory to extreme amenability), to universal algebraists via the study of CSPs, and to combinatorialists interested in Ramsey theory.

Complete graphs and unstructured sets are trivial examples of ultrahomogeneous structures; more interesting examples include the rational numbers as an ordered set, the countable universal homogeneous K_n-free graphs and the universal homogeneous partial order (for more information and examples, see [3]). The theory of an ultrahomogeneous structure has only one countable model, and for this reason they are often assumed to be countable. Although the same is not true for homomorphism-homogeneous structures (the example from Corollary 2.2 in [4] is a non-ω-categorical HH-homogeneous graph), we will restrict our attention to countable homomorphism-homogeneous structures to avoid set-theoretic complications.

The classic Fraïssé theorem (see [5]) establishes a correspondence between ultrahomogeneous structures and classes of finite structures that satisfy three

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fairly common properties and the amalgamation property (AP): if M is ultrahomogeneous, then its *age* (the class of finite structures that embed into M) has the four properties mentioned before, and if $\mathcal{C}$ is a class with those properties, then there exists a countable ultrahomogeneous $M_{\mathcal{C}}$ such that $\mathrm{Age}(M_{\mathcal{C}}) = \mathcal{C}$. Additionally, Fraïssé's theorem says that any two (countable) ultrahomogeneous structures with the same age are isomorphic.

In [4], Cameron and Nešetřil introduced the notion of *homomorphism-homogeneity*, a variation on ultrahomogeneity in which any homomorphism between finite substructures of M is restriction of an endomorphism of M. In that same paper, they isolated an amalgamation property for the subclass in which injective homomorphisms between finite substructures are restrictions of injective endomorphisms (MM in the notation introduced below) and proved an analogue of Fraïssé's theorem with it.

Some time later, Lockett and Truss [2] made finer distinctions between homomorphism-homogeneous structures. They introduced new notions of homomorphism-homogeneity, bringing the total to 18. These notions are labelled by two letters XY, where $\mathrm{X} \in \{\mathrm{H}, \mathrm{M}, \mathrm{I}\}$ and $\mathrm{Y} \in \{\mathrm{H}, \mathrm{M}, \mathrm{E}, \mathrm{B}, \mathrm{I}, \mathrm{A}\}$. A structure is XY-homogeneous if every local homomorphism (that is, a homomorphism with finite domain) of type X is restriction of an endomorphism of type Y. The symbols X and Y are interpreted as follows:

- H: homomorphism/endomorphism
- M: monomorphism (injective)
- E: epimorphism (surjective)
- B: bimorphism (bijective)
- I: self-embedding
- A: automorphism

For example, the definition of homomorphism-homogeneity mentioned above corresponds to HH-homogeneity in this language, and ultrahomogeneity corresponds to IA-homogeneity.

For a while, homomorphism-homogeneity was a popular subject, and some papers were written that attempted to develop the theory of homomorphism-homogenous structures along the lines of the theory of ultrahomogeneous structures, with moderate success (see [6], [7], [8]). However, no analogue of Fraïssé's theorem was known for most of the classes defined by Lockett and Truss until Coleman's work in [1]. In that paper, Coleman isolated amalgamation properties and proved analogues of Fraïssé's theorem for 12 of the 18 classes, in a more or less uniform way.

The six remaining classes (IH, IE, IM, IB, MH, ME) are characterised by a mismatch between the type of local homomorphism X and endomorphism Y. To understand what we mean by this, consider any morphism-extension class not listed in the parenthesis, say MB. Every restriction of a bijective endomorphism (class B) to a finite substructure is a monomorphism (class M), and similar statements hold for the 12 classes for which Fraïssé theorems are known. On the other hand, the restrictions of Y-endomorphisms to finite subsets are not necessarily X-morphisms for the six $\mathrm{XY} \in \{\mathrm{IH}, \mathrm{IE}, \mathrm{IM}, \mathrm{IB}, \mathrm{MH}, \mathrm{ME}\}$.

In the six mismatched classes, it is not immediately obvious what the amalgamation property should look like. It turns out that two almost trivial observations are the key to find the appropriate amalgamation property in IM: first, if we extend a local isomorphism i to a global monomorphism I, then some of the restrictions of I may not be isomorphisms; and second, if f is a monomorphism obtained as restriction of a global monomorphism in an IM-homogeneous structure, then all local monomorphisms that 'do the same as f' (we will call these *manifestations* of a monomorphism in the age of M) are also restrictions of global monomorphisms.

In this paper, we develop the idea in the preceding paragraph and prove Fraïssé theorems for IB and IM. Most of the conceptual work goes into the Fraïssé theorem for IM (section 3), and then we adapt the same argument for IB (section 4).

2. Represented monomorphisms

The age of a relational structure M is the class of all finite relational structures that embed into M (warning: we will use an alternative definition in this paper). A central piece in our understanding of ultrahomogeneous structures is Fraïssé's theorem, which we present below.

Theorem 1 (Fraïssé 1953)**.** *If M is an ultrahomogeneous structure, then* $\mathrm{Age}(M)$ *satisfies the following properties:*

1. $\mathrm{Age}(M)$ *contains only countably many distinct structures up to isomorphism,*
2. $\mathrm{Age}(M)$ *is closed under isomorphism and (induced) substructures,*
3. *for all $A, B \in \mathrm{Age}(M)$ there exists $C \in \mathrm{Age}(M)$ and embeddings $A \to C, B \to C$ (Joint Embedding Property, or JEP), and*
4. *for all $A, B_1, B_2 \in \mathrm{Age}(M)$ and embeddings $e\colon A \to B_1,\ j\colon A \to B_2$ there exists $D \in \mathrm{Age}(M)$ and embeddings $e'\colon B_1 \to D,\ j'\colon B_2 \to D$ such that $e' \circ e = j' \circ j$.*

Conversely, if $\mathcal{C}$ is a class of finite structures that satisfies the four conditions above, then there exists an ultrahomogeneous $M_{\mathcal{C}}$ such that $\mathrm{Age}(M_{\mathcal{C}}) = \mathcal{C}$. Any two ultrahomogeneous structures with the same age are isomorphic.

Analogues of this theorem were first proved for MM in [4] , and later for HH in [6]. In [1], Coleman proved Fraïssé theorems for all the morphism-extension classes introduced in [2] except IH, IE, IM, IB, MH, and ME. Our main contributions in this paper are Fraïssé theorems for IM (Theorem 21) and IB (Theorem 36).

We prefer to see the age of a relational structure as a category with countably many objects, rather than as an unorganised proper class containing multiple copies of the same structure. An effect of this is that the concept of substructure changes, (compare the second condition in Theorem 1 with the third condition in Definition 2 below), but this is more something to keep in mind than a real problem.

Our focus in this paper is on relational structures. Throughout, L will denote a relational language with finitely many n-ary relations for each n.

Definition 2. *A* pre-Fraïssé class *is a category* $(\mathcal{C}, \mathcal{H})$ *whose objects are finite* L*-structures and whose arrows* $\mathcal{H}$ *are homomorphisms, which satisfies*

1. $\mathcal{H}$ *includes all embeddings,*
2. $|\mathcal{C}| \leq \aleph_0$ *and* $\mathcal{C}$ *does not contain distinct but isomorphic structures,*
3. $\mathcal{C}$ *is hereditary: for all finite* L*-structures* A *and all* $B \in \mathcal{C}$*, if there exists an embedding* $e\colon A \to B$*, then there exists* $A' \in \mathcal{C}$ *isomorphic to* A*, and*
4. *the Joint Embedding Property: for all* $A, B \in \mathcal{C}$*, there exists* $C \in \mathcal{C}$ *and embeddings* $A \to C, B \to C$*.*

When we speak of the age of M, we will usually mean the object set of a pre-Fraïssé class whose objects are representatives of the isomorphism types of finite structures that embed into M. If the arrows are relevant, we will mention them.

Pre-Fraïssé classes are simply the ages of countable relational structures (modulo isomorphism of finite substructures), equipped with some homomorphisms.

Given a set of finite structures $\mathcal{C}$, $\mathrm{Hom}(\mathcal{C})$ denotes the family of all homomorphisms between elements of $\mathcal{C}$, and similarly $\mathrm{Mon}(\mathcal{C})$ denotes all monomorphisms. We will use different types of arrows to distinguish different types of homomorphism. In this paper, $g\colon A \hookrightarrow B$ means that g is an embedding.

