

The Tutte polynomial and related polynomials

Lecture notes 2010, 2012, 2014

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The following notes derive from three related series of lectures given for the Selected Chapters in Combinatorics course (*Vybrané kapitoly z kombinatoriky I*) at the Computer Science Institute (IÚUK) and the Department of Applied Mathematics (KAM) of the Faculty of Mathematics and Physics (MFF) at Charles University, Prague: “Many facets of the Tutte polynomial” (2010), “Graph invariants, homomorphisms and the Tutte polynomial” (2012), and “Duality in combinatorics: the examples of Tutte, Erdős, and Ramsey” (2014).¹ I have tried to make the notation as uniform as possible throughout and to avoid repetitions arising from overlaps between the three courses. However, I have left some background to cycles and cuts in Section 6 as it stands, rather than asking the reader to find the relevant material in Section 2.

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¹The courses in 2012 and 2014 were complemented by lectures given by Prof. Jaroslav Nešetřil, but the content of these lectures is not covered by the notes presented here.

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1 The chromatic polynomial

1.1 Graph-theoretic preliminaries

Let $G = (V, E)$ be a graph. A *spanning subgraph* is a subgraph (V, A) with $A \subseteq E$, and is denoted by G_A . An *induced subgraph* is a subgraph (U, A) , where $A = \{uv \in E : u \in U, v \in U\}$, and is denoted by $G[U]$. The number of connected components of G is denoted by $c(G)$. A *maximal spanning forest* F is a forest which is a spanning subgraph of G with the property that F is contained in no other spanning forest of G , i.e., no edge of G can be added to F without creating a cycle of G . A maximal spanning forest of a connected graph is a spanning tree.

The *rank* $r(G) = |V| - c(G)$ is the size of maximal spanning forest of G . Conversely, a spanning subgraph G_A with $c(G_A) = c(G)$ is a maximal spanning forest of G . For $A \subseteq E$ we often identify A with the spanning subgraph (V, A) and write $r(A)$ for $r(G_A)$. So $r(A) = |A|$ if and only if G_A is a spanning forest; $r(A) = r(E)$ if and only if G_A has the same number of connected components as G ; and $r(A) = |A| = r(E)$ if and only if G_A is a maximal spanning forest of G . The *nullity* $n(G) = |E| - r(G)$ is the dimension of the cycle space of G (for a plane graph, this is the number of faces of G excepting the outer face). For $A \subseteq E$ we set $n(A) = n(G_A)$.

Deleting an edge $e \in E$ forms the spanning subgraph $G \setminus e = (V, E \setminus \{e\})$. *Contracting* an edge $e = uv$ forms the graph G/e obtained by deleting e and then identifying the endpoints of e .

An edge e is a *bridge* (*isthmus*, *cut-edge*, *coloop*) of G if $r(G \setminus e) < r(G)$, i.e., the number of connected components is increased upon removing e . An edge e is a bridge if and only if $r(\{e\}) = 1$. An edge $e = uv$ is a *loop* of G if $u = v$. Contracting a loop is the same as deleting it. An edge e is loop if and only if $n(\{e\}) = 1$. An edge e is *ordinary* if it is neither a bridge nor a loop.

1.2 The chromatic polynomial and proper colourings

There are various ways to define the chromatic polynomial $P(G; z)$ of a graph G . Perhaps the first that springs to mind is to define it to be the graph invariant $P(G; k)$ with the property that when k is a positive integer $P(G; k)$ is the number of colourings of the vertices of G with k or fewer colours such that adjacent vertices receive different colours. One then has to prove that $P(G; k)$ is indeed a polynomial in k . This can be done for example by an inclusion-exclusion argument, or by establishing that $P(G; k)$ satisfies a deletion-contraction recurrence and using induction.

However, we shall take an alternative approach and define a polynomial $P(G; z)$ by specifying its coefficients as graph invariants that count what are called colour-partitions of the vertex set of G . It immediately emerges that $P(G; k)$ does indeed count the proper vertex k -colourings of G . A further aspect of this approach is that we choose a basis different to the usual basis $\{1, z, z^2, \dots\}$ for polynomials in z . This basis, $\{1, z, z(z-1), \dots\}$, has the advantage that we are able to calculate the chromatic polynomial very easily for

many graphs, such as complete multipartite graphs.

The chromatic polynomial has been the subject of intensive study ever since Birkhoff introduced it in 1912 [8], perhaps with an analytic approach to the Four Colour Conjecture in mind. Although such an approach has not led to such a proof of the Four Colour Conjecture being found, study of the chromatic polynomial has led to many advances in graph theory that might not otherwise have been made. The chromatic polynomial played a significant role historically in Tutte's elucidation of tension-flow duality. (Later we look at Tutte's eponymous polynomial, introduced as simultaneous generalization of the chromatic and flow polynomials.)

More about graph colourings can be found in e.g. [10, ch. V], [17, ch. 5], and more about the chromatic polynomial in e.g. [7, ch. 9] and [20].

We approach the chromatic polynomial via the key property that vertices of the same colour in a proper colouring of G form an independent (stable) set in G .

Definition 1.1. *A colour-partition of a graph $G = (V, E)$ is a partition of V into disjoint non-empty subsets, $V = V_1 \cup V_2 \cup \dots \cup V_k$, such that the colour-class V_i is an independent set of vertices in G , for each $1 \leq i \leq k$ (i.e., each induced subgraph $G[V_i]$ has no edges).*

The chromatic number $\chi(G)$ is the least natural number k for which such a partition is possible.

If G has a loop then it has no colour-partitions. Adding or removing edges in parallel to a given edge makes no difference to what counts as a colour-partition, since its definition depends only on whether vertices are adjacent or not.

We denote the falling factorial $z(z-1)\cdots(z-i+1)$ by $z^{\underline{i}}$.

Definition 1.2. *Let $G = (V, E)$ be a graph and let $a_i(G)$ denote the number of colour-partitions of G into i colour-classes. The chromatic polynomial of G is defined by*

$$P(G; z) = \sum_{i=1}^{|V|} a_i(G) z^{\underline{i}}.$$

For example, when G is the complete graph on n vertices,

$$P(K_n; z) = z^{\underline{n}} = z(z-1)\cdots(z-n+1),$$

with $a_i(K_n) = 0$ for $1 \leq i \leq n-1$ and $a_n(K_n) = 1$.

If G has n vertices then $a_n(G) = 1$ so that $P(G; z)$ has leading coefficient 1. The constant term $P(G; 0)$ is zero since z is a factor of $z^{\underline{i}}$ for each $1 \leq i \leq n$. If E is non-empty then $P(G; 1) = 0$, so that $z-1$ is a factor of $P(G; z)$. More generally, the integers $0, 1, \dots, \chi(G)-1$ are all roots of $P(G; z)$, and $\chi(G)$ is the first positive integer that is not a root of $P(G; z)$.

Proposition 1.3. *If $G = (V, E)$ is a simple graph on n vertices and m edges then the coefficient of z^{n-1} in $P(G; z)$ is equal to $-m$.*

Proof. A partition of n vertices into $n - 1$ subsets necessarily consists of $n - 2$ singletons and one pair of vertices $\{u, v\}$. This is a colour-partition if and only if $uv \notin E$. Hence $a_{n-1}(G) = \binom{n}{2} - m$, where m is the number of pairs of adjacent vertices, equal to the number of edges of G when there are no parallel edges. Then

$$[z^{n-1}]P(G; z) = -(1 + 2 + \cdots + n-1)a_n(G) + a_{n-1}(G) = -m.$$

□

The *join* $G_1 + G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \cup V_2$ and edge set

$$E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}.$$

For example the join of two cocliques $\overline{K}_r + \overline{K}_s$ is a complete bipartite graph $K_{r,s}$.

Proposition 1.4. *The chromatic polynomial of the join $G_1 + G_2$ is given by*

$$P(G_1 + G_2; z) = P(G_1; z) \circ P(G_2; z),$$

where the \circ operation is defined by $z^i \circ z^j = z^{i+j}$, extended linearly to polynomials.

Proof. The number of colour-partitions of $G = G_1 + G_2$ is given by

$$a_k(G) = \sum_{i+j=k} a_i(G_1)a_j(G_2),$$

since every vertex of G_1 is adjacent in G to every vertex of G_2 , so that any colour-class of vertices in G is either a colour class of G_1 or a colour class of G_2 . □

The operation \circ treats falling factorials $z^{\underline{i}}$ as though they were usual powers z^i when multiplying together the polynomials $\sum_i a_i(G_1)z^{\underline{i}}$ and $\sum_j a_j(G_2)z^{\underline{j}}$. This is part the shadowy world of “umbral calculus”...

Question 1

- (i) Find the chromatic polynomial of the wheel $C_n + K_1$ on $n+1$ vertices.
- (ii) Find an expression for the chromatic polynomial of the complete bipartite graph $K_{r,s}$ relative to the factorial basis $\{z^{\underline{n}}\}$ (leaving your answer in the form of a double sum).

Definition 1.5. A proper k -colouring of the vertices of $G = (V, E)$ is a function $f : V \rightarrow [k]$ with the property that $f(u) \neq f(v)$ whenever $uv \in E$.

Note that the vertices of a graph are regarded as labelled and colours are distinguished: colourings are different even if equivalent up to an automorphism of G or a permutation of the colour set.

Proposition 1.6. *If $k \in \mathbb{N}$ then $P(G; k)$ is the number of proper vertex k -colourings of G .*

Proof. To every proper colouring in which exactly i colours are used there corresponds a colour-partition into i colour classes. Conversely, given a colour-partition into i classes there are k^i ways to assign colours to them. Hence the number of proper k -colourings is $\sum a_i(G)k^i = P(G; k)$. \square

The fact that the polynomial $P(G; z)$ can be interpolated from its evaluations at positive integers gives a method of proving identities satisfied by $P(G; z)$ generally. Namely, check the truth of the identity when $z = k \in \mathbb{N}$ by verifying a combinatorial property of proper k -colourings. We finish this section with some examples.

Proposition 1.7. *Suppose G' is obtained from G by joining a new vertex to each vertex of an r -clique in G . Then $P(G'; z) = (z - r)P(G; z)$.*

Proof. The identity holds when z is equal to a positive integer k , for to each proper k -colouring of G there are exactly $k - r$ colours available for the new vertex to extend to a proper colouring of G' . \square

Consequently, if G is a tree on n vertices then $P(G; z) = z(z - 1)^{n-1}$ (every tree on $n \geq 2$ vertices has a vertex of degree 1 attached to a 1-clique in a tree on $n - 1$ vertices).

A *chordal graph* is a graph such that every cycle of length four or more contains a chord, i.e., there are no induced cycles of length four or more. A chordal graph can be constructed by successively adding a new vertex and joining it to a clique of the existing graph [19]. This ordering of vertices is known as a *perfect elimination ordering*. By Proposition 1.7, for a chordal graph G we have $P(G; z) = z^{c(G)}(z-1)^{k_1} \cdots (z-s)^{k_s}$, where $k_1 + \cdots + k_s = |V| - c(G)$ and $s = \chi(G) - 1$.

Question 2

(i) Show that if G is the disjoint union of G_1 and G_2 then $P(G; z) = P(G_1; z)P(G_2; z)$.