An embedding $e\colon A \hookrightarrow B$ is an isomorphism between A and its image in B. We will abuse notation and write e^{-1} for the left inverse of an embedding, even when e is not surjective.

Definition 3. *Let* M, N *be* L*-structures with* $\mathrm{Age}(M) = \mathrm{Age}(N) = \mathcal{C}$*,* $A, B \in \mathcal{C}$*, and* $f\colon A \to B$ *a homomorphism. Given* $e\colon A \hookrightarrow M$ *and* $j\colon B \hookrightarrow N$*, we use* $f\langle e, j\rangle$ *to denote* $j \circ f \circ e^{-1}\colon e[A] \to N$*.*

When $M = N$*, we call* $f\langle e, j\rangle$ *a* manifestation *of* f *in* M*.*

Of course, every local homomorphism of a structure M is manifestation of a homomorphism in $\mathrm{Mon}(\mathrm{Age}(M))$.

Definition 4. *Let* M *be an* L*-structure and* $\mathbb{M} \subseteq \mathrm{End}(M)$ *a monoid of endomorphisms. The relation* $\mathrm{Rep(f;e,j)}^{\mathbb{M}}_{\mathrm{M}}$ *holds for a homomorphism* $f\colon A \to B$ *and embeddings* $e\colon A \hookrightarrow M$*,* $j\colon B \hookrightarrow M$ *if there exists* $F \in \mathbb{M}$ *such that* $f\langle e, j\rangle = F\big|_{e[A]}$*.*

When $F\big|_{e[A]} = f\langle e, j\rangle$ *as above, we will say that* F represents f *in* $\mathbb{M}$ *over* e *and* j*.*

Remark 1. The condition $F\big|_{e[A]} = f\langle e, j\rangle$ is equivalent to $F \circ e = j \circ f$.

Definition 5. *Let M be an L-structure and $A \in \mathrm{Age}(M)$. Two embeddings $e, e' \colon A \hookrightarrow M$ are* domain-equivalent*, in symbols $e \sim e'$, if there exists $\sigma \in \mathrm{Aut}(A)$ such that $e' = e \circ \sigma$.*

Domain-equivalent embeddings have the same image, and domain-equivalence is intended to formalise the notion of considering a substructure of M as a subset with the structure induced on it, rather than as the image of a particular embedding.

There is a second equivalence relation that is relevant here. Suppose that an endomorphism F represents a homomorphism $f \colon A \to B$ over e and j, and we have embeddings $e' \sim e$, $j' \sim j$, so that $j = j' \circ \tau$ and $e' = e \circ \sigma$ for some automorphisms σ, τ. Then the same F represents $\tau \circ f \circ \sigma^{-1}$. This motivates the following definition.

Definition 6. *Let M be an L-structure and $A, B \in \mathrm{Age}(M)$. Two homomorphisms $h, h' \colon A \to B$ are* equivalent*, in symbols $h \approx h'$, if there exists $\sigma \in \mathrm{Aut}(A)$ and $\tau \in \mathrm{Aut}(B)$ such that $h' = \tau \circ h \circ \sigma$.*

Just as domain-equivalence was about substructures, equivalence is intended to capture the effect of a homomorphism $f \colon A \to B$, in the sense that two equivalent homomorphisms will produce isomorphic substructures that are embedded in B in the same way (i.e., their images are translates of one another under the natural action of the automorphism group of B).

Proposition 7. *Let M be an L-structure, $A, B \in \mathrm{Age}(M)$, $\mathbb{M}$ a monoid of endomorphisms of M, and $f \colon A \to B$ a homomorphism. Suppose $g \approx f$ and there exist $e \colon A \hookrightarrow M, j \colon B \hookrightarrow M$ such that $\mathrm{Rep(f; e, j)}^{\mathbb{M}}_{\mathrm{M}}$; then there exist $e' \sim e, j' \sim j$ such that $\mathrm{Rep(g; e', j')}^{\mathbb{M}}_{\mathrm{M}}$.*

Proof. Suppose $g \approx f$, say, $g = \tau \circ f \circ \sigma$. Take embeddings e', j' with $e' = e \circ \sigma$ and $j = j' \circ \tau$. Now

$$F \circ e' = F \circ e \circ \sigma = j \circ f \circ \sigma = j' \circ \tau \circ f \circ \sigma = j' \circ g.$$

$\square$

Proposition 7 tells us that the statement $\mathrm{Rep(f; e, j)}^{\mathbb{M}}_{\mathrm{M}}$ is really about the $\approx$-class of f and the $\sim$-classes of e and j. Although we will not say it explicitly in our statements or introduce notation for this fact, the reader should keep it in mind.

In the following theorem, I$\mathbb{M}$ means that every local isomorphism is restriction of some element of $\mathbb{M}$.

Theorem 8. *Let M be an I$\mathbb{M}$-homogeneous L-structure, where $\mathbb{M} \subseteq \mathrm{End}(M)$ is a monoid. If $\mathrm{Rep(f; e, e')}^{\mathbb{M}}_{\mathrm{M}}$ holds for some $e \colon A \hookrightarrow M, e' \colon B \to M$, then every manifestation of f in M is restriction of an element of $\mathbb{M}$.*

Proof. Let F be a function in $\mathbb{M}$ satisfying $F \circ e = e' \circ f$.

Take local isomorphisms $i\colon \mathrm{im}(j) \to \mathrm{im}(e)$ and $i'\colon \mathrm{im}(e') \to \mathrm{im}(j')$. By $\mathbb{IM}$-homogeneity, i and i' are restrictions of endomorphisms $I, J \in \mathbb{M}$ respectively. Then $J \circ F \circ I \in \mathbb{M}$ represents f over j, j'. $\square$

The preceding theorem allows us to simply write 'f is represented in $\mathbb{M}$,' without mentioning the embeddings, when M is $\mathbb{IM}$-homogeneous.

Corollary 9. *Let M be an $\mathbb{IM}$-homogeneous structure, where $\mathbb{M}$ is a monoid of endomorphisms of M, and $f\colon A \to B$ a homomorphism in* $\mathrm{Age}(M)$. *If* $\mathrm{Rep(f;e,e')}_{\mathrm{M}}^{\mathbb{M}}$ *holds, and $f \approx g$, then* $\mathrm{Rep(g;j,j')}_{\mathrm{M}}^{\mathbb{M}}$ *holds for all $j\colon A \to M$, $j'\colon B \to M$.*

Proof. Follows directly from Proposition 7 and Theorem 8. $\square$

We will write $\mathrm{Rep(f)}_{\mathrm{M}}^{\mathbb{M}}$ to mean 'there exist e, e' such that $\mathrm{Rep(f;e,e')}_{\mathrm{M}}^{\mathbb{M}}$.'

Definition 10. *Let $g\colon A \to C$ be a homomorphism. We say that g is* associated with *$f\colon A \to B$ if f is a surjective homomorphism and there exists an embedding $B \hookrightarrow C$ such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle 1_A}\uparrow & & \uparrow \\ A & \xrightarrow{f} & B \end{array}$$

For example, embeddings are associated with automorphisms and monomorphisms are associated with surjective monomorphisms. We can say that two monomorphisms g and h are effect-equivalent if there exists a surjective f such that both g and h are associated with f.

Now, a simple observation: we can restrict our attention to surjective homomorphisms under any notion of homogeneity.

Proposition 11. *Let M be an $\mathbb{IM}$-homogeneous structure for some monoid $\mathbb{M} \subseteq \mathrm{End}(M)$. For all homomorphisms $g\colon A \to C$,* $\mathrm{Rep(g)}_{\mathrm{M}}^{\mathbb{M}}$ *holds iff* $\mathrm{Rep(f)}_{\mathrm{M}}^{\mathbb{M}}$, *where f is a surjective representative of the effect-equivalence class of g.*

Proof. Direct consequence of Theorem 8. $\square$

Proposition 11 will be applied freely without reference throughout the text.

3. Extensible monomorphisms and IM-homogeneity

In this section we isolate the monomorphisms over which a restricted version of the mono-amalgamation property from [4] holds in IM-homogeneous structures, and prove an analogue of Fraïssé's theorem for IM.

Notation 1. Let $\mathcal{C}$ be the object set of a pre-Fraïssé class and $A \in \mathcal{C}$.

1. $\mathcal{C}^{(n)}$ is the set of structures in $\mathcal{C}$ with n elements,

2. $\mathcal{C}^{(\leq n)}$ is the set of structures in $\mathcal{C}$ with at most n elements,

3. $\mathcal{C}_A$ is the set of structures $B \in \mathcal{C}$ for which there exist embeddings $A \hookrightarrow B$.

We can mix these symbols; for example, $\mathcal{C}_A^{(\leq n)}$ is the set of elements of $\mathcal{C}$ that contain isomorphic copies of A and have at most n elements.

Definition 12. *Let $(\mathcal{C}, \mathrm{Mon}(\mathcal{C}))$ be a pre-Fraïssé class, $A, B \in \mathcal{C}$, and $f\colon A \to B$ a surjective monomorphism. We will say that f is*

1. extensible over $e\colon A \hookrightarrow A'$ *if there exists $B' \in \mathcal{C}_B^{(|A'|)}$, $j\colon B \hookrightarrow B'$, and a surjective monomorphism $g\colon A' \to B'$ such that $j \circ f = g \circ e$. In this case, g is called an* $(|A'| - |A|)$-extension *of f.*

2. 1-extensible *in $\mathcal{C}$ if it is extensible over all $e\colon A \hookrightarrow A^+$, for all $A^+ \in \mathcal{C}_A^{(|A|+1)}$.*

3. $n+1$-extensible *in $\mathcal{C}$ if f is n-extensible and every n-extension of f is 1-extensible.*

4. $\mathcal{C}$-extensible *if f is $n+1$-extensible for all $n \in \omega$.*

If g is a non-surjective monomorphism, we will say that g is 1-extensible (or n-extensible, or age-extensible) if it is associated with a surjective 1-extensible (or n-extensible, or age-extensible) monomorphism.

The idea behind this definition is to simulate in $\mathrm{Age}(M)$ the steps that one would naturally take to prove that a local monomorphism f can be extended to an injective endomorphism in a countable structure M, namely choosing any vertex outside the domain of f and finding a suitable image for it outside the image of f.

Proposition 13. *Let M be an* IM*-homogeneous structure and suppose $j\colon A \hookrightarrow B$ is an embedding between elements of* $\mathrm{Age}(M)$*. If $e\colon A \hookrightarrow M$ is an embedding, then there exists a monomorphism $m\colon B \to M$ such that $m \circ j = e$.*

Proof. Let $h\colon B \to M$ be an embedding, and $A' = h \circ j[A]$. Take any local isomorphism $i\colon A' \to e[A]$ and apply IM-homogeneity: there exists an injective endomorphism I of M that extends i. Now $m := I \circ h$ satisfies the conclusions of the statement. □

We will later give a more detailed version Proposition 13, (Theorem 15) which characterises IM-homogeneity.

The proofs from this point on involve a great deal of selecting embeddings with the appropriate image and naming isomorphism types corresponding to finite substructures of some structure. To abbreviate these steps, we introduce a couple of symbols.

Notation 2.

1. If M is an L-structure and X is a finite induced substructure of M, then $\mathrm{isotp}(X)$ denotes the unique object of $\mathrm{Age}(M)$ isomorphic to X.

2. If $C \in \mathrm{Age}(M)$ and $X \subset M$ is isomorphic to C, then E_X^C is a representative of the $\sim$-class of an embedding $C \hookrightarrow M$ with image X.

The actual identity of E_X^C is not relevant, but only its $\sim$-class, so the definition above is good enough for our purposes.

Theorem 14. *Let M be a countable IM-homogeneous L-structure. A monomorphism $f\colon A \to B$ from* $\mathrm{Mon}(\mathrm{Age}(M))$ *is* $\mathrm{Age}(M)$*-extensible if and only if* $\mathrm{Rep(f)}_{\mathrm{M}}^{\mathrm{Mon(M)}}$.

Proof. Suppose first that $\mathrm{Rep(f)}_{\mathrm{M}}^{\mathrm{Mon(M)}}$ holds, so there exist embeddings e, j such that $f\langle e,j\rangle$ is restriction of a monomorphism $M \to M$. It follows from IM-homogeneity (Theorem 8) that all manifestations of f are extensible.

Let $f_n\colon A_n \to B_n$ be an n-extension of f (we allow $n = 0$). We will prove that g is $n+1$-extensible.

Given any $A^+ \in \mathrm{Age}(M)_{A_n}^{(|A_n|+1)}$ and an embedding $s\colon A \hookrightarrow A^+$, let $i\colon A^+ \hookrightarrow M$ and $j\colon B_n \hookrightarrow M$ be embeddings. By Theorem 8, the manifestation $f_n\langle i \circ s, j\rangle$ is restriction of some injective endomorphism $F\colon M \to M$.

Let B^+ be $\mathrm{isotp}(F \circ i[A^+])$ and $t\colon B_n \to B^+$ any embedding with $E_{F\circ i[A^+]}^{B^+} \circ t = j$. The sets $i[A^+] \setminus i \circ s[A_n]$ and $F \circ i[A^+] \setminus j[B_n]$ contain one vertex each, say u and v respectively. The choices we have made ensure that $f_n[i,j] \cup \{(u,v)\}$ is a local surjective monomorphism, or equivalently, manifestation of some $f_{n+1}\colon A^+ \to B^+$ via the embeddings i and $F \circ i$. This proves that f is $n+1$-extensible, and by induction $\mathrm{Age}(M)$-extensible.

For the converse, suppose that f is $\mathrm{Age}(M)$-extensible, and let $f\langle e,j\rangle$ be a manifestation of f. Given any $a \notin e[A]$, let A^+ be $\mathrm{isotp}(e[A] \cup \{a\})$ and $s\colon A \to A^+$ an embedding satisfying $E_{e[A]\cup\{a\}}^{A^+} \circ s = e$. By 1-extensibility of f, there exists B^+, an embedding $t\colon B \to B^+$ and a monomorphism $f^+\colon A^+ \to B^+$ with $t \circ f = f^+ \circ s$. By Proposition 13, there exists a monomorphism $k\colon B^+ \to M$ with $k \circ t = j$. All the injectivity conditions imply that $k[B] \setminus j[B]$ contains exactly one element, say b. Tracing the arrows, one can see that $f\langle e,j\rangle \cup \{(a,b)\}$ is a monomorphism. Since M is countable, we can run this argument over an enumeration to obtain an injective endomorphism that extends $f\langle e,j\rangle$. $\square$

Next comes a strengthening of Proposition 13, the IM-analogue of the embedding-extension property of ultrahomogeneous structures.

Theorem 15. *A countable structure M is* IM*-homogeneous if and only if for all $A, B \in$* $\mathrm{Age}(M)$ *and all embeddings $s\colon A \hookrightarrow B, i\colon A \hookrightarrow M$ there exists $C \in$* $\mathrm{Age}(M)$*, an* $(\mathrm{Age}(M), \mathrm{Mon}(\mathrm{Age}(M)))$*-extensible $f\colon B \to C$, and $k\colon C \hookrightarrow M$ such that $k \circ f \circ j = e$.*

Proof. The "only if" part is essentially Proposition 13 (in the proof, we can take C to be the isomorphism type of $m[B]$ with an appropriate embedding;

m is Age(M)-extensible by Theorem 14, since it is restriction of an injective endomorphism).

Now let σ be an automorphism of $A \in \text{Age}(M)$, and consider the local isomorphism $\sigma\langle i, j\rangle$.

Given any $a \notin i[A]$, let A' be isotp$(A \cup \{a\})$. We can apply the hypothesis to an embedding $s\colon A \to A'$ that satisfies $E^{A'}_{i[A]\cup\{a\}} \circ s = i$ and j to find $c \notin j[A]$ such that $f_1 \coloneqq \sigma\langle i, j\rangle \cup \{(a, c)\}$ is a monomorphism.

The induction step is slightly more complicated. Suppose that we have succeeded in finding a local monomorphism $f_n\langle i_n, j_n\rangle$ extending $\sigma\langle i, j\rangle$ such that $f_n\colon A_n \to B_n$ is Age(M)-extensible. Consider $b \notin i_n[A_n]$, and let A_{n+1} be isotp$(A_n \cup \{b\})$ with an embedding $u_n\colon A_n \to A_{n+1}$ that satisfies $E^{A_{n+1}}_{i_n[A_n]\cup\{b\}} \circ u_n = i_n$.