(ii) Prove that

$$P(G; x + y) = \sum_{U \subseteq V} P(G[U]; x)P(G[V \setminus U]; y).$$

Proposition 1.8. *Suppose $G = (V, E)$ has the property that $V = V_1 \cup V_2$ with $G[V_1 \cap V_2]$ complete and no edges joining $V_1 \setminus (V_1 \cap V_2)$ to $V_2 \setminus (V_1 \cap V_2)$. Then*

$$P(G; z) = \frac{P(G[V_1]; z)P(G[V_2]; z)}{P(G[V_1 \cap V_2]; z)}.$$

In particular, if G is a connected graph with 2-connected blocks G_1, \dots, G_ℓ then

$$P(G; z) = z^{1-\ell} P(G_1; z) P(G_2; z) \cdots P(G_\ell; z).$$

Proof. It suffices to prove the first identity when z is a positive integer k . Each proper colouring of the clique $G[V_1 \cap V_2]$ extends to $P(G[V_1]; k)/P(G[V_1 \cap V_2]; k)$ proper colourings of $G[V_1]$, and independently to $P(G[V_2]; k)/P(G[V_1 \cap V_2]; k)$ proper colourings of $G[V_2]$. Seeing that such a proper colouring of the clique $G[V_1 \cap V_2]$ also extends to $P(G; k)/P(G[V_1 \cap V_2]; k)$ proper colourings of G , we have

$$\frac{P(G; k)}{P(G[V_1 \cap V_2]; k)} = \frac{P(G[V_1]; k)}{P(G[V_1 \cap V_2]; k)} \frac{P(G[V_2]; k)}{P(G[V_1 \cap V_2]; k)}.$$

□

1.3 Deletion and contraction

Proposition 1.9. *The chromatic polynomial of a graph G satisfies the relation*

$$P(G; z) = P(G \setminus e; z) - P(G/e; z),$$

for any edge e .

Proof. When e is a loop we have $P(G; z) = 0 = P(G \setminus e; z) - P(G/e; z)$ since $G \setminus e \cong G/e$. When e is parallel to another edge of G we have $P(G; z) = P(G \setminus e; z)$ and $P(G/e; z) = 0$ since G/e has a loop.

Suppose then that e is not a loop or parallel to another edge. Consider the proper vertex k -colourings of $G \setminus e$. Those which colour the ends of e differently are in bijective correspondence with proper k -colourings of G , while those that colour the ends the same are in bijective correspondence with proper k -colourings of G/e . Hence $P(G \setminus e; k) = P(G; k) + P(G/e; k)$ for each positive integer k . □

Proposition 1.9 provides the basis for a possible inductive proof of any given statement about the chromatic polynomial for a minor-closed class of graphs (such as planar graphs). We shall see a few such examples in the sequel.

Question 3

Use the deletion–contraction recurrence of Proposition 1.9 to

- (i) give another proof that the chromatic polynomial of a tree on n vertices is given by $z(z-1)^{n-1}$;
- (ii) find the chromatic polynomial of the cycle C_n .

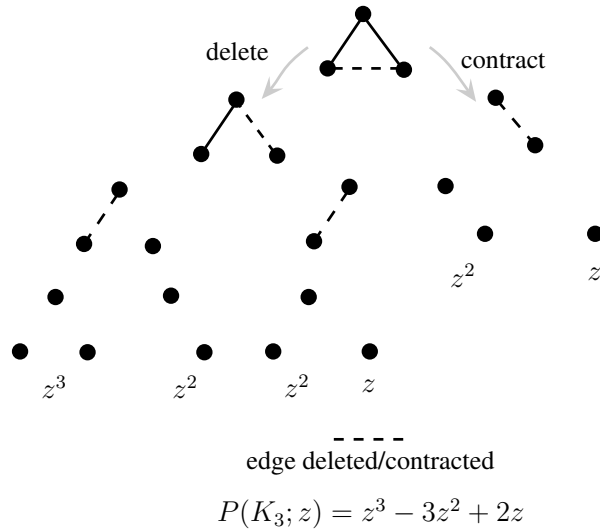


Figure 1: Deletion-contraction computation tree for the chromatic polynomial of K_3 . Parallel edges produced by contraction/identification are omitted since they do not affect the value of the chromatic polynomial. Leaf nodes are empty graphs.

We can use the recurrence given by Proposition 1.9 to compute the chromatic polynomial of a graph G recursively. A convenient way to record this computation is to draw a binary tree rooted at G whose nodes are minors of G and where the children of a node are the two graphs obtained by the deletion and contraction of an edge. Along each branch of the computation tree it does not matter in which order we choose the edges to delete or contract. If we continue this computation tree until no edges remain to delete and contract then the leaves of the computation tree are edgeless graphs \overline{K}_i on $1 \leq i \leq n$ vertices, whose chromatic polynomial is given by z^i . The sign of this term in its contribution to the chromatic polynomial of G is positive if an even number of contractions occur on its branch, and negative otherwise. See Figure 1 for an example.

For a simple graph $G = (V, E)$ a binary deletion-contraction tree of depth $|E|$ is required to reach cliques at all the leaves. When multiple edges appear they can be deleted to leave simple edges (in other words, contraction of an edge parallel to another edge gives a loop and this contributes zero to the chromatic polynomial).

Question 4

Suppose G is a simple connected graph on n vertices.

- (i) Prove that the number of edge contractions along a branch of the computation tree for the chromatic polynomial of G whose leaf node is a coclique of i vertices is equal to $n - i$.
- (ii) Prove that for each $1 \leq i \leq n$ we can always obtain a coclique on i vertices by deleting/contracting edges in some appropriate order. Deduce that

$$P(G; z) = \sum_{0 \leq i \leq n-1} (-1)^i c_i(G) z^{n-i},$$

where $c_i(G) > 0$ is the number of cocliques of order $n - i$ occurring as leaf node in the computation tree for G . (A formal proof of the fact that the coefficients of $P(G; z)$ alternate in sign is given in Proposition 1.10 below. A combinatorial interpretation for $c_i(G)$ in terms of spanning forests of G is given by Theorem 1.3.)

If we start with a connected graph G in building the computation tree we can always choose an edge whose deletion leaves the graph connected, so that the children of a node are both connected graphs. In this way we end up with trees (at which point deleting any edge disconnects the tree). Seeing that we know that the chromatic polynomial of a tree on i vertices is given by $z(z - 1)^{i-1}$ we could stop the computation tree at this point when we reach trees as leaf nodes. The sign of the term $z(z - 1)^{i-1}$ contributed to $P(G; z)$ by a leaf node tree on i vertices is positive if there are an even number of edge contractions on its branch, and otherwise it has negative sign in its contribution. See the left-hand diagram of Figure 2 for an example with $G = K_4^-$ (K_4 minus an edge).

Question 5

- (i) Show in a similar way to the previous question that if G is a connected graph on n vertices then each leaf of the deletion-contraction computation tree for G which is a tree on i vertices contributes $(-1)^{n-i}z(z-1)^{i-1}$ to $P(G; z)$.
- (ii) Deduce that when G is connected

$$P(G; z) = z \sum_{1 \leq i \leq n} (-1)^{n-i} t_i(G) (z-1)^{i-1},$$

where $t_i(G)$ is the number of trees of order i occurring as leaf nodes in the computation tree for G . (We shall see later that the coefficients $t_i(G)$ have a combinatorial interpretation in terms of spanning trees of G .)

If we write the recurrence given in Proposition 1.9 as $P(G \setminus e; z) = P(G; z) + P(G/e; z)$, we can by adding edges between non-adjacent vertices or identifying such non-adjacent vertices “fill out” a dense connected graph to complete graphs. Add the edge e to $G \setminus e$ to make G , and if G/e has parallel edges these can be removed without affecting the value of $P(G/e; z)$: in any event, the number of non-edges in both G and (the simplified graph) G/e is one less than in $G \setminus e$. Hence, starting with a simple connected graph $G = (V, E)$, $\binom{|V|}{2} - |E|$ addition-identification steps are required to reach complete graphs. See the right-hand diagram in Figure 2 for a small example.

Question 6 By considering the definition of the chromatic polynomial (Definition 1.2), prove that

$$z^n = \sum_{1 \leq i \leq n} S(n, i) z^i,$$

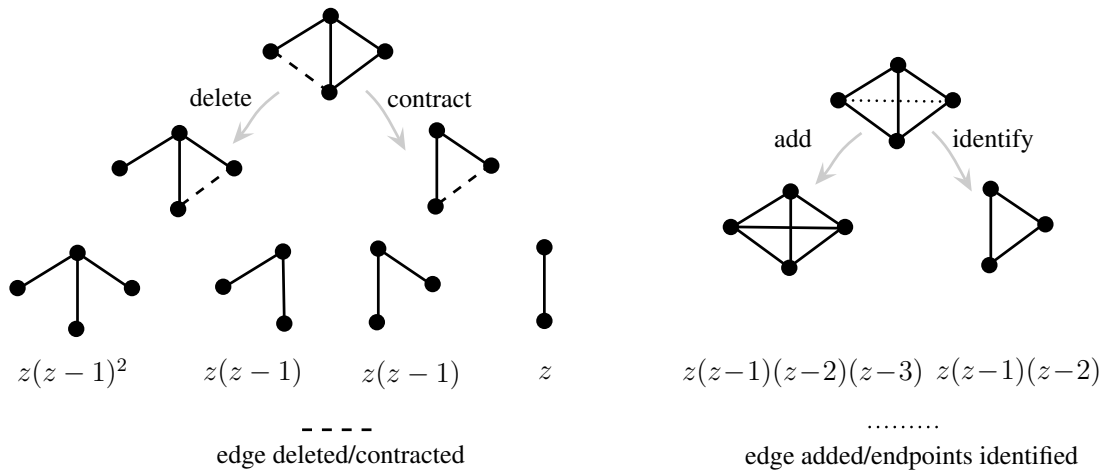
where $S(n, i)$ is equal to the number of partitions of an n -set into i non-empty sets. (These are known as the *Stirling numbers of the second kind*.)

To move from the basis $\{z^n\}$ to the basis $\{z^n\}$ for polynomials in z we have the identity

$$z^n = \sum_{1 \leq i \leq n} s(n, i) z^i,$$

where $s(n, i)$ are the signed Stirling numbers of the first kind, defined recursively by

$$s(n, i) = s(n-1, i-1) - (n-1)s(n-1, i),$$



$$P(K_4^-; z) = z(z-1)^2 - 2z(z-1) + z \quad P(K_4^-; z) = z(z-1)(z-2)(z-3) + z(z-1)(z-2)$$

Figure 2: Deletion-contraction and addition-identification computation tree for the chromatic polynomial of K_4^- . Parallel edges produced by contraction/identification are omitted since they do not affect the value of the chromatic polynomial. Leaf nodes for deletion-contraction are trees, leaf nodes for addition-identification are complete graphs.

$$\begin{cases} s(r, 0) = 0 & r = 1, 2, \dots \\ s(r, r) = 1 & r = 0, 1, 2, \dots \end{cases}$$

The number $(-1)^{n-i}s(n, i)$ counts the number of permutations of an n -set that have exactly i cycles. By Question 4 it is also the number of cliques of order i occurring as leaves in the computation tree for $P(K_n; z)$, and by Theorem 1.3 below it also has an interpretation in terms of forests on n vertices.

Question 7

- (i) Explain why $P(G; z) > 0$ when $z \in (-\infty, 0)$, provided G has no loops.
- (ii) Show that if G is connected and without loops then $P(G; z)$ is non-zero with sign $(-1)^{|V|-1}$ when $z \in (0, 1)$.

Remark 1.1. Let $z^{\bar{i}}$ denote the rising factorial $z(z+1)\cdots(z+i-1)$. Brenti [12] proved that

$$P(G; z) = \sum_{1 \leq i \leq |V|} (-1)^{|V|-i} b_i(G) z^{\bar{i}},$$

where $b_i(G)$ is the number of set partitions $V_1 \cup V_2 \cup \cdots \cup V_i$ of V into i blocks paired with an acyclic orientation of $G[V_1] \cup G[V_2] \cup \cdots \cup G[V_i]$. See [94] for expressions for the

coefficients of the chromatic polynomial relative to any polynomial basis $\{e_i(z)\}$ of binomial type (meaning it satisfies $e_j(x+y) = \sum_{0 \leq i \leq j} \binom{j}{i} e_i(x) e_{j-i}(y)$).