We know that $f_n\colon A_n \to B_n$ is Age(M)-extensible, so there exist $K \in \text{Age}(M)^{(|B_n|+1)}_{B_n}$, $w\colon B_n \hookrightarrow K$, and an Age$(M)$-extensible $g\colon A_{n+1} \to K$ such that $g \circ u_n = w \circ f_n$. Now we can apply the hypothesis to find $B \in \text{Age}(M)$, an Age(M)-extensible monomorphism $m\colon K \to B$, and an embedding $r\colon B \to M$ such that $r \circ m \circ w = j_n$; take B_{n+1} as isotp$(r[B])$ and f_{n+1} to be $(m \circ f_n)\langle E^{A_{n+1}}_{i_n[A_n]\cup\{b\}}, r\rangle$.

If we argue as above over an enumeration of M, then we find a monomorphism $M \to M$ extending $\sigma\langle i, j\rangle$ after ω steps. $\square$

We will use wavy arrows to represent $\mathcal{C}$-extensible monomorphisms, to underscore their special status.

Lemma 16. *Let M be an IM-homogeneous structure. Then the following amalgamation property holds in* Age(M)*:*

> **(IMAP)** *for all $A, B_1, B_2 \in$ Age(M), if $f\colon A \to B_1$ is* Age(M)*-extensible and $g\colon A \hookrightarrow B_2$ is an embedding, then there exist $C \in$ Age(M), an embedding $f'\colon B_1 \hookrightarrow C$, and an* Age$(M)$*-extensible $g'\colon B_2 \to C$ such that $f' \circ f = g' \circ g$.*

$$\begin{tikzcd} & B_1 \arrow[dr, hook, "f'"] & \\ A \arrow[ur, squiggly, "f"] \arrow[dr, hook, "g"'] & & C \\ & B_2 \arrow[ur, squiggly, "g'"'] & \end{tikzcd}$$

Proof. Given f, g, A, B_1, B_2 as in the statement, take $e\colon B_2 \hookrightarrow M$ and $j\colon B_1 \hookrightarrow M$, and consider $f\langle e \circ g, j\rangle$. Since f is Age(M)-extensible, we know by Theorem 14 that $\mathrm{Rep(f; e \circ g, j)^{Mon(M)}_{M}}$ holds, so there exists an injective endomorphism F that extends $f\langle e \circ g, j\rangle$. Let C be isotp$(F \circ e[B_2] \cup j[B_1])$, and k an embedding in the $\sim$-class of $E^{C}_{F\circ e[B_2]\cup j[B_1]}$.

Now $f' \coloneqq k^{-1} \circ j$, $g' \coloneqq k^{-1} \circ F \circ e$, and C witness the conclusions of the lemma. $\square$

Proposition 17. *In any pre-Fraïssé class $(\mathcal{C}, \text{Mon}(\mathcal{C}))$, if f and g are $\mathcal{C}$-extensible monomorphisms and* $\text{dom}(g) = \text{codom}(f)$*, then $g \circ f$ is $\mathcal{C}$-extensible.*

Proof. We may assume that f and g are surjective.

Suppose $f\colon A \to B$ and $g\colon B \to C$. Given $A^+ \in \mathcal{C}_A^{(|A|+1)}$ and $e\colon A \hookrightarrow A^+$, 1-extensibility of f yields $j\colon B \hookrightarrow B^+$ and $f^+\colon A^+ \to B^+$. Now we can apply 1-extensibility of g with respect to j to obtain g^+ and an embedding $k\colon C \hookrightarrow C^+$. Since f^+ and g^+ are composable monomorphisms, the monomorphism $g^+ \circ f^+$, together with k and C^+, witnesses 1-extensibility.

This argument can be repeated to prove inductively that $g \circ f$ is $n+1$-extensible for all $n \in \omega$. $\square$

A quintuple (A, B_1, B_2, f, e) where A, B_1, B_2 are elements of a pre-Fraïssé class $\mathcal{C}$, $f\colon A \to B_1$ is age-extensible in $(\mathcal{C}, \mathrm{Mon}(\mathcal{C}))$, and $e\colon A \hookrightarrow B_2$ is an *IM-amalgamation problem.* A structure $C \in \mathcal{C}$ for which there exist $j\colon B_1 \hookrightarrow C$ and age-extensible $g\colon B_2 \to C$ is a *solution* to (A, B_1, B_2, f, e).

Proposition 18. *Let $(\mathcal{C}, \mathrm{Mon}(\mathcal{C}))$ be the age of an IM-homogeneous structure, and suppose that $f\colon A \to P$ is $\mathcal{C}$-extensible and $e\colon A \hookrightarrow B$ is an embedding. If $i\colon P \to Q$ is an embedding, then a solution to the problem $(A, Q, B, i \circ f, e)$ is a solution to (A, P, B, f, e).*

Proof. Suppose that S is a solutioin to $(A, Q, B, i \circ f, e)$, witnessed by u and g, so that $u \circ (i \circ f) = g \circ e$. Then S is a solution to (A, P, B, f, e), as witnessed by $u \circ i$ and g. $\square$

Thus, we can 'transfer' an amalgamation problem over P to one over Q in order to enrich a given finite structure with images under age-extensible monomorphisms of some structure.

Lemma 19. *Let $(\mathcal{C}, \mathrm{Mon}(\mathcal{C}))$ be a pre-Fraïssé class. If all embeddings are $\mathcal{C}$-extensible and the IMAP holds, then there exists a countable IM-homogeneous structure $M_{\mathcal{C}}$ with $\mathrm{Age}(M_{\mathcal{C}}) = \mathcal{C}$.*

Proof. Enumerate $\mathcal{C}$ as $\{C_i : i \in \omega\}$. Let $M_0 = C_0$; each time we select an M_i, we produce an enumeration of all IM-amalgamation problems of the form

$$\begin{tikzcd} & M_i \\ A \arrow[ur, rightsquigarrow, "f"] \arrow[dr, hook, "e"'] & \\ & B \end{tikzcd}$$

as tuples, that is $\{(A_{i,j}, B_{i,j}, f_{i,j}, e_{i,j}) : j \in \omega\}$. For bookkeeping purposes, we will also need a bijective function $\varphi\colon \{2n : n \in \mathbb{N}\} \to \{(i,j) : i, j \in \omega\}$ such that if $(i,j) = f(m)$, then $i \leq m$.

Suppose that we have produced M_k, and embeddings $s_i\colon M_{i-1} \hookrightarrow M_i$ for $1 \leq i \leq k$.

1. If $k = 2n$, then consider $\varphi(k)$. This is a pair (m, n) corresponding to some amalgamation problem that we have already enumerated. If $m = k$, then we take any solution of the amalgamation problem as M_{k+1} and

$s_k \colon M_k \hookrightarrow M_{k+1}$ as the embedding given by the IMAP; and if $m < k$, then we solve the problem

$$\begin{tikzcd}
 & M_k \\
A_{m,n} \arrow[ur, hook, squiggly, "s_k \circ s_{k-1} \circ \ldots \circ s_m \circ f_{m,n}"] \arrow[dr, hook, "e_{m,n}"'] & \\
 & B
\end{tikzcd}$$

By Proposition 18, a solution to this problem is also a solution to the original one, and moreover, we enrich the current M_k. Let s_{k+1} the embedding $M_k \to M_{k+1}$ given by the IMAP.

2. If $k = 2n+1$, then let M_{k+1} be a structure obtained by applying the JEP to M_k and C_{n+1}, which also yields an embedding $s_{k+1} \colon M_k \to M_{k+1}$.

Now, to construct the countable structure, let ξ_0 be any bijective function $M_0 \to |M_0|$; if we have enumerated M_k by ξ_k, then let ξ_{k+1} to be any bijection $M_{k+1} \to |M_{k+1}|$ extending the function $\xi'_{k+1} \colon s_{k+1}[M_k] \to |M_k|$ given by $\xi'_{k+1}(s_{k+1}(v)) = \xi_k(v)$. This guarantees $\xi_{k+1} \circ s_{k+1} = \xi_k$.

Given any tuple of natural numbers $(a_1, \ldots, a_{\mathrm{ar}(R)})$, there exists k such that $|M_k| \geq \max\{a_1, \ldots, a_{\mathrm{ar}(R)}\}$. We declare

$$R^M(a_1, \ldots, a_{\mathrm{ar}(R)}) \Leftrightarrow R^{M_k}(\xi_k^{-1}(a_1), \ldots, \xi_k^{-1}(a_{\mathrm{ar}(R)})).$$

Now we claim that $\mathrm{Age}(M) = \mathcal{C}$ and M is IM-homogeneous.

$\mathrm{Age}(M) \subseteq \mathcal{C}$ is clear because any subset of M embeds in M_k for sufficiently large k, and M_k is an element of the hereditary class $\mathcal{C}$. The odd steps of the construction ensure that every element of $\mathcal{C}$ embeds into some M_k, and therefore into M.

To prove IM-homogeneity, we will show that the condition from Theorem 15 is satisfied by M. Let $e \colon A \to B$ and $i \colon A \to M$ be embeddings. By construction, there is a natural number k such that i is an embedding $A \hookrightarrow M_k$. At some point we solved a problem whose solution is also a solution of

$$\begin{tikzcd}
 & M_k \\
A \arrow[ur, hook, "i"] \arrow[dr, hook, "e"'] & \\
 & B
\end{tikzcd}$$

(embeddings are $\mathcal{C}$-extensible by hypothesis), which ensures the existence of an $\mathcal{C}$-extensible monomorphism $B \to M_{k+1} \subset M$, and so M is IM-homogeneous. This concludes our proof. $\square$

Definition 20. *Two structures M_0, M_1 are EM-equivalent (from Extensible and Monomorphism) if $\mathrm{Age}(M_0) = \mathrm{Age}(M_1)$ and for all $\mathrm{Age}(M_0)$-extensible $f \colon A \to B$ and all embeddings $e_i \colon A \to M_i, e_{1-i} \colon B \to M_{1-i}$, the function $f\langle e_i, e_{1-i}\rangle$ is restriction of a monomorphism $M_i \to M_{1-i}$.*

Our next result is the analogue of Fraïssé's theorem for IM-homogeneous structures. We use $\mathrm{Ext}(\mathcal{C})$ to denote the set of all $\mathcal{C}$-extensible monomorphisms.

Theorem 21 (IM Fraïssé)**.** *Let M be an IM-homogeneous structure. Then all embeddings are $\mathrm{Age}(M)$-extensible, $(\mathrm{Age}(M), \mathrm{Ext}(\mathrm{Age}(M)))$ is a pre-Fraïssé class with the IMAP and local monomorphism is represented in $\mathrm{Mon}(M)$ iff it is a manifestation of an $\mathrm{Age}(M)$-extensible monomorphism.*

Conversely, if $(\mathcal{C}, \mathrm{Ext}(\mathcal{C}))$ is a pre-Fraïssé class with the IMAP in which all embeddings are $\mathcal{C}$-extensible, then there exists a countable IM-homogeneous structure $M_{\mathcal{C}}$ such that $\mathrm{Age}(M_{\mathcal{C}}) = \mathcal{C}$ and in which a local monomorphism is M-extensible iff it is a manifestation of a $\mathcal{C}$-extensible monomorphism.

Finally, any two countable IM-homogeneous structures with the same age are EM-equivalent.

Proof. Theorem 14 and Lemma 16 prove the first statement; its converse is Lemma 19.

We need a bit of scaffolding for the proof of uniqueness of the limits up to EM-equivalence. Take $u \in \{0, 1\}$ and two IM-homogeneous structures M_u, M_{1-u} with the same age $\mathcal{C}$. Let $f_0 \colon A_0 \to B_0$ be any $\mathcal{C}$-extensible monomorphism in $\mathrm{Mon}(\mathcal{C})$ and $e_0 \colon A_0 \to M_u, j_0 \colon B_0 \to M_{1-u}$ two embeddings with images X_0, Y_0 respectively, so that $h_0 \coloneqq f_0\langle e_0, j_0\rangle$ is a surjective monomorphism $X_0 \to Y_0$.

Enumerate $M_u \setminus X_0$ as $\{x_i : i \in \omega\}$, and let A_i be $\mathrm{isotp}(X_i)$, where $X_i \coloneqq X \cup \{x_j : j < i\}$. Write e_i for $E^{A_i}_{X_i}$ and take embeddings $v_i \colon A_i \to A_{i+1}$ such that $\mathrm{id}\big|_{X_i} \circ e_i = e_{i+1} \circ v_i$.

For each $i \in \omega$, we will find finite subsets $Y_{i+1} \supset Y_n$ contained in M_{1-u}, $B_i \in \mathcal{C}$, and all the functions necessary to make the following diagram commutative for any finite k:

$$
\begin{tikzcd}
 & A_0 \arrow[rr, hook, "v_0"] \arrow[dl, "e_0"'] \arrow[dd, squiggly, "f_0"] & & A_1 \arrow[rr, hook, "v_1"] \arrow[dl, "e_1"'] \arrow[dd, squiggly, "f_1"] & & A_2 \arrow[rr, hook, "v_2"] \arrow[dl, "e_2"'] \arrow[dd, squiggly, "f_2"] & & \dots \arrow[rr, hook, "v_{k-1}"] & & A_k \arrow[dl, "e_k"'] \arrow[dd, squiggly, "f_k"] \\
X_0 \arrow[rr, hook, "\mathrm{id}"] \arrow[dd] & & X_1 \arrow[rr, hook, "\mathrm{id}"] \arrow[dd] & & X_2 \arrow[rr, hook, "\mathrm{id}"] \arrow[dd] & & \dots \arrow[rr, hook, "\mathrm{id}"] & & X_k \arrow[dd] & \\
 & B_0 \arrow[rr, hook, "w_0"] \arrow[dl, "j_0"'] & & B_1 \arrow[rr, hook, "w_1"] \arrow[dl, "j_1"'] & & B_2 \arrow[rr, hook, "w_2"] \arrow[dl, "j_2"'] & & \dots \arrow[rr, hook, "w_{k-1}"] & & B_k \arrow[dl, "j_2"'] \\
Y_0 \arrow[rr, hook, "\mathrm{id}"] & & Y_1 \arrow[rr, hook, "\mathrm{id}"] & & Y_2 \arrow[rr, hook, "\mathrm{id}"] & & \dots \arrow[rr, hook, "\mathrm{id}"] & & Y_k &
\end{tikzcd}
$$

The unmarked arrows $X_i \to Y_i$ are $f_i\langle e_i, j_i\rangle$.

Once Y_{i+1} has been selected (this will be explained below), take B_{i+1} as $\mathrm{isotp}(Y_{i+1})$ and embeddings $w_i \colon B_i \hookrightarrow B_{i+1}$, $j_{i+1} \colon B_{i+1} \hookrightarrow Y_{n+1}$ such that $j_{i+1} \circ w_i = j_i$; the $\mathcal{C}$-extensible $f_{i+1} \colon A_{i+1} \to Bi + 1$ will come from $\mathcal{C}$-extensibility of f_i. Since M_u is countable, $H \coloneqq \bigcup\{f_i\langle e_i, j_i\rangle : i \in \omega\}$ is a monomorphism $M_u \to M_{1-u}$.

Suppose f_k, Y_k, B_k, j_k, w_k have been defined for $k \leq t$. We will now describe how to extend the sequences to $t + 1$.

Consider A_{t+1}. Let $s \colon A_{t+1} \to M_{1-u}$ be any embedding. Let g be the manifestation of f_t over the embeddings $s \circ v_t$ and f_t, so that $g \circ s \circ v_t = j_t \circ f_t$.

We know that M_{1-u} is IM-homogeneous and f_t is $\mathcal{C}$-extensible, so g restriction of an injective endomorphism $F\colon M_{1-u} \to M_{1-u}$.

Define Y_{t+1} as $Y_t \cup \{F(v)\}$, where v is the unique element of $s[A_{t+1}] \setminus s \circ v_t[A_t]$, and let B_{t+1} be isotp(Y_{t+1}). The embeddings j_{t+1} and w_t can now be chosen so that $j_{t+1} \circ w_t = j_t$, and we can take f_{t+1} to be the monomorphism of which $F\big|_{s[A_{t+1}]}$ is a manifestation over the embeddings s and j_{t+1}. Applying the equivalence between being restriction of an injective endomorphism of M_{1-u} and $\mathcal{C}$-extensibility, we see that f_{t+1} is $\mathcal{C}$-extensible.

Continuing in this manner ω steps, we find a monomorphism $M_u \to M_{1-u}$ as described before.

□

4. IB-homogeneity

IB-homogeneity is a special case of IM-homogeneity in which the extensions of local isomorphisms are guaranteed to be bijective.

Conceptually, there is no great difference between this section and the preceding one, and we will sometimes refer the reader to the corresponding result in section 3 when a proof is a simple modification of the IM case.