From the computation tree for finding the chromatic polynomial of a connected graph G it can be argued (Question 5) that the coefficients of $P(G; z)$ alternate in sign. Let us formalize this argument and prove it for graphs in general:

Proposition 1.10. *Suppose G is a loopless graph and that*

$$P(G; z) = \sum_{0 \leq i \leq |V|} (-1)^i c_i(G) z^{|V|-i}.$$

Then $c_i(G) > 0$ for $0 \leq i \leq r(G)$, and $c_i(G) = 0$ for $r(G) < i \leq |V|$.

Proof. We shall show that

$$(-1)^{|V|} P(G; -z) = \sum_{0 \leq i \leq r(G)} c_i(G) z^{|V|-i}$$

has strictly positive coefficients. (When G has loops $P(G; z) = 0$.) By the deletion-contraction formula, and using the fact that $|V(G \setminus e)| = |V(G)|$ and $|V(G/e)| = |V(G)| - 1$ when e is not a loop,

$$(-1)^{|V(G)|} P(G; -z) = (-1)^{|V(G \setminus e)|} P(G \setminus e; -z) + (-1)^{|V(G/e)|} P(G/e; -z).$$

Hence

$$c_i(G) = c_i(G \setminus e) + c_{i-1}(G/e).$$

Assume inductively on the number of edges that $c_i(G) > 0$ for $0 \leq i \leq r(G)$, and that $c_i(G) = 0$ otherwise. As a base for induction, $(-1)^n P(\overline{K}_n; -z) = z^n$.

By inductive hypothesis, for $0 \leq i \leq r(G \setminus e)$ we have $c_i(G \setminus e) > 0$ and for $0 \leq i - 1 \leq r(G/e)$ we have $c_{i-1}(G/e) > 0$. When e is not a bridge $r(G \setminus e) = r(G)$ and so $c_i(G \setminus e) > 0$ for $0 \leq i \leq r(G)$, otherwise for a bridge $r(G \setminus e) = r(G) - 1$ and in this case $c_i(G) > 0$ for $0 \leq i \leq r(G) - 1$. Since e is not a loop $r(G/e) = r(G) - 1$, so we have $c_{i-1}(G/e) > 0$ for $1 \leq i \leq r(G)$. Together these inequalities imply $c_i(G) > 0$ for $0 \leq i \leq r(G)$.

Clearly z divides $P(G; z)$ for a connected graph. It follows that $z^{c(G)}$ is a factor of $P(G; z)$ by multiplicativity of the chromatic polynomial over disjoint unions. Hence $c_i(G) = 0$ for $r(G) < i \leq |V(G)|$. Also, the degree of $P(G; z)$ is $|V(G)|$ by its definition, so there are no remaining non-zero coefficients. \square

We shall see below in Whitney's Broken Circuit Theorem that the numbers $c_i(G)$ have a combinatorial interpretation in terms of spanning forests of G .

Question 8

- (i) Prove that the only rational roots of $P(G; z)$ are $0, 1, \dots, \chi(G) - 1$. (It may help to remind oneself that a monic polynomial with integer coefficients cannot have rational roots that are not integers.)
- (ii) Show that the root 0 has multiplicity $c(G)$ and that the root 1 has multiplicity equal to the number of blocks of G .

Jackson [40] proved that $P(G; z)$ can have no root in $(1, 32/27]$. Thomassen [80] a few years later proved that in any other interval of the real line there is a graph whose chromatic polynomial has a root contained in it.

Earlier in the history of the chromatic polynomial, Birkhoff and Lewis [9] showed that the chromatic polynomial of a plane triangulation cannot have a root in the intervals $(1, 2)$ or $[5, 8)$. Tutte [82] observed that for planar graphs there is often a root of the chromatic polynomial close to τ^2 where $\tau = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio, and proved that if G is a triangulation of the plane with n vertices then $P(G; \tau^2) \leq \tau^{5-n}$. See e.g. [20, ch. 12-14] and [41] for more about chromatic roots.

Here is another illustration of how deletion-contraction arguments can be used to give simple inductive proofs. On the other hand, as with inductive proofs generally, the art is knowing what to prove. We shall shortly see that the coefficients of $P(G; z)$ have a general expression, given by Whitney's Broken Circuit Theorem, of which Proposition 1.3 and the following are particular instances.

Proposition 1.11. *For a simple graph G on n vertices and m edges the coefficient of z^{n-2} in $P(G; z)$ is equal to $\binom{m}{2} - t$, where t is the number of triangles in G .*

Proof. The assertion is true when $m = 0, 1, 2$. Suppose G has n vertices and $m \geq 3$ edges. For a non-loop e , $c_2(G) = c_2(G \setminus e) - c_1(G/e)$. Inductively, $c_2(G \setminus e) = \binom{m-1}{2} - t_0$, where t_0 is the number of triangles in G not containing the edge e , the graph $G \setminus e$ being simple. In a triangle $\{e, e_1, e_2\}$ of G containing e , the edges e_1, e_2 do not appear in any other triangle of G containing e , since G is simple. When e is contracted the edges e_1 and e_2 become parallel edges in G/e , and moreover there are no other edge parallel to these. Hence for each triangle $\{e, e_1, e_2\}$ of G we remove one parallel edge in G/e in order to reduce it to a simple graph. So $c_1(G/e) = (m-1) - t_1$, where t_1 is the number of triangles of G containing e . With $t_0 + t_1 = t$ equal to the number of triangles in G , the result now follows by induction. \square

Proposition 1.12. *If $P(G; z) = z(z-1)^{n-1}$ then G is a tree on n vertices, and more generally $P(G; z) = z^c(z-1)^{n-c}$ implies G is a forest on n vertices with c components.*

Proof. The degree of $P(G; z)$ is n so G has n vertices. The coefficient of z^c is non-zero but z^{c-1} has zero coefficient, hence by Proposition 1.10 G has c connected components. Finally, reading off the coefficient of z^{n-1} tells us that the number of edges is $n - c$, so that G is a forest on n vertices with c components. \square

Question 9 Prove that if $P(G; z) = P(K_n; z)$ then $G \cong K_n$ and that if $P(G; z) = P(C_n; z)$ then $G \cong C_n$.

1.4 Subgraph expansions

Theorem 1.2. *The chromatic polynomial of a graph $G = (V, E)$ has subgraph expansion*

$$P(G; z) = \sum_{F \subseteq E} (-1)^{|F|} z^{c(F)},$$

where $c(A)$ is the number of connected components in the spanning subgraph (V, A) .

Proof. We prove the identity when z is a positive integer k .

For an edge $e = uv$ let $M_e = \{\kappa : V \rightarrow [k] : \kappa(u) = \kappa(v)\}$. Then

$$\bigcap_{e \in E} \overline{M}_e = \{\kappa : V \rightarrow [k] : \forall_{uv \in E} \kappa(u) \neq \kappa(v)\}$$

is the set of proper k -colourings of G . By the principle of inclusion-exclusion,

$$\left| \bigcap_{e \in E} \overline{M}_e \right| = \sum_{F \subseteq E} (-1)^{|F|} \left| \bigcap_{f \in F} M_f \right|.$$

But $\left| \bigcap_{f \in F} M_f \right| = k^{c(F)}$, since a function $\kappa : V \rightarrow [k]$ monochrome on each edge of F is constant on each connected component of (V, F) , and conversely assigning each connected component a colour independently yields such a function κ . \square

In the subgraph expansion for the chromatic polynomial given in Theorem 1.2 there are many cancellations. If $f \in F$ belongs to a cycle of (V, F) then the sets F and $F \setminus \{f\}$ have contributions to the sum that cancel. Whitney's Broken Circuit expansion results by pairing off subgraphs in a systematic way.

Let $G = (V, E)$ be a simple graph whose edge set has been ordered $e_1 < e_2 < \dots < e_m$. A *broken circuit* is the result of removing the first edge from some circuit, i.e., a subset $B \subseteq E$ such that for some edge e_l the edges $B \cup \{e_l\}$ form a circuit of G and $i > l$ for each $e_i \in B$.

Theorem 1.3. Whitney [91]. *Let G be a simple graph on n vertices with edges totally ordered, and let $P(G; z) = \sum (-1)^i c_i(G) z^{n-i}$. Then $c_i(G)$ is the number of subgraphs of G which have i edges and contain no broken circuits.*

Proof. Suppose B_1, \dots, B_t is a list of the broken circuits in lexicographic order based on the ordering of E . Let f_j ($1 \leq j \leq t$) denote the edge which when added to B_j completes a circuit. Note that $f_j \notin B_k$ when $k \geq j$ (otherwise B_k would contain in f_j an edge smaller than any edge in B_j , contrary to lexicographic ordering).

Define \mathcal{S}_0 to be the set of subgraphs of G containing no broken circuit and for $1 \leq j \leq t$ define \mathcal{S}_j to be the set of subgraphs containing B_j but not B_k for $k > j$. Then $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_t$ is a partition of the set of all subgraphs of G .

If $A \subseteq E$ does not contain f_j , then A contains B_j if and only if $A \cup \{f_j\}$ contains B_j . Further, A contains B_k ($k > j$) if and only if $A \cup \{f_j\}$ contains B_k , since f_j is not in B_k either. If one of the subgraphs A and $A \cup \{f_j\}$ are in \mathcal{S}_j then both are, and since $c(A) = c(A \cup \{f_j\})$ the contributions to the alternating sum cancel.

The only terms remaining are contributions from subsets in \mathcal{S}_0 : a subset of size i spans a forest with $n - i$ components, thus contributing $(-1)^i z^{n-i}$ to the sum. \square

Proposition 1.13. *Suppose G is a simple connected graph on n vertices and m edges and having girth g , and that $P(G; z) = \sum (-1)^i c_i(G) z^{n-i}$. Then*

$$c_i(G) = \binom{m}{i}, \text{ for } i = 0, 1, \dots, g-2,$$

and

$$c_{g-1}(G) = \binom{m}{g-1} - t,$$

where t is the number of circuits of size g in G .

Question 10

Show that if G is a simple connected graph on n vertices and m edges and $P(G; z) = \sum (-1)^i c_i(G) z^{n-i}$ then, for $0 \leq i \leq n-1$,

$$\binom{n-1}{i} \leq c_i(G) \leq \binom{m}{i}.$$

Proposition 1.14. *If G is a simple connected graph on n vertices and m edges and $P(G; z) = \sum (-1)^i c_i(G) z^{n-i}$ then,*

$$c_{i-1}(G) \leq c_i(G) \text{ for all } 1 \leq i \leq \frac{1}{2}(n-1).$$

Proof. In terms of the coefficients relative to the tree basis $\{z(z-1)^{n-1}\}$,

$$P(G; z) = \sum_{i=1}^n (-1)^{n-i} t_i(G) z(z-1)^{i-1},$$

we have

$$c_i(G) = \sum_{0 \leq j \leq i} t_{n-j}(G) \binom{n-1-j}{n-1-i} = \sum_{j=0}^i t_{n-j}(G) \binom{n-1-j}{i-j}.$$

If $i \leq \frac{1}{2}(n-1)$ then $i-j \leq \frac{1}{2}(n-1-j)$ for all $j \geq 0$. By unimodality of the binomial coefficients,

$$\binom{n-1-j}{i-j} \geq \binom{n-1-j}{i-1-j} \text{ for } i \leq \frac{1}{2}(n-1), j \geq 0.$$

Since each $t_{n-j}(G)$ is a non-negative integer, it follows that $c_i(G) \geq c_{i-1}(G)$ for $i \leq \frac{1}{2}(n-1)$. □

Question 11

Recall that if G is a forest then $P(G; z) = z^{c(G)}(z-1)^{r(G)}$. Also $(-1)^{|V(G)|}P(G; -z) = \sum_i c_i(G)z^{|V(G)|-i}$, where $c_i(G) = c_i(G \setminus e) + c_{i-1}(G/e)$.

- (i) Simplify the proof of Proposition 1.10, that $c_i(G) > 0$ for $0 \leq i \leq r(G)$ and $c_i(G) = 0$ otherwise, by using as base for induction the truth of the statement for forests and choosing a non-bridge edge in the deletion-contraction induction step.
- (ii) Likewise, prove that $c_{i-1}(G) < c_i(G)$ for $0 \leq i \leq \frac{1}{2}r(G)$ (Proposition 1.14 for not necessarily connected graphs G) by using base for induction the fact that this statement is true for forests and using deletion-contraction of a non-bridge edge.
- (iii) Re-prove Theorem 1.3 that $c_i(G)$ is the number of i -subsets of $E(G)$ not containing a broken circuit by showing that this quantity satisfies the recurrence $c_i(G) = c_i(G \setminus e) + c_{i-1}(G/e)$. (For this induction on number of edges the base case is $c_0(\overline{K}_n) = 1$ and $c_i(\overline{K}_n) = 0$ for $i > 0$, for which the assertion is trivially satisfied. To move by induction to an arbitrary graph G , with total order on $E(G)$ used to define broken circuits, choose the edge e to be the greatest.)

Proposition 1.14 is the easy half of a long-standing conjecture first made by Read in 1968 that the coefficients $c_i(G)$ of the chromatic polynomial are *unimodal*. An even stronger conjecture of *log-concavity* was later made, i.e., that $c_{i-1}(G)c_{i+1}(G) \leq c_i(G)^2$. Both conjectures fell simultaneously in 2010 when J. Huh [39] proved log-concavity as a corollary of a more general theorem in algebraic geometry.

A theorem due to Newton states that if a polynomial $\sum_i c_i z^{n-i}$ has strictly positive coefficients and all of its roots are real then the sequence (c_i) of coefficients is log-concave (and hence unimodal). If it were the case that the chromatic polynomial always had real roots then log-concavity of the sequence of absolute values of its coefficients would therefore follow by this result. However, not only is it true that there are some graphs whose chromatic polynomial has complex roots that are not real, but Sokal [75] showed that the set of complex numbers that are roots of some chromatic polynomial are dense in the whole complex plane. (This in contradistinction to when we restrict attention to the

real line itself, where no chromatic roots can lie on $(-\infty, \frac{32}{27}]$.) Can you think of a family of graphs $\{G_n\}$ with the property that $P(G_n; z)$ has non-real roots?

1.5 Some other deletion–contraction invariants.

We have seen that the chromatic polynomial $P(G; z)$ satisfies the recurrence relation

$$P(G; z) = P(G \setminus e; z) - P(G/e; z), \quad (1)$$

for any edge e of G . Together with boundary conditions

$$P(\overline{K}_n; z) = z^n, \quad n = 1, 2, \dots \quad (2)$$

this suffices to determine $P(G; z)$ on all graphs. A slight variation on giving the boundary conditions (2) is to supplement the recurrence (1) with the property of multiplicativity over disjoint unions

$$P(G_1 \cup G_2; z) = P(G_1; z)P(G_2; z), \quad (3)$$

and then to give the single boundary condition $P(K_1; z) = z$.

Define

$$B(G; k, y) = \sum_{f: V(G) \rightarrow [k]} y^{\#\{uv \in E(G): f(u)=f(v)\}},$$

where $k \in \mathbb{Z}_{>0}$ and y is an indeterminate. This polynomial in y is a generating function for colourings of G (not necessarily proper) counted according to the number of monochromatic edges, i.e., edges receiving the same colour on their endpoints. (Edges are taken with their multiplicity when counting the number of monochromatic edges in the exponent of y .) Note that $B(G; k, 0) = P(G; k)$.

Proposition 1.15. *For each edge e of G ,*

$$B(G; k, y) = (y - 1)B(G/e; k, y) + B(G \setminus e; k, y).$$

Together with the boundary conditions $B(\overline{K}_n; k, y) = k^n$, for $n = 1, 2, \dots$, this determines $B(G; k, y)$ as a polynomial in k and y .

Proof. Given $e = st$,

$$\begin{aligned} B(G; k, y) &= y \sum_{\substack{f: V(G) \rightarrow [k] \\ f(s)=f(t)}} y^{\#\{uv \in E \setminus e: f(u)=f(v)\}} + \sum_{\substack{f: V(G) \rightarrow [k] \\ f(s) \neq f(t)}} y^{\#\{uv \in E \setminus e: f(u)=f(v)\}} \\ &= yB(G/e; k, y) + [B(G \setminus e; k, y) - B(G/e; k, y)]. \end{aligned}$$

The fact that $B(G; k, y)$ is a polynomial follows by induction of the number of edges and the given boundary condition $B(\overline{K}_n; k, y) = k^n$. Further, it has degree $|V(G)|$ as a polynomial in k and degree $|E(G)|$ as a polynomial in y (again by induction on number of edges by tracking the relevant coefficient in the recurrence $B(G; k, y) = (y - 1)B(G/e; k, y) + B(G \setminus e; k, y)$). \square

An *acyclic orientation* of a graph is an orientation that has no directed cycles. A loop has no acyclic orientation, but any loopless graph does (for example, if its vertices are labelled by $1, \dots, n$ and an edge is directed from the smaller to the higher number).

Theorem 1.4. [Stanley, 1973] *The number of acyclic orientations of a graph G with at least one edge is given by $(-1)^{|V(G)|}P(G; -1)$.*

Proof. Let $Q(G)$ denote the number of acyclic orientations of G . When G is a single edge $Q(G) = 2$ and when G is a loop $Q(G) = 0$. If e is parallel to another edge of G then $Q(G) = Q(G \setminus e)$, since parallel edges must have the same direction in an acyclic orientation. Also, Q is multiplicative over disjoint unions, i.e., $Q(G_1 \cup G_2) = Q(G_1)Q(G_2)$.

To prove then that $Q(G) = (-1)^{|V(G)|}P(G; -1)$ it suffices to show that when e is not a loop or parallel to another edge of G we have

$$Q(G) = Q(G \setminus e) + Q(G/e). \quad (4)$$

Let $e = uv$ be a simple edge of G and consider an acyclic orientation \mathcal{O} of $G \setminus e$. There is always one direction $u \rightarrow v$ or $u \leftarrow v$ possible so that \mathcal{O} can be extended to an acyclic orientation of G : if both directions were to produce directed cycles then there would have to be a directed path from u to v and a directed path from v to u , which together would make a directed cycle in \mathcal{O} .

Those acyclic orientations of $G \setminus e$ that permit exactly one direction of e are in bijective correspondence with the subset of acyclic orientations of G where the direction of e cannot be reversed while preserving the property of being acyclic. Such an orientation of G induces an orientation that has a directed cycle in G/e , and contributes 1 to $Q(G)$ and $1 + 0 = 1$ to $Q(G \setminus e) + Q(G/e)$.

Those acyclic orientations of $G \setminus e$ where the direction of e can be reversed to make another acyclic orientation of G are in bijective correspondence with those orientations of G that induce acyclic orientations on the contracted graph G/e . Such a pair of acyclic orientations of G differing just on the direction of e contribute 2 to $Q(G)$ and $1 + 1 = 2$ to $Q(G \setminus e) + Q(G/e)$.

This establishes the recurrence (4). □

In [83] Tutte describes how he was led to define his polynomial (he called it the dichromate) by observing how graph invariants such as the chromatic polynomial and the number of spanning trees of a graph shared the property of satisfying a deletion–contraction recurrence.

Question 12 Suppose $f(G)$ is a graph invariant that for a connected graph G counts one of the following:

- (i) the number of spanning trees of G ,
- (ii) the number of spanning forests of G ,
- (iii) the number of connected spanning subgraphs of G .

Further suppose we stipulate that f is multiplicative over disjoint unions, $f(G_1 \cup G_2) = f(G_1)f(G_2)$.

Show that in each case f satisfies the recurrence

$$f(G) = f(G \setminus e) + f(G/e),$$

for each edge e of G that is not a loop or bridge. How do these three invariants differ for bridges and loops?

2 Flows and tensions

2.1 Orientations

An undirected graph $G = (V, E)$ can be made into a digraph in $2^{|E|}$ ways: for each edge $uv \in E$ we decide to direct u towards v , or to direct v towards u . If the edge is a loop, i.e. $u = v$, then we still think of there being two opposite ways to orient the loop – this is a matter of convenience for later definitions (and makes sense when we talk about orienting plane graphs, where the two possible directions can indeed be distinguished).

We orient a graph in order to extract structural properties of the underlying undirected graph, but the orientation that is chosen is arbitrary: the results obtained are independent of this choice. (The reader may recall the rôle played by an orientation of G in proving Kirchhoff's Matrix Tree Theorem, which gives an expression for the number of spanning trees of G .)

Suppose then we are given an orientation ω of $G = (V, E)$. By this we mean that ω assigns a direction to each edge $uv \in E$, either $u \xrightarrow{\omega} v$ or $u \xleftarrow{\omega} v$. We write G^ω for the digraph so obtained. For $U \subset V$ let $\omega^+(U)$ denote the set of the edges which begin in U and end outside U in the digraph G^ω , i.e., $\omega^+(U) = \{uv \in E : u \in U, v \in V \setminus U, u \xrightarrow{\omega} v\}$. The set $\omega^-(U) = \omega^+(V \setminus U)$ comprises edges which in G^ω begin outside U and end in U .

For a vertex $v \in V$ the set $\omega^+(\{v\})$ consists of those edges directed out of v by the orientation ω and $\omega^-(\{v\})$ is the set of edges directed into v . The *indegree* of a vertex v in G^ω is $|\omega^-(\{v\})|$ and its *outdegree* is $|\omega^+(\{v\})|$.

If G is a plane graph then each orientation ω of G determines an orientation ω^* of its dual G^* . This orientation is obtained by giving an edge e^* of G^* the orientation that is obtained from that of e by rotating it 90° clockwise: the edge e^* travels from the face to the left of e to the face to the right of e .

More formally, given an orientation ω of the plane graph $G = (V, E)$ we define the orientation ω^* of $G^* = (V^*, E^*)$ as follows. Let V^* be the set of faces of the embedded graph G . For each arc $u \xrightarrow{\omega} v$ of G^ω , suppose uv lies on the boundary of faces X and Y . Suppose further that X is the face that would be traversed anticlockwise if the direction of $u \xrightarrow{\omega} v$ were followed all the way round it (so that Y would be traversed in a clockwise direction following the direction given by $u \xrightarrow{\omega} v$). Then under the orientation ω^* we direct edge XY of G^* as the arc $X \xrightarrow{\omega^*} Y$.

See Fig.3

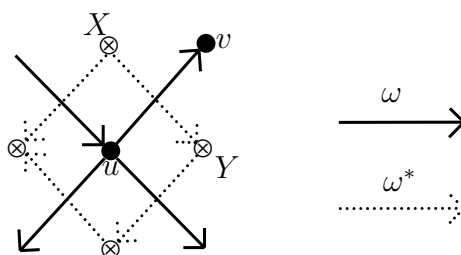


Figure 3: Dual orientation ω^* of an orientation ω of a plane graph

Question 13

- (i) What is the dual orientation of ω^* ?
- (ii) If C is a circuit of G all of whose edges follow the same direction under orientation ω (i.e. it is cyclically oriented) then what is the dual of C and how is it oriented under orientation ω^* ?

2.2 Circuits and cocircuits

We use terminology for graphs here that corresponds to viewing a graph $G = (V, E)$ in terms of its cycle matroid; the sense of “circuit” and “cycle” therefore differs from traditional graph theoretical usage.

A *cycle* is a set of edges defining a spanning subgraph of G all of whose vertex degrees are even (i.e., an Eulerian subgraph). A *circuit* is an inclusion-minimal cycle (i.e., a connected 2-regular subgraph). A cycle is the edge-disjoint union of circuits. A subset of edges is *dependent* in a graph $G = (V, E)$ if it contains a cycle and *independent* otherwise. An independent set of edges forms a forest. A maximal independent set of edges (add an edge and a cycle is formed) of a connected graph is a spanning tree.