Definition 22. *Let $L = \{R_i : i \in I\}$ be a relational language. If $\mathfrak{C} = (C, S_i)_{i\in I}$ is an L-structure, then its* complement *is the structure*

$$\overline{\mathfrak{C}} = (C, C^{\mathrm{ar}(R_i)} \setminus S_i)_{i\in I}.$$

In other words, a tuple $(c_1, \dots c_{\mathrm{ar}(R_k)})$ satisfies R_k in $\overline{\mathfrak{C}}$ exactly when $\mathfrak{C} \models \neg R_k(c_1, \dots c_{\mathrm{ar}(R_k)})$. We will usually not distinguish between a structure and its domain.

Definition 23. *Let A, B be relational structures in the language L. An* antimonomorphism *is an injective function $f\colon a \to B$ such that*

$$A \models \neg R(a_1, \dots, a_{\mathrm{ar}(R)}) \Rightarrow B \models \neg R(f(a_1), \dots, f(a_{\mathrm{ar}(R)})),$$

for all $R \in L$.

Antimonomorphisms will be used to find preimages for elements outside the image of a local monomorphism. The following two results are straightforward generalisations of results from [9].

Observation 24. *Let A, B be relational structures. A function $f\colon A \to B$ is a bijective homomorphism iff f^{-1} is a bijective homomorphism $\overline{B} \to \overline{A}$.*

Proof. Suppose that f is a bijective homomorphism. If $R(b_1, \dots, b_{\mathrm{ar}(R)})$ holds in $\overline{B}$, then $\neg R(b_1, \dots, b_{\mathrm{ar}(R)})$ in B. Since f is bijective and preserves the relations from L, but may map a tuple that does not satisfy R to a tuple that does, the preimage of a tuple that does not satisfy R is always a tuple that does not satisfy R, so $\neg R(f^{-1}(b_1), \dots, f^{-1}(b_{\mathrm{ar}(R)}))$ holds in A, or, equivalently, $R(f^{-1}(b_1), \dots, f^{-1}(b_{\mathrm{ar}(R)}))$ in $\overline{A}$. □

In the case of injective but not surjective homomorphisms, the proof of Observation 24 says that the left inverse of a monomorphism is a partial injective homomorphism between the complement structures. Using $(f^{-1})^{-1} = f$ and $\overline{\overline{A}} = A$, the four conditions below are equivalent.

Corollary 25. *Let A, B be relational structures and suppose that $f : A \to B$ is a function. The following are equivalent:*

1. *f is a bijective homomorphism $A \to B$,*
2. *f^{-1} is a bijective homomorphism $\overline{B} \to \overline{A}$,*
3. *f is a bijective antihomomorphism $\overline{A} \to \overline{B}$, and*
4. *f^{-1} is a bijective antihomomorphism $B \to A$.*

Bijective endomorphism of a structure M are called *bimorphisms*; $\mathrm{Bi}(M)$ is the bimorphism monoid of M.

Corollary 26. *Let M be an L structure. For all monomorphisms $f \in \mathrm{Mon}(\mathrm{Age}(M))$, $\mathrm{Rep(f)}_{\mathrm{M}}^{\mathrm{Bi(M)}}$ iff $\mathrm{Rep(f^{-1})}_{\overline{\mathrm{M}}}^{\mathrm{Bi}(\overline{\mathrm{M}})}$.*

Proof. Follows from Corollary 25. □

Proposition 27. *If M is an IB-homogeneous L-structure, then $\overline{M}$ is also IB-homogeneous.*

Proof. Suppose M is an IB-homogeneous L-structure. By definition, $\overline{M}$ is also an L-structure.

Let $i\colon A \to B$ be a local isomorphism in $\overline{M}$. Since isomorphisms preserve relations and their negations, $i^{-1}\colon \overline{B} \to \overline{A}$ is a local isomorphism of M. By IB-homogeneity, there exists a bimorphism $J\colon M \to M$ that extends i^{-1}. Now by Corollary 25, $J^{-1}\colon \overline{M} \to \overline{M}$ is a bimophism of $\overline{M}$, and since J extends i^{-1}, it follows that J^{-1} extends i. □

Now we will proceed by analogy with the IM case. The first step is to identify those monomorphisms in the age whose manifestations will be restrictions of bimorphisms in an IB-limit.

Notation 3. If $(\mathcal{C}, \mathrm{Mon}(\mathcal{C}))$ is a pre-Fraïssé class, then $\overline{\mathcal{C}}$ is $\{\overline{A} : A \in \mathcal{C}\}$.

Because we only allow total functions as arrows in a pre-Fraïssé class, the left inverse of a monomorphism is not necessarily an arrow in $(\overline{\mathcal{C}}, \mathrm{Mon}(\overline{\mathcal{C}}))$, but there is a function that is just as good for our purposes.

Definition 28. *Let $(\mathcal{C}, \mathrm{Mon}(\mathcal{C}))$ be a pre-Fraïssé class and $f\colon A \to B$ an element of $\mathrm{Mon}(\mathcal{C})$. Then f^{-} denotes a surjective antimonomorphism $C \to A$ for which an embedding $C \to B$ exists so that the following diagram commutes.*

$$
\begin{tikzcd}
A \arrow[r, "f"] & B & B \\
A \arrow[u, "\mathrm{id}_A"] & C \arrow[l, "f^-"] \arrow[u, hook]
\end{tikzcd}
$$

One can verify that the inverse of a surjective representative of the effect-equivalence class of f satisfies the conditions from Definition 28.

Definition 29. *Let $(\mathcal{C}, \mathrm{Mon})$ be a pre-Fraïssé class, $A, B \in \mathcal{C}$, and $f\colon A \to B$ a surjective monomorphism. We will say that f is*

1. 1-biextensible *if f is 1-extensible in $(\mathcal{C}, \mathrm{Mon}(\mathcal{C}))$ and f^{-1} is 1-extensible in $(\overline{\mathcal{C}}, \mathrm{Mon}(\overline{\mathcal{C}}))$.*
2. $n+1$-biextensible *if every n-extension of f in $(\mathcal{C}, \mathrm{Mon}(\mathcal{C}))$ is 1-biextensible and every n-extension of f^{-1} in $(\overline{\mathcal{C}}, \mathrm{Mon}(\overline{\mathcal{C}}))$ is 1-biextensible.*
3. $\mathcal{C}$-biextensible *if f is $n+1$-biextensible for all $n \in \omega$.*

If f is not surjective, we will say that f is 1-biextensible if it is associated with a 1-biextensible surjective monomorphism. We also define $n+1$-biextensibility and $\mathcal{C}$-biextensibility for non-surjective monomorphisms in this manner.

As a particular case of Theorem 8, we get that if any manifestation of a monomorphism f in an IB-homogeneous M is restriction of a bimorphism of M, then all manifestations of f in M are restrictions of a bimorphism of M.

In diagrams, antimonomorphisms will be represented by dashed arrows.

Theorem 30. *Let M be an IB-homogeneous structure and suppose $j\colon A \to B$ is an embedding between elements of $\mathrm{Age}(M)$. If $e\colon A \to M$ is an embedding, then there exist $C, D \in \mathrm{Age}(M)$, a surjective monomorphism $m\colon B \to C$, a surjective antimonomorphism $n\colon B \to D$, and embeddings $k\colon C \hookrightarrow M$, $t\colon D \hookrightarrow M$ such that $k \circ m \circ j = e = t \circ n \circ j$.*

$$
\begin{tikzcd}
 & B \arrow[dl, dashed, "n"'] \arrow[dr, squiggly, "m"] & \\
D \arrow[dr, "t"'] & A \arrow[u, hook, "j"] \arrow[d, hook, "e"] & C \arrow[dl, hook, "k"] \\
 & M &
\end{tikzcd}
$$

Moreover, m, n can be chosen so that $\mathrm{Rep(m)}_{\mathrm{M}}^{\mathrm{Bi(M)}}$ and $\mathrm{Rep(n)}_{\overline{\mathrm{M}}}^{\mathrm{Bi}(\overline{\mathrm{M}})}$ hold.

Proof. We can proceed as in the proof of Theorem 15. Let $i\colon B \hookrightarrow M$ be any embedding, and $s\colon i \circ j[A] \to e[A]$ an isomorphism. By IB-homogeneity, there exists a bimorphism $S\colon M \to M$ such that $S\big|_{i \circ j[A]} = s$.