A *cutset* K is a subset of edges defined by a bipartition of V , i.e., $K = \{uv \in E : u \in U, v \in V \setminus U\}$ where $U \subseteq V$. A *cocircuit* (or *bond*) is an inclusion-minimal cutset of $G = (V, E)$. A cocircuit of a connected graph is a cutset $\{uv \in E : u \in U, v \in V \setminus U\}$

with the additional property that the induced subgraphs $G[U]$ and $G[V \setminus U]$ are both connected. The rank of a graph decreases when removing a cutset. A cutset K is a cocircuit if and only if deleting K produces exactly one extra connected component, i.e., in this case $r(G \setminus K) = r(G) - 1$.

The nullity of the graph G is defined by $n(G) = |E| - r(G)$. The nullity of G decreases when contracting the edges of a cycle of G into a single vertex, and for a circuit C we have $n(G/C) = n(G) - 1$. In terms of the cycle matroid of G , a circuit C is a minimal set of dependent edges: removing an edge from C destroys the cycle that makes C dependent.

Question 14

A bridge in a graph G forms a cutset of G by itself. Dually, a loop in G forms a cycle of G by itself. Show that

- (i) an edge e is a bridge in G if and only if e does not belong to any circuit of G .
- (ii) an edge e is a loop in G if and only if e does not belong to any cocircuit of G .

A subset B is a cocircuit of a connected graph G if and only if contracting all edges not in B (and deleting any isolated vertices that result) produces a “bond-graph”, consisting of two vertices joined by $|B|$ parallel edges. (A subset K is a cutset of G if and only if the result is a graph whose blocks are bond-graphs – the vertices in this graph correspond to the connected components of $G \setminus K$.) Likewise, a subset C is a circuit of G if and only if deleting all the edges not in C (and deleting any isolated vertices that result) produces a cycle-graph (2-regular) on $|C|$ edges.

Question 15

Show that if $G = (V, E)$ is a plane graph and G^* is its dual then a subset of edges B is a cocircuit of G if and only if B is a circuit of G^* . (Assume the Jordan Curve Theorem: a simple closed curve – such as that bounding a circuit in a plane graph – partitions the plane minus the curve into an interior region bounded by the curve and an exterior region.)

A *spanning tree* of a connected graph G is a maximal set of independent edges: adding an edge creates a cycle. A spanning tree of G is a basis of the cycle matroid of G . More generally, when G is not connected, a maximal set of independent edges is a maximal spanning forest of G (add an edge and it is no longer a forest).

Suppose $G = (V, E)$ is connected and T is a spanning tree of G . Then

- (i) for each $e \in E \setminus T$ there is a unique circuit of G contained in $T \cup \{e\}$, which we shall denote by $C_{T,e}$, and

(ii) for each $e \in T$ there is a unique cocircuit contained in $E \setminus T \cup \{e\}$, which we shall denote by $B_{T,e}$.

Let C be a circuit of G . The two possible cyclic orderings of the edges of C define two cyclic orientations of the edges of C . Choose one of these orientations arbitrarily, making a directed cycle \vec{C} . Define C^+ to be the set of edges whose orientation in G^ω is the same as that in \vec{C} , and define C^- to be the set of those edges directed in G^ω in the opposite direction to that in \vec{C} .

This signing extends to cycles (Eulerian subgraphs) more generally, since any cycle is a disjoint union of circuits: when the cycle is the union of k edge-disjoint circuits there are 2^k choices for signing it.

Similarly, for a cocircuit (bond) B of G , defined by $U \subset V$ such that $B = \{uv \in E : u \in U, v \notin U\}$, we orient the bond B by directing edges from U to $V \setminus U$ to make \vec{B} . (Again there are two choices of orientation, depending on which side of the cut we nominate to be U and which side $V \setminus U$.) We then define B^+ and B^- in an analogous way to circuits. Clearly this procedure of signing cocircuits extends to cutsets more generally by directing edges from one side of the cut to the other. (Alternatively, a cutset is a disjoint union of cocircuits (why?), so in a similar way to cycles we can sign a cutset by signing its constituent cocircuits.)

We have already encountered signed cutsets in Section 2.1: for a subset $U \subset V$ the set $\omega^+(U)$ of edges that begin in U and terminate outside U comprise the positive elements of the cocircuit defined by U , and $\omega^-(U) = \omega^+(V \setminus U)$ the negative elements.

In this way, for a given orientation of G as a digraph G^ω , we have separated the edge sets of (co)circuits into positive and negative elements. In fact, given G and its set of (co)circuits, if the partition of each (co)circuit into positive and negative elements is given, then we can recover the orientation of edges (provided the way the (co)circuits have been signed is consistent with some orientation – what conditions are required for this to be the case?).

Question 16

A matroid is *regular* if there is an orientation of its circuits and cocircuits such that for all circuits C and all cocircuits B

$$|C^+ \cap B^+| + |C^- \cap B^-| = k \Leftrightarrow |C^+ \cap B^-| + |C^- \cap B^+| = k.$$

Explain why this statement holds for graphic matroids.

Definition 2.1. Let C be a signed circuit of an oriented graph G^ω on edge set E . The signed characteristic vector $\vec{\chi}_C \in \{0, \pm 1\}^E$ of C is defined by

$$\vec{\chi}_C(e) = \begin{cases} 1 & \text{if } e \in C^+, \\ -1 & \text{if } e \in C^-, \\ 0 & \text{if } e \notin C. \end{cases}$$

The signed characteristic vector $\vec{\chi}_B$ of a signed cocircuit B is similarly defined.

A fundamental relationship between signed characteristic vectors of circuits and cocircuits (and of cycles and cutsets more generally) is given by the following:

Proposition 2.2. *The signed characteristic vector of a circuit C is orthogonal to the signed characteristic vector of a cocircuit B :*

$$\sum_{e \in E} \vec{\chi}_B(e) \vec{\chi}_C(e) = 0.$$

Proof. Given a cocircuit K with positive elements $\omega^+(U)$ and negative elements $\omega^-(U)$, the inner product $\sum_{e \in E} \vec{\chi}_K(e) \vec{\chi}_C(e)$ is the number of edges of the circuit C going from U to $V \setminus U$ in its circuit-orientation, minus the number of edges going from $V \setminus U$ to U , and this is equal to zero. (In the simple closed walk that follows the edges of the circuit, for each edge followed in the direction from U to $V \setminus U$ there is a corresponding edge followed in the reverse direction from $V \setminus U$ to U .) \square

2.3 The incidence matrix of an oriented graph

We suppose still that we are given an orientation ω of the graph $G = (V, E)$.

Definition 2.3. *The incidence matrix of an oriented graph G^ω is the matrix $D = (d_{v,e}) \in \{0, \pm 1\}^{V \times E}$ whose (v, e) -entry is defined by*

$$d_{v,e} = \begin{cases} +1 & \text{if } e \text{ is directed out of } v \text{ by } \omega, \\ -1 & \text{if } e \text{ is directed into } v \text{ by } \omega, \\ 0 & \text{if } e \text{ is not incident with } v, \text{ or } e \text{ is a loop on } v. \end{cases}$$

A loop e corresponds to a zero column of D indexed by e (the fact that under any orientation the loop e is both going out of and going into v implies any flow along this edge is self-cancelling); each column of D indexed by an ordinary edge or bridge contains one entry $+1$, one entry -1 , and remaining entries all 0.

The row of D indexed by u is equal to $\vec{\chi}_{\omega^+(\{u\}) \cup \omega^-(\{u\})}$ (regarded as a row vector). If G is connected then if we delete any row of D the remaining rows form a basis for the signed characteristic vectors of cutsets. This is because

$$\vec{\chi}_{\omega^+(U) \cup \omega^-(U)} = \sum_{u \in U} \vec{\chi}_{\omega^+(\{u\}) \cup \omega^-(\{u\})},$$

and we may choose U to not contain the vertex whose row has been deleted. More generally, for any graph G there are $r(G)$ rows of D spanning signed characteristic vectors of cutsets, which can be obtained by deleting, for each component of G , one row indexed by a vertex in the component.

Let A be an additive Abelian group (for us A will either be \mathbb{Z} or finite). Scalar multiples of a $\{0, \pm 1\}$ -vector by an element of A are defined by using the identities $0a = 0, 1a = a$

and $(-1)a = -a$ for each $a \in A$. The Abelian group A is a \mathbb{Z} -module, with the action of \mathbb{Z} defined inductively by $ta = (t-1)a + a$ for integer $t > 0$ and $ta = -|t|a$ (inverse of $|t|a$ in A) for integer $t < 0$.

The set of vectors with entries in A indexed by E is denoted by A^E , and likewise A^V those vectors with entries indexed by V . We shall think of elements of A^E interchangeably as elements of the additive group formed by taking the $|E|$ -fold direct sum of A with itself, as vectors indexed by E , or as functions $\phi : E \rightarrow A$.

The incidence matrix defines a homomorphism $D : A^E \rightarrow A^V$ between additive groups, and its transpose likewise a homomorphism $D^T : A^V \rightarrow A^E$. For each $\phi : E \rightarrow A$,

$$(D\phi)(v) = \sum_{\substack{e=uv \\ u \xleftarrow{\omega} v}} \phi(e) - \sum_{\substack{e=uv \\ u \xrightarrow{\omega} v}} \phi(e).$$

The map $D : A^E \rightarrow A^V$ is called the *boundary*, assigning the net flow to each vertex from the given mapping $\phi : E \rightarrow A$.

For $\kappa : V \rightarrow A$ and edge $e = uv$,

$$(D^T\kappa)(e) = \begin{cases} \kappa(v) - \kappa(u) & \text{if } u \xleftarrow{\omega} v \\ \kappa(u) - \kappa(v) & \text{if } u \xrightarrow{\omega} v. \end{cases}$$

By the first isomorphism theorem for groups we have $\text{im } D \cong A^E / \ker D$ and $\text{im } D^T \cong A^V / \ker D^T$.

Proposition 2.4. *Let G be a graph with connected components on vertex sets $V_1, \dots, V_{c(G)}$.*

(i) *The incidence mapping $D : A^E \rightarrow A^V$ has image*

$$\text{im } D = \left\{ \kappa : V \rightarrow A; \sum_{v \in V_i} \kappa(v) = 0, \text{ for each } 1 \leq i \leq c(G) \right\} \cong A^{r(G)}.$$

(ii) *The transpose $D^T : A^V \rightarrow A^E$ has kernel*

$$\ker D^T = \left\{ \kappa : V \rightarrow A; \kappa \text{ constant on } V_i, \text{ for each } 1 \leq i \leq c(G) \right\} \cong A^{c(G)}.$$

Proof. (i) Given $\phi : E \rightarrow A$ we have

$$\begin{aligned} \sum_{v \in V_i} (D\phi)(v) &= \sum_{v \in V_i} \sum_{e \in E} d_{v,e} \phi(e) \\ &= \sum_{e \in E} \phi(e) \sum_{v \in V_i} d_{v,e} \\ &= 0, \end{aligned}$$

the last line since the entries $\{d_{v,e} : v \in V_i\}$ in the column c_e of D are either all zero (when e is not an edge in the component on V_i), or contain precisely $+1$ and -1 as non-zero elements.

Conversely, suppose that $\kappa : V \rightarrow A$ is such that $\sum_{v \in V_i} \kappa(v) = 0$ for each $1 \leq i \leq c(G)$. For given i , choose any $u \in V_i$. Then, letting κ_i denote the restriction of κ to V_i ,

$$\kappa_i = \sum_{v \in V_i} \kappa(v) \delta_v = \sum_{v \in V_i \setminus \{u\}} \kappa(v) (\delta_v - \delta_u)$$

where δ_v is defined by $\delta_v(w) = 1$ if $w = v$ and $\delta_v(w) = 0$ otherwise. Since $G[V_i]$ is connected, for each $v \in V_i$ there is a path from u to v , say $u = v_0, e_1, v_1, \dots, v_{\ell-1}, e_\ell, v_\ell = v$ and

$$\delta_v - \delta_u = (\delta_{v_\ell} - \delta_{v_{\ell-1}}) + \dots + (\delta_{v_1} - \delta_{v_0}) = D(\pm \delta_{e_\ell}) + \dots + D(\pm \delta_{e_1}),$$

where $\delta_e(f) = 1$ if $e = f$ and 0 otherwise, and the signs are chosen according to whether the directed path from u to v follows the orientation ω or goes against it. Hence $\delta_v - \delta_u \in \text{im } D$ for each $v \in V_i$, whence $\kappa_i \in \text{im } D$ also. This implies finally that κ itself belongs to $\text{im } D$.

(ii) Suppose that $\kappa : V \rightarrow A$ is such that $D^T \kappa = 0$. For an edge $e = uv$ with orientation $u \xrightarrow{\omega} v$, $(D^T \kappa)(e) = \kappa(v) - \kappa(u) = 0$, so that κ takes the same value on the endpoints of any edge. If u and w are in the same component of G then there is a walk starting at u and finishing at w and so $\kappa(u) = \kappa(w)$.

Conversely, if κ is constant on every component then $D^T \kappa = 0$.

□

The subgroups $\ker D^T$ and $\text{im } D$ of A^V are of less interest from the point of view of their relationship to the combinatorial properties of the graph G than the subgroups $\ker D$ and $\text{im } D^T$ of A^E . From Proposition 2.4 we know that as additive groups $\ker D \cong A^{n(G)}$ and $\text{im } D^T \cong A^{r(G)}$. The combinatorial interest comes from the fact that there are generating sets for these groups associated with circuits and cocircuits of G , and that further structural properties of $\ker D$ and $\text{im } D^T$ (namely properties of the intersections $\ker D \cap B^E$ and $\text{im } D \cap B^E$ where $B \subset A$) correspond to combinatorial features of the graph. We shall be concerned in particular with the case $B = A \setminus \{0\}$, and when $A = \mathbb{R}$ also with the case $B = \mathbb{Z}$.

2.4 A -flows and A -tensions

From now on we assume that A is a commutative ring and we consider A^E and its subgroups $\ker D$ and $\text{im } D^T$ as modules over A .

When $A = \mathbb{Z}$ we take ordinary integer multiplication. When A is finite, by the classification theorem for finite Abelian groups A takes the form $\mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \oplus \dots \oplus \mathbb{Z}_{k_r}$, where $2 \leq k_1 \mid k_2 \mid \dots \mid k_r$ (the notation $a \mid b$ meaning that a divides b), where k_r is the least common multiple of the orders of the elements of A . Componentwise multiplication then endows A with the structure of a commutative ring $R \cong \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \oplus \dots \oplus \mathbb{Z}_{k_r}$. Note however that if A is the r -fold direct sum of \mathbb{Z}_p for prime p then there is another natural choice of multiplication, namely that which makes A the finite field \mathbb{F}_{p^r} .

Let us start by defining flows² on a graph in what is the usual way, by stipulating that the Kirchhoff condition holds at each vertex. We shall then derive various other equivalent definitions.

Definition 2.5. *An A -flow of G is a mapping $\phi : E \rightarrow A$ such that*

$$\sum_{e \in \omega^+(\{v\})} \phi(e) - \sum_{e \in \omega^-(\{v\})} \phi(e) = 0 \quad \text{for each } v \in V.$$

A nowhere-zero A -flow is an A -flow $\phi : E \rightarrow A$ with the additional property that $\phi(e) \neq 0$ for every $e \in E$.

In other words, an A -flow ϕ as a vector is an element of $\ker D$, since the signed characteristic vectors $\vec{\chi}_{\omega^+(\{v\}) \cup \omega^-(\{v\})}$ are the rows of D .

For any $U \subseteq V$ we have

$$\sum_{u \in U} \vec{\chi}_{\omega^+(\{u\}) \cup \omega^-(\{u\})} = \vec{\chi}_{\omega^+(U) \cup \omega^-(U)},$$

since when $e = uv \in E$ has both $u \in U$ and $v \in U$ we have $e \in \omega^+(\{u\})$ and $e \in \omega^-(\{v\})$, or vice versa, so that $\vec{\chi}_{\omega^+(\{u\}) \cup \omega^-(\{u\})}(e) + \vec{\chi}_{\omega^+(\{v\}) \cup \omega^-(\{v\})}(e) = 0$.

For any $U \subseteq V$ we have

$$\begin{aligned} \sum_{u \in U} \left(\sum_{e \in \omega^+(\{u\})} \phi(e) - \sum_{e \in \omega^-(\{u\})} \phi(e) \right) \\ = \sum_{e \in \omega^+(U)} \phi(e) - \sum_{e \in \omega^-(U)} \phi(e) \end{aligned}$$

since when $e = uv \in E$ has both $u \in U$ and $v \in U$ we have $e \in \omega^+(\{u\})$ and $e \in \omega^-(\{v\})$ so that cancellation of $\phi(e)$ with $-\phi(e)$ occurs for the edge e .

Hence it is equivalent to define an A -flow as a mapping $\phi : E \rightarrow A$ such that

$$\sum_{e \in B^+} \phi(e) - \sum_{e \in B^-} \phi(e) = 0 \quad \text{for every cocircuit } B \text{ of } G.$$

Introduce a bilinear form \langle , \rangle on A^E by setting

$$\langle \phi, \psi \rangle = \sum_{e \in E} \phi(e)\psi(e).$$

²In other sources what we call a “nowhere-zero flow” is often just called a “flow”, while what we have chosen to call a “flow” is called a “circulation”. Compare too a “proper colouring” of vertices if a graph, which is conventionally just called a “colouring”, while an arbitrary assignment of colours to vertices is given some other name or “colouring” is qualified by a parenthetical “not necessarily proper”.

In this notation, Proposition 2.2 says that $\langle \vec{\chi}_B, \vec{\chi}_C \rangle = 0$ for each signed bond B and signed circuit C .

Let \mathcal{B} and \mathcal{C} denote respectively the set of signed bonds and signed circuits of the oriented graph G^ω on edge set E .

Since the signed characteristic vectors $\vec{\chi}_{B_v}$ of the signed bonds $B_v = \omega^+(v) \cup \omega^-(v)$ span the characteristic vectors $\vec{\chi}_B$ of all bonds $B \in \mathcal{B}$, it is equivalent to define ϕ to be an A -flow if and only if

$$\langle \phi, \vec{\chi}_B \rangle = 0 \quad \text{for each } B \in \mathcal{B}.$$

Since signed characteristic vectors of bonds are orthogonal to signed characteristic vectors of circuits, ϕ is an A -flow of G if

$$\phi = \sum_{C \in \mathcal{C}} a_C \vec{\chi}_C \quad \text{for some } a_C \in A \text{ indexed by } C \in \mathcal{C}.$$

The converse is immediate when A is finite or a field: in the first case by counting (we know that $\ker D \cong A^{n(G)}$ and there are $n(G)$ linearly independent signed characteristic vectors $\vec{\chi}_C$) and in the second case by orthogonal decomposition of vector spaces. When $A = \mathbb{Z}$ the fact that all flows take the form $\phi = \sum_{C \in \mathcal{C}} a_C \vec{\chi}_C$ for $a_C \in \mathbb{Z}$ amounts to the fact that the signed characteristic vectors $\vec{\chi}_C$ form an integral basis for $\ker D$ as a lattice in \mathbb{R}^E .

The set of A -flows of G is given by

$$\mathcal{Z}_A = \left\{ \sum_{C \in \mathcal{C}} a_C \vec{\chi}_C \mid a_C \in A \right\}.$$

A graph G has a *nowhere-zero k -flow* if there is a flow $\phi \in \mathcal{Z}_{\mathbb{Z}}$ such that $0 < |\phi(e)| < k$ for all $e \in E$, and a *nowhere-zero A -flow* if there is a flow $\phi \in \mathcal{Z}_A$ such that $\phi(e) \neq 0$ for all $e \in E$.

The row space of the incidence matrix of G^ω is spanned by the signed characteristic vectors of cocircuits of G . The dual notion to flows is that of tensions (also known as *coflows*), which are defined as elements of $\text{im } D^T$:

Definition 2.6. Let \mathcal{B} denote the set of signed bonds of an oriented graph G^ω on edge set E . The set of A -tensions is defined by

$$\mathcal{K}_A = \left\{ \sum_{B \in \mathcal{B}} a_B \vec{\chi}_B \mid a_B \in A \right\}.$$

By Proposition 2.2 the signed characteristic vectors of circuits and cocircuits are orthogonal (as vectors over \mathbb{Z}). Once multiplication is defined making A into a ring, this extends to the following key relationship between flows and tensions:

Theorem 2.7. Suppose A is a commutative ring. If ϕ is an A -flow of a graph G and θ is an A -tension of G then ϕ and θ are orthogonal as vectors over A :

$$\sum_{e \in E} \phi(e)\theta(e) = 0.$$

An A -flow or A -tension whose value on each edge of G belongs to $B \subseteq A$ is called a B -flow or B -tension respectively. In the next section we shall be particularly interested in the case $B = A \setminus 0$.

Proposition 2.8. *A vector is a \mathbb{Z}_2 -flow if and only if it is the characteristic function of an Eulerian subgraph of G and is a \mathbb{Z}_2 -tension if and only if it is the characteristic vector of a cutset of G .*

When working over \mathbb{Z}_2 the signed characteristic functions of signed (co)circuits become characteristic functions of (co)circuits. The set of \mathbb{Z}_2 -flows is a binary vector space called the *cycle space* of G , comprising characteristic vectors of Eulerian subgraphs of G , and the set of \mathbb{Z}_2 -tensions is called the *cocycle space* of G , comprising characteristic vectors of cutsets of G .

Question 17

- (i) To what does an integer 2-flow of G correspond? When does G have a nowhere-zero 2-flow?
- (ii) Dually, when does G have a nowhere-zero 2-tension?
- (iii) Is it true that G has a nowhere-zero 2-flow if and only if G has a nowhere-zero \mathbb{Z}_2 -flow? And dually, what is the analogous statement for nowhere-zero 2-tensions and nowhere-zero \mathbb{Z}_2 -tensions?

2.5 Tensions and colourings

An A -potential of G is a mapping $\kappa : V \rightarrow A$ and can be thought of as a (not necessarily proper) vertex colouring of G with colours the elements of A . Given an orientation ω of G , the mapping $D^T \kappa$ is called the *potential difference* or *coboundary* of κ .

We identify colourings of the vertices of G , where the colours are taken in A , with the corresponding A -potential of G . An A -tension of G corresponds to $|A|^{c(G)}$ different A -colourings of G : to each $\theta \in \mathcal{K}_A$ corresponds $|A|^{c(G)}$ colourings $\kappa : V \rightarrow A$ with $D^T \kappa = \theta$. This relationship of tensions to vertex colourings is what underlies the duality between colourings and flows, as we shall see.

For a proper vertex A -colouring the corresponding A -tension is nowhere-zero. This is a basic observation linking flows and colourings and leads to the following:

Proposition 2.9. *Let G be a graph and let A be an Abelian group of order $k \geq 2$. Then $\chi(G) \leq k$ if and only if G admits a nowhere-zero A -tension.*

Question 18

- (i) Prove Proposition 2.9.
- (ii) Explain why a nowhere-zero A -tension of $G = (V, E)$ remains a nowhere-zero A -tension of $G \setminus e$, where e is any edge of G .
- (iii) Dually, show that if ϕ is a nowhere-zero A -flow of G then, with no change in its values on $E \setminus \{e\}$, it is also a nowhere-zero A -flow of G/e .