Let C be $\mathrm{isotp}(S \circ i[B])$. Then $m = (E^C_{S \circ i[B]})^{-1} \circ S \circ i\colon B \to C$ is a surjective monomorphism, and $m\langle i, E^C_{S \circ i[B]}\rangle = S\big|_{i \circ j[A]}$, so $\mathrm{Rep(m)}_{\mathrm{M}}^{\mathrm{Bi(M)}}$ holds.

Since s is an isomorphism, s^{-1} is also an isomorphism, so it is restriction of a bimorphism $T\colon M \to M$. Let D be $\mathrm{isotp}(T^{-1} \circ i[B])$. Now $i^{-1} \circ T \circ E^{D}_{T^{-1}\circ i[B]}$ is a surjective monomorphism $D \to B$, so by Proposition 27, $(i^{-1} \circ T \circ E^{D}_{T^{-1}\circ i[B]})^{-1}$ is a local surjective antimonomorphism, or, equivalently, manifestation of the inverse of a surjective monomorphism, say $(n\langle i, E^{D}_{T^{-1}\circ i[B]}\rangle)^{-1}$. Again by Proposition 27, $n\langle i, E^{D}_{T^{-1}\circ i[B]}\rangle$ is restriction of a bimorphism of $\overline{M}$, and we get $\mathrm{Rep(n)}^{\mathrm{Bi}(\overline{\mathrm{M}})}_{\overline{\mathrm{M}}}$. □

We will now show that if M is IB-homogeneous, then the local monomorphisms that are represented in $\mathrm{Bi}(M)$ are exactly the manifestations of $\mathrm{Age}(M)$-biextensible monomorphisms.

Theorem 31. *Let M be a countable IB-homogeneous L-structure. A surjective monomorphism $f_0\colon A \to B$ in $\mathrm{Mon}(\mathrm{Age}(M))$ is $\mathrm{Age}(M)$-biextensible iff $\mathrm{Rep(f_0)}^{\mathrm{Bi(M)}}_{\mathrm{M}}$.*

Proof. Suppose that f_0 is $\mathrm{Age}(M)$-biextensible, and let $f_0\langle e_0, j_0\rangle$ be any manifestation of f_0.

Enumerate $M \setminus e[A] = \{x_i : i \in \omega\}$ and $M \setminus j[B] = \{y_i : i \in \omega\}$, and let $X_i \coloneqq e[A] \cup \{x_k : k < i\}$ $Y_i \coloneqq e[A] \cup \{y_k : k < i\}$.

Suppose that $f_r\langle e_r, j_r\rangle = g\colon S \to Y_m$ is a surjective monomorphism extending $f_0\langle e_0, j_0\rangle$. Consider x_n, where n is the least value such that x_n is not in S. Let P be $\mathrm{isotp}(S \cup \{x_n\})$, and $m\colon \mathrm{isotp}(S) \to P$ an embedding such that $E^{P}_{S\cup\{x_n\}} \circ m = e_r$. Apply $\mathrm{Age}(M)$-extensibility to find $C \in \mathrm{Age}(M)^{(|\mathrm{isotp}(Y_m)|+1)}_{\mathrm{isotp}(Y_m)}$, an embedding $k\colon \mathrm{isotp}(Y_m) \to C$, and an $\mathrm{Age}(M)$-biextensible $h\colon P \to C$. Now take any embedding $\ell\colon C \hookrightarrow M$, and any isomorphism $n\colon \ell \circ k[\mathrm{isotp}(Y_n)] \to j_r[\mathrm{isotp}(Y_r)]$. By IB-homogeneity, n is restriction of a bimorphism N.

Now we can take $B = \mathrm{isotp}(N \circ k[C])$ and f_{r+1} as a monomorphism such that $f_{r+1}\langle E^{P}_{S\cup\{x_n\}}, N \circ k\rangle = g \cup \{(a_0, F(c))\}$. This is a monomorphism extending g.

To ensure surjectivity for the extension, suppose that $f_i\langle e_i, j_i\rangle = h\colon X_t \to Z$ is an extension of $f_0\langle e_0, j_0\rangle$. Consider y_n, where n is the least value such that $y_n \notin Z$. Observe h^{-1} is a surjective local monomorphism in $\overline{M}$ by Proposition 24, so we can argue as in the preceding paragraph to find a vertex z such that $h^{-1} \cup \{(y_n, z)\}$ is a local monomorphism of $\overline{M}$. By Proposition 24 again, $g \cup \{(z, y_n)\}$ is a monomorphism that extends $f_i\langle e_i, j_i\rangle$.

Alternating the arguments of the two preceding paragraphs, we produce a bimorphism $F = \bigcup\{f_i\langle e_i, j_i\rangle : i \in \omega\}$.

For the converse, suppose that all manifestations of $f_0\colon A \to B$ are represented in $\mathrm{Bi}(M)$, as are all n-extensions of f_0 up to $n = k$ (we get 0 for free). We will prove that f_0 is $k+1$-$\mathrm{Age}(M)$-biextensible and the result will then follow by induction.

Let $g\colon A_k \to B_k$ be any k-extension of f. Let A_{k+1} be any element of $\mathrm{Age}(M)^{(|A_k|+1)}_{A_k}$, $j\colon A_k \hookrightarrow A_{k+1}$, and $e\colon A_{k+1} \hookrightarrow M$ any embedding. Consider any embedding $i\colon B_k \to M$. By hypothesis, every manifestation of g is M-extensible, so in particular $g\langle e \circ j, i\rangle$ is M-extensible, and we can take B_{k+1} to be the isomorphism type of the image of $e[A_{k+1}]$ under a bimorphism that extends $g\langle e\circ j, i\rangle$ to prove that g is $k+1$-extensible in $(\mathrm{Age}(M), \mathrm{Mon}(\mathrm{Age}(M)))$. Viewing the inverse of $g\langle e \circ j, i\rangle$ as a local monomorphism in M, and applying the same reasoning, we get that g^{-1} is $k+1$-biextensible in $(\overline{\mathrm{Age}(M)}, \mathrm{Mon}(\overline{\mathrm{Age}(M)}))$. $\square$

Our next goal is to describe the amalgamation property for IB-homogeneous structures.

Lemma 32. *Let M be an IB-homogeneous structure. Then the following amalgamation property holds in* $\mathrm{Age}(M)$*:*

> **(IBAP)** *for all $A, B_1, B_2 \in \mathrm{Age}(M)$ and all $f\colon A \to B_1$, $g\colon A \to B_2$, where f is $\mathrm{Age}(M)$-biextensible and g is an embedding, there exists $C \in \mathrm{Age}(M)$, an embedding $f'\colon B_i \to C$, and an $\mathrm{Age}(M)$-biextensible $g'\colon B_2 \to C$ such that $f' \circ f = g' \circ g$.*

Proof. Take A, B_1, B_2, f, g as in the statement, and let $e_1\colon B_1 \to M, e_2\colon B_2 \to M$ be embeddings. Now $f[e_2\circ g, e_1\circ f]$ is manifestation of an $\mathrm{Age}(M)$-biextensible monomorphism, so there exists a bimorphism $F\colon M \to M$ with $F\big|_{e_2\circ g[A]} = f[e_2 \circ g, e_1 \circ f]$.

Let C be the element of $\mathrm{Age}(M)$ isomorphic to $F \circ e_2[B_2] \cup e_1[B_1]$, and let d be an embedding $C \to M$ with image $F \circ e_2[B_2] \cup e_1[B_1]$. Now take $f'\colon B_1 \to C$ as an embedding with $d \circ f' = e_1$, and let g' be a monomorphism satisfying $g'[e_2, d] = F\big|_{e[B_2]}$.

Then $F\big|_{e[B_2]}$ is manifestation of g', and since F is a bimorphism extending $F_{e[B_2]}$, it follows from Theorem 31 that g' is $\mathrm{Age}(M)$-biextensible. This completes the proof. $\square$

Remark 2. In [1], all classes with XY where Y represents a surjective class of endomorphisms have two amalgamation properties. The same is true here, although it is not completely obvious: the IBAP can be broken into an amalgamation property for age-extensible monomorphisms whose inverses are associated with age-extensible monomorphisms in the age of complements, and an amalgamation property for age-extensible antimonomorphisms whose inverses are associated with age-extensible monomorphisms.

The IB-analogue of the embedding-extension property of ultrahomogeneous structures is our next result.