Although flows and tensions are defined relative to an orientation of G , the structure of \mathcal{Z}_A and \mathcal{K}_A (in particular, their size) is independent of the choice of orientation. Given an A -flow ϕ under orientation ω , by replacing $\phi(e)$ by $-\phi(e)$ for each edge e on which ω and ω' differ we obtain an A -flow of G under orientation ω' . A similar observation can be made for A -tensions.

The *support* of $\phi \in \mathcal{C}_A$ is defined by $\text{supp}(\phi) = \{e \in E : \phi(e) \neq 0\}$. A subset $S \subseteq E$ is a *minimal support* if $S = \text{supp}(\phi)$ for some flow ϕ and the only flow whose support is properly contained in S is the zero flow. The set of A -flows with a given minimal support (together with the zero flow) form a one-dimensional space of flows, namely of the form $a\vec{\chi}_C$ for some $a \in A$ and circuit C . A *primitive* A -flow is a flow ϕ with minimal support and for which each $\phi(e)$ is 0, 1 or -1 . In other words, ϕ is equal to $\pm\vec{\chi}_C$ for some circuit C . A \mathbb{Z} -flow π *conforms* to a \mathbb{Z} -flow ϕ if $\text{supp}(\pi) \subseteq \text{supp}(\phi)$ and $\pi(e)\phi(e) > 0$ for $e \in \text{supp}(\pi)$.

Question 19

- (i) Explain why for a given \mathbb{Z} -flow ϕ there is a primitive \mathbb{Z} -flow π which conforms to ϕ . Show that any \mathbb{Z} -flow ϕ is the sum of integer multiples of primitive \mathbb{Z} -flows, each of which conforms to ϕ .
- (iii) Prove that if ϕ is a nowhere-zero \mathbb{Z}_k -flow then there is a nowhere-zero \mathbb{Z} -flow ψ for which $\psi(e) \equiv \phi(e) \pmod{k}$ and $-k < \psi(e) < k$.
- (iv) Deduce that if G has a nowhere-zero \mathbb{Z}_k -flow then it has a nowhere-zero \mathbb{Z}_{k+1} -flow.

Theorem 2.10. *Let G be a graph with an orientation of its edges. For every $k \geq 2$, the following conditions are equivalent:*

- (i) *There exists a nowhere-zero \mathbb{Z}_k -flow in G .*
- (ii) *For any Abelian group A of order k , there exists a nowhere-zero A -flow in G .*
- (iii) *There exists a nowhere-zero k -flow in G .*

2.6 Duality of bases for A -tensions and A -flows

For a connected graph we have seen the signed characteristic vectors of cocircuits are spanned by the linearly independent set of vectors $\{\vec{\chi}_{\omega^+(\{u\})\cup\omega^-(\{u\})} : u \in V \setminus \{v\}\}$, where v is an arbitrary vertex.

A pair of bases, one for cocircuits and the other for circuits, can be defined relative to a fixed spanning tree of the graph. These bases are, in a sense we shall make precise shortly, dual to each other.

Proposition 2.11. *Let G be a connected graph, D its incidence matrix (for some orientation of G), and T a spanning tree of G .*

The signed characteristic vectors of the circuits $\{C_{T,e} : e \in E \setminus T\}$ form a basis for the set of A -flows of G . The signed characteristic vectors of the cocircuits $\{K_{T,e} : e \in E\}$ form a basis for the space of A -tensions of G .

Proof. A given edge $e \in E \setminus T$ belongs to $C_{T,e}$ but no other cycle $C_{T,f}$ for $f \neq e$. Hence the signed characteristic vectors $\{\vec{\chi}_{C_{T,e}} : e \in E \setminus T\}$ are linearly independent, and form a basis since there are $|E \setminus T| = n(G)$ of them.

Likewise, a given edge $e \in T$ belongs to $K_{T,e}$ but to no other $K_{T,f}$ for $f \neq e$, so the $|T| = r(G)$ signed indicator vectors of these cocircuits are linearly independent. \square

We now come to an abstract expression of the fact that we have already encountered that A -tensions of a planar graph correspond to A -flows of its dual:

Proposition 2.12. *Let G be a connected plane graph with orientation ω and G^* its dual graph with dual orientation ω^* . Let D denote the incidence matrix of G^ω and D^* the incidence matrix of $(G^*)^{\omega^*}$. Then $D^*D^T = O$. Also, $\ker(D^*) = \text{im}(D^T)$ and $\text{im}((D^*)^T) = \ker(D)$.*

Proof. Given a vertex $v \in V$ and face X incident with v , there are exactly two edges e, f belonging to X and with v as an endpoint. Then

$$(D^*D^T)_{X,v} = (D^*)_{X,e}(D)_{v,e} + (D^*)_{X,f}(D)_{v,f}. \quad (5)$$

Note that reversing the orientation of edge e does not change the value of $(D^*)_{X,e}(D)_{v,e}$ since both signs are flipped. Likewise for reversing the orientation of e . Taking the orientation that directs e into v and f out of v (for example), we calculate that (5) is equal to $(+1)(+1) + (+1)(-1) = 0$. Hence $D^*D^T = O$, so that $\text{im}(D^*)$ is orthogonal to $\text{im}(D^T)$. Since D has rank $r(G)$ and D^* has rank $r(G^*) = n(G)$ it follows that $\text{im}((D^*)^T) = \ker(D)$ and $\ker(D^*) = \text{im}(D^T)$. \square

Thus we have it formalized in stone what we already by now know: A -tensions of G are precisely A -flows of G^* . Moreover A -tensions of G with minimal support are A -flows of G^* with minimal support. In particular, circuits of G^* are cocircuits of G , and cocircuits of G^* are circuits of G . A defining property of planar graphs is that the dual of the cycle

matroid of a planar graph G is also a graphic matroid, namely the cycle matroid of the planar dual graph G^* (Theorem 6.6).

Since faces of G correspond to vertices of G^* , another natural basis for circuits of a connected plane graph G consists of the characteristic vectors of all but one of the face boundaries (say all but the outer face). This corresponds to the cocircuit basis of G^* obtained by taking the characteristic vectors of the edges incident with a common vertex, for all but one vertex of G^* .

Call a graph G^* the *abstract dual* of a graph G if $E(G) = E(G^*)$ and the cocircuits of G^* are precisely the circuits of G . This is to say that the cutset space of G^* is the cycle space of G : the cycle matroids of G and G^* are dual. We have seen that a connected planar graph has an abstract dual, equal to its geometric dual when it is embedded in the plane. This is a defining property of planar graphs:

Theorem 2.13. (Whitney, 1933) *A graph is planar if and only if it has an abstract dual.*

For a proof see for example [17, ch. 4].

2.7 Examples of nowhere-zero flows

We saw earlier in Proposition 2.8 that if ϕ is a \mathbb{Z}_2 -flow of a graph G then the support of ϕ (the set of edges where it is non-zero) is an edge-disjoint union of circuits. The reader is invited to deduce the following corollary:

Proposition 2.14. *The faces of a plane graph can be properly coloured with two colours if and only if all the vertices have even degree.*

Nowhere-zero \mathbb{Z}_3 -flows are in general difficult customers (there is a longstanding conjecture of Tutte concerning them), but by restricting attention to 3-regular graphs things become easier:

Proposition 2.15. *A cubic graph G has a nowhere-zero \mathbb{Z}_3 -flow if and only if it is bipartite.*

Proof. Given a nowhere-zero \mathbb{Z}_3 -flow of G , choose the orientation of G so that the value on each edge is $+1$. Then in this orientation every vertex is either a source or sink and this yields a proper vertex 2-colouring of G . Conversely, if G has a proper 2-colouring κ with colours $0, 1 \in \mathbb{Z}_3$ then, directing vertices coloured 0 towards vertices coloured 1, the potential difference $\delta\kappa$ is equal to 1 everywhere and so is not only a nowhere-zero \mathbb{Z}_3 -tension but also a nowhere-zero \mathbb{Z}_3 -flow, since G is cubic. \square

When translated to planar graphs this gives a theorem of Heawood from 1890:

Proposition 2.16. *A plane triangulation G has a proper vertex 3-colouring if and only if it has a proper face 2-colouring (equivalently, G is Eulerian).*

Question 20

Prove Proposition 2.16.

Thus we have found examples of graphs with a nowhere-zero 3-flow. What about a nowhere-zero 4-flow? Let us try to give some examples. You have probably heard of this one:

Proposition 2.17. *A simple cubic planar graph has a edge 3-colouring if and only if its faces can be properly coloured with four colours.*

(A graph G is said to be edge k -colourable if we can colour the edges of G with k colours such that any two incident edges have different colours.)

This is Tait's theorem (from 1880)[78] which was isolated in order to give one of the first proofs of the Four Colour Conjecture. It also led to study of *Hamiltonian graphs* and to the *Petersen graph*. This text wouldn't be complete without its picture.

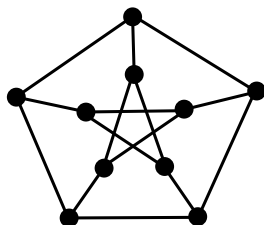


Figure 4: Petersen Graph

The same argument that Tait used to prove Proposition 2.17 can be generalized to non-planar graphs:

Proposition 2.18.

A cubic graph G has a nowhere-zero 4-flow if and only if it has a proper edge 3-colouring.

Proof. By Theorem 2.10 a graph has a nowhere-zero 4-flow if and only if it has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow. Let the non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ be a, b, c . We have $a+b+c=0$ and $a+a=b+b=c+c=0$. From this it is easy to see that a mapping $f : E(G) \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$ is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow if and only if it is a proper edge 3-colouring using the colours a, b, c . \square

Question 21 Using the equivalence of Proposition 2.18, show that the Petersen graph does not have a nowhere-zero 4-flow. (Hint: consider edges of a fixed colour in a putative edge 3-colouring, at least one of which must occur on the outer 5-cycle in Figure 4. What does this imply about the number of occurrences of this colour on the inner 5-cycle?)

The Petersen graph does however have nowhere-zero 5-flows, as shown in Figure 5. The following is an alternative characterization of graphs with a nowhere-zero 4-flow:

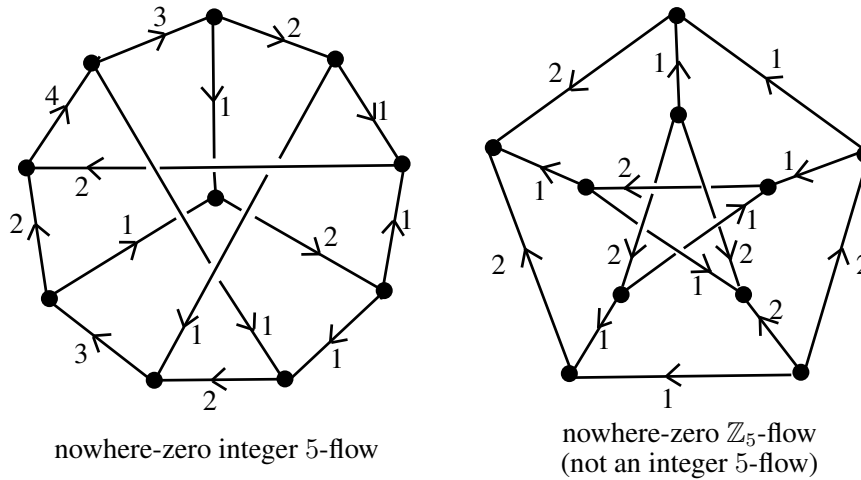


Figure 5: The Petersen graph (in two of its guises) with on the left a nowhere-zero 5-flow (also a nowhere-zero \mathbb{Z}_5 -flow) and on the right a nowhere-zero \mathbb{Z}_5 -flow.