Theorem 33. *A countable structure M is IB-homogeneous if and only if for all $A, B \in \mathrm{Age}(M)$ and all embeddings $j\colon A \hookrightarrow B$, $e\colon A \hookrightarrow M$ there exists $C \in \mathrm{Age}(M)$, a surjective $\mathrm{Age}(M)$-biextensible $f\colon B \to C$ and an embedding $k\colon C \hookrightarrow M$ such that $k \circ f \circ j = e$.*

Proof. Suppose first that M is IB-homogeneous. Let $i\colon B \hookrightarrow M$ be an embedding, and $s\colon i\circ j[A] \to e[A]$ an isomorphism. By IB-homogeneity, s is restriction of a bimorphism $S\colon M \to M$. Let C be the isomorphism type of $S \circ i[B]$, and

f a monomorphism such that $S\big|_{i[B]}$ is manifestation of f. Then f is $\mathrm{Age}(M)$-biextensible by Theorem 31.

Now suppose that the condition from the statement holds, let $m\colon A \to A$ be an automorphism, and $e, j\colon A \hookrightarrow M$ embeddings. Follow the argument from the first part of the proof of Theorem 31 to extend the local isomorphism $m\langle e, j\rangle$ to a bimorphism of M. $\square$

Lemma 34. *Let $(\mathcal{C}, \mathrm{Mon}(\mathcal{C}))$ be a pre-Fraïssé class. If all embeddings are $\mathcal{C}$-biextensible and IBAP holds, then there exists a countable IB-homogeneous structure $M_{\mathcal{C}}$ with $\mathrm{Age}(M_{\mathcal{C}}) = \mathcal{C}$.*

Proof. Suppose that M is produced by the construction from the proof of Lemma 19, but using IBAP instead of IMAP. The proof of $\mathrm{Age}(M_{\mathcal{C}}) = \mathcal{C}$ is the same as in Lemma 19.

We claim that M is IB-homogeneous. We will prove that the condition from Theorem 33 holds in M. Suppose then that $e\colon A \hookrightarrow B$ and $i\colon A \hookrightarrow M$ are embeddings. By construction, there is a natural number k such that i is an embedding $A \hookrightarrow M_k$, and at some point we solved an amalgamation problem whose solution is also a solution of

$$\begin{tikzcd}[row sep=small] & M_k \\ A \arrow[ur, hook, "i"] \arrow[dr, hook, "e"'] & \\ & B \end{tikzcd}$$

so there exists a $\mathcal{C}$-biextensible $f\colon B \to M_{k+1}$, and M is IB-homogeneous. $\square$

Definition 35. *Two L-structures M_0, M_1 are* BB-equivalent *(from Biextensible and Bimorphism) if they have the same age $\mathcal{C}$ and for all $\mathcal{C}$-biextensible $f\colon A \to B$ and embeddings $e\colon A \to M_i, j\colon B \to M_{1-i}$, the function $f\langle e, j\rangle$ is restriction of a bijective homomorphism $M_i \to M_{1-i}$ ($i \in \{0, 1\}$).*

Theorem 36 (IB Fraïssé)**.** *Let M be an IB-homogeneous structure. Then $(\mathrm{Age}(M), \mathrm{Ext}(\mathrm{Age}(M)))$ is a pre-Fraïssé class with the IBAP, and for all $f \in \mathrm{Mon}(\mathrm{Age}(M))$ we have $\mathrm{Rep(f)}_{\mathrm{M}}^{\mathrm{Bi(M)}}$ if and only if f is $\mathrm{Age}(M)$-biextensible (in particular, all embeddings are $\mathrm{Age}(M)$-biextensible).*

Conversely, if $(\mathcal{C}, \mathrm{Ext}(\mathcal{C}))$ is a pre-Fraïssé class with the IBAP in which all embeddings are $\mathcal{C}$-biextensible, then there exists a countable IB-homogeneous structure $M_{\mathcal{C}}$ such that $\mathrm{Age}(M_{\mathcal{C}}) = \mathcal{C}$.

Finally, any two countable IB-homogeneous structures with the same age are BB-equivalent.

Proof. The first two statements have already been proven.

We will only sketch the proof of the last one, as the argument is very similar to that from Theorem 21.

Suppose that M_0, M_1 are IB-homogeneous structures with the same age $\mathcal{C}$, and $f_0\colon A_0 \to B_0$ is a $\mathcal{C}$-biextensible monomorphism. Take $u \in \{0, 1\}$ and embeddings $e_0\colon A_0 \hookrightarrow M_u, j_0\colon B_0 \hookrightarrow M_{1-u}$. We will show how to extend $g_0 \coloneqq f_0\langle e_0, j_0\rangle$ to a bijective homomorphism $M_u \to M_{1-u}$.

Enumerate $M_u \setminus e[A] = \{x_i : i \in \omega\}$, $M_{1-u} = \{y_i : i \in \omega\}$, and let $X_i = e[A] \cup \{x_k : k < i\}$, $Y_i = j[B] \cup \{y_k : k < i\}$.

Suppose that $g_k = f_k\langle e_k, j_k\rangle$ is a bijective monomorphism between X_r and some $Z \subset M_{1-u}$. Take the first element of M_{1-u} not in Z, say y_s, and note that g_k^{-1} is a monomorphism $\overline{Z} \to \overline{X_r}$, which moreover is manifestation of the $\mathcal{C}$-biextensible f_k^{-1}. Let C be the isomorphism type of $Z \cup \{y_s\}$, and $m\colon C \hookrightarrow \overline{M_u}$. Then we can manifest f_k^{-1} over $m \circ j_k[B_k]$ and $e_k[A_k]$ to find a vertex x_t such that $g_k^{-1} \cup \{(y_s, x_t)\}$ is a partial monomorphism $\overline{M_{1-u}} \to \overline{M_u}$. By Proposition 27, $g_{k+1} \coloneqq (g_k^{-1} \cup \{(y_s, x_t)\})^{-1}$ is a monomorphism that includes y_s in its image, and by Theorem 31, g_{k+1} is $\mathcal{C}$-biextensible.

And if $g_k = f_k\langle e_k, j_k\rangle$ is a bijective monomorphism whose image Y_r, then consider the vertex x_n with minimal n such that x_n is not in the domain of g_k. Let A_{i+1} be the isomorphism type of $e[A] \cup \{x_n\}$, and let $\ell\colon A_{i+1} \hookrightarrow M_{1-u}$ be an embedding. Now $f_k\langle \ell, j_k\rangle$ is M_{1-u}-biextensible, and we can use an extension to find an image for x_n.

Continuing in this manner, we find that $f_0\langle e_0, j_0\rangle$ is restriction of a bijective homomorphism of $M_u \to M_{1-u}$. $\square$

5. Final comments and questions

The method to find amalgamation properties and prove Fraïssé theorems for XY-homogeneous structures that we used is fairly simple:

1. Determine the smallest class of local homomorphism $\mathrm{X} \in \{\mathrm{H}, \mathrm{M}, \mathrm{I}\}$ that contains all restrictions of Y-endomorphisms,

2. define a suitable notion of age-extensibility,

3. verify that XY-homogeneous structures satisfy an amalgamation property with respect to the age-extensible homomorphisms,

4. verify that extensibility of embeddings to age-extensible homomorphisms characterises XY-homogeneous structures (that is, find analogues of theorems 15 and 33),

5. carry out the Fraïssé construction,

6. verify that limits are unique up to the relation that holds when two structures M, N have the same age and for every every age-extensible f and embeddings $i\colon \mathrm{dom}(f) \to M$, $j\colon \mathrm{codom}(f) \to N$, the homomorphism $f\langle i, j\rangle$ is restriction of a Y-morphism $M \to N$.

Of these steps, only #2 seems to be a problem for IH, IE, MH, and ME, because the inverse relation of a homomorphism is not only not necessarily a homomorphism (between complement structures), but often not even a function. A possible way to work around this issue is to define age-extensibility in terms of *all* the right inverses of the homomorphism, so that a homomorphism is extensible if each of the injective functions that can be obtained as a right inverse of it is an extensible monomorphism in the sense of the present paper (the requirement of the extension being injective would be dropped).

Although the idea above makes sense, I have not worked out the details yet, so the question arises:

Question 37. *Are the methods in this paper flexible enough to prove Fraïssé theorems for IH, IE, MH, and ME?*

As mentioned in the introduction, ultrahomogeneity is associated with quantifier elimination. The question of partial quantifier elimination (where the elimination set is not the set of quantifier-free formulas) was explored in [10], but, to my knowledge, not completely settled. In particular,

Question 38. *Given a morphism-extension class XY, what is the elimination set of its first-order theory?*

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