Proposition 2.19. *A graph $G = (V, E)$ has a nowhere-zero 4-flow if and only if $E = E_1 \cup E_2$ and each of the graphs (V, E_1) and (V, E_2) is Eulerian.*

Proof. By Theorem 2.10 has a nowhere-zero 4-flow if and only if it has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow ϕ . Write $\phi = (\phi_1, \phi_2)$ and observe that the ϕ_i 's are \mathbb{Z}_2 -flows which are nowhere-zero \mathbb{Z}_2 -flows on the support E_i of ϕ_i . However, as we observed at the beginning of this section, this happens if and only if the subgraph on edge set E_i has all vertex degrees even. Moreover ϕ is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow if and only if $E = E_1 \cup E_2$. \square

Proposition 2.16 gives a nowhere-zero \mathbb{Z}_2 -flow condition for a plane triangulation to have a proper 3-colouring of its vertices (a nowhere-zero \mathbb{Z}_3 -tension). Underlying this is the dual version of Proposition 2.15.

Using Proposition 2.19 and tension-flow duality, a planar graph has a proper 4-colouring of its vertices if and only if its dual is the union of two of its Eulerian subgraphs. This criterion for a graph to have a nowhere-zero 4-flow emerges by using the fact that a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow is supported in each component on an Eulerian subgraph. If we had considered nowhere-zero \mathbb{Z}_4 -flows rather than $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flows then what criterion would we obtain instead? For a cubic graph we would find a perfect matching (edges receiving the value 2) together with a collection of oriented circuits (edges with value ± 1), each of which has the property that relative to the circuit orientation the values assigned to its edges alternate between 1 and -1 (i.e., the circuit is even and edge 2-coloured). This is effectively Proposition 2.18, which gives an edge 3-colouring equivalent to the existence of a nowhere-zero 4-flow of a cubic graph.

Question 22

Characterize graphs G that have a nowhere-zero \mathbb{Z}_2^r -flow in terms of Eulerian subgraphs. What is the dual version: when does a graph have a nowhere-zero \mathbb{Z}_2^r -tension?

An Eulerian orientation of a graph G is an orientation of G with the property that the indegree at a vertex is equal to its outdegree. Clearly G must be Eulerian, and by decomposing G into an edge-disjoint union of cycles there exist Eulerian orientations of G in this case.

Proposition 2.20. *Let G be a 4-regular graph. Then there is a one-to-one correspondence between nowhere-zero \mathbb{Z}_3 -flows of G and Eulerian orientations of G .*

Proof. For a given nowhere-zero \mathbb{Z}_3 -flow of G , arrange the orientation σ of G so that each flow value is equal to 1. Then the only way to obtain net flow zero at a vertex is to have two edges directed out and two edges directed in. In other words, the orientation σ is Eulerian. (Put alternatively, keep the fixed orientation σ of G and for a given nowhere-zero \mathbb{Z}_3 -flow of G preserve the orientation when flow value is +1 and reverse the orientation when flow value is -1: the result is an Eulerian orientation, uniquely defined by the flow values and σ .) \square

Nowhere-zero \mathbb{Z}_3 -flows of a graph G more generally correspond to orientations of G in which every vertex has indegree congruent to outdegree modulo 3.

We move on now to nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flows, whose significance in the history of attempts at proving the Four Colour Theorem we shall briefly describe. First a lemma which is not only of immediate use, but also to the problem of counting nowhere-zero A -flows that we consider in the next section.

Lemma 2.21. *Let $G = (V, E)$ be a connected graph and T a spanning tree of G . Let A be an Abelian group and $\phi_0 : E \setminus T \rightarrow A$. Then there is a unique A -flow ϕ of G such that $\phi(e) = \phi_0(e)$ for $e \in E \setminus T$.*

Proof. The vector

$$\phi = \sum_{e \in E \setminus T} \phi_0(e) \vec{\chi}_{C_{T,e}}$$

as a linear combination of basis vectors for \mathcal{Z}_A is an A -flow and since $e \notin C_{T,f}$ when $f \neq e$ the value of ϕ at e is given by $\phi(e) = \phi_0(e)$. Conversely, if an A -flow takes value $\phi_0(e)$ at each $e \in E \setminus T$ then it is equal to ϕ as defined above, since any vector in \mathcal{Z}_A has a unique expression as a linear combination of basis vectors. \square

Theorem 2.22. *A graph with a Hamiltonian circuit (a circuit traversing all vertices of G) has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow.*

Proof. Let H be a Hamiltonian circuit of G and T a spanning tree (a path) obtained from H by removing one of its edges. Let ϕ_1 be a \mathbb{Z}_2 -flow of G with support containing $E \setminus T$, which exists by Lemma 2.21 (taking $\phi_0(e) = 1$ for $e \in E \setminus T$). Let ϕ_2 be the \mathbb{Z}_2 -flow with support the circuit H . Then (ϕ_1, ϕ_2) is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow of G . \square

In the early years of trying to prove the Four Colour Conjecture, Tait conjectured in 1884 that every 3-connected planar graph was Hamiltonian (an example of 2-connected planar non-Hamiltonian was known, consisting of 20 vertices and 12 pentagonal faces). Tutte in 1956 gave a counterexample with 46 vertices. (See e.g. [73] for diagrams and a succinct historical account of variations on the Four Colour Conjecture.) See also the Herschel graph depicted in Figure 12.

We have found many graphs that have a nowhere-zero A -flow when $|A| \leq 4$. In the dual problem, no matter how large we choose $|A|$ there will always be graphs that do not have a nowhere-zero A -tension, namely those graphs with chromatic number exceeding $|A|$. The simplest obstruction to a proper k -colouring is an induced clique on $k + 1$ vertices. Is there an obstruction to a nowhere-zero A -flow when $|A| \geq 5$? Certainly not cliques, as we shall see shortly. But which graphs do not have a nowhere-zero \mathbb{Z}_5 -flow? Tutte (again!) had thoughts upon this matter, and no one has yet resolved the question. But let us keep to simple things for the moment and see off the complete graphs as being quite tame creatures when it comes to nowhere-zero flows.

The complete graph K_2 is a bridge and therefore does not have a nowhere-zero flow. K_3 is Eulerian and so has a nowhere-zero \mathbb{Z}_2 -flow. K_4 has a proper edge 3-colouring and hence has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow. On the other hand, K_4 does not have a nowhere-zero \mathbb{Z}_3 -flow since it is a non-bipartite cubic graph and does not have a nowhere-zero \mathbb{Z}_2 -flow since it is not Eulerian.

Proposition 2.23. *K_n has a nowhere-zero \mathbb{Z}_2 -flow when $n \geq 3$ is odd. K_n has a nowhere-zero \mathbb{Z}_3 -flow when $n \geq 6$ is even.*

Proof. The case of odd n follows since K_n is Eulerian. For $n = 6$ we have K_6 is the edge-disjoint union of two copies of K_3 and one copy of $K_{3,3}$. Each of these graphs has a nowhere-zero \mathbb{Z}_3 -flow ($K_{3,3}$ since it is a cubic bipartite graph). The union of these flows makes a nowhere-zero \mathbb{Z}_3 -flow of K_6 .

Consider now even $n > 6$ and assume the assertion of the theorem holds for $n - 2$. The graph K_n is the edge-disjoint union of K_{n-2} and $K_{2,n}^+$, where the latter is $K_{2,n}$ with an edge e added between the vertices of degree n . By hypothesis K_{n-2} has a nowhere-zero \mathbb{Z}_3 -flow. To make a nowhere-zero \mathbb{Z}_3 -flow of $K_{2,n}^+$ take the sum of nowhere-zero \mathbb{Z}_3 flows on each of the n triangles: this is non-zero on all but possibly the edge e . If necessary, make the value on e non-zero by adding in the flow again from a single (arbitrary) triangle of edges e, e_1, e_2 : this makes the value on e non-zero, and reverses the sign of the flow on e_1 and e_2 . We have thus constructed a nowhere-zero \mathbb{Z}_3 -flow of K_n . \square

2.8 The flow polynomial

We turn to the problem of *counting* nowhere-zero A -flows.

Theorem 2.24. (Tutte [81].) *Let A be a finite Abelian group of order k and G a graph with an orientation of its edges. Then the number of nowhere-zero A -flows of G is*

$$F(G; k) = \sum_{F \subseteq E} (-1)^{|E|-|F|} k^{n(F)}.$$

Proof. By Lemma 2.21 the number of A -flows of any subgraph (V, F) of $G = (V, E)$ is equal to $k^{|F|-r(F)}$, since a maximal spanning forest of (V, F) has $r(F)$ edges. Equivalently, $k^{n(F)}$ is the number of A -flows of G whose support is contained in F . The result follows by the inclusion-exclusion principle. \square

The polynomial $F(G; k)$ is called the *flow polynomial* of G . Theorem 2.24 implies that the number of nowhere-zero A -flows depends only on $|A|$, not on the structure of A as a group. In particular, the existence of an A -flow only depends on $|A|$, i.e., if A and A' are Abelian groups with $|A| = |A'|$ then G has a nowhere-zero A flow if and only if G has a nowhere-zero A' -flow. As a consequence, the existence of a nowhere-zero A -flow implies the existence of a nowhere-zero A' -flow when $|A'| > |A|$. This is because (as Tutte first showed in 1950 – see e.g. [44], [64], [17] for details, and Theorem 2.10 above) a nowhere-zero k -flow exists if and only if a nowhere-zero \mathbb{Z}_k -flow exists, whence if $k' > k$ then there is a nowhere-zero $\mathbb{Z}_{k'}$ -flow whenever there is a nowhere-zero \mathbb{Z}_k -flow. (Thinking of A -flows as duals of A -tensions, it is obvious that if G has a nowhere-zero A -tension then it has a nowhere-zero A' -tension, by using the correspondence of nowhere-zero A -tensions with proper A -colourings.)

Proposition 2.25. *The flow polynomial satisfies*

$$F(G; k) = \begin{cases} F(G/e; k) - F(G \setminus e; k) & e \text{ ordinary,} \\ 0 & e \text{ a bridge,} \\ (k-1)F(G \setminus e) & e \text{ a loop,} \\ 1 & E = \emptyset. \end{cases}$$

Proof. When $E = \emptyset$ the subgraph expansion for $F(G; k)$ gives $F(G; k) = 1$. When G has a bridge e it does not have a nowhere-zero flow, for $\{e\}$ is a cut of G . If e is a loop, on the other hand, then we can freely assign any non-zero value to it and still have a nowhere-zero flow. When e is ordinary, we have a bijection between nowhere-zero flows of $G \setminus e$ and flows of G that are zero only at e , and between nowhere-zero flows of G/e and flows of G that are nowhere-zero except possibly at e . (This argument also works when e is a bridge, but it needs to be shown that in this case $F(G \setminus e; k) = F(G/e; k)$, which amounts to showing that $F(G; k) = 0$.) \square

Question 23

Suppose $G = (V, E)$ is a connected graph and A a finite Abelian group of order k .

- (i) Given a spanning tree T and $\theta_0 : T \rightarrow A$, prove there is a unique A -tension θ of G such that $\theta(e) = \theta_0(e)$.
- (ii) Deduce that the number of nowhere-zero A -tensions of G is given by

$$F^*(G; k) = \sum_{F \subseteq E} (-1)^{|E|-|F|} k^{r(F)}.$$

- (iii) Formulate and prove a deletion-contraction recurrence satisfied by the polynomial $F^*(G; k)$.

Kochol [50] shows that the number of nowhere-zero k -flows is also a polynomial in k (*not* the same as the flow polynomial $F(G; k)$) - this polynomial counting integer flows does not satisfy a deletion-contraction recurrence.

For any finite Abelian group A there are loopless graphs G that do not have a nowhere-zero A -tension (take G with $\chi(G) > |A|$). The situation for nowhere-zero A -flows is quite different, where bridges are the only obstruction to having a nowhere-zero A -flow once $|A|$ is sufficiently large. In fact Seymour showed that $|A| \geq 6$ will do, and it is a famous conjecture of Tutte that in fact $|A| \geq 5$ suffices. Within the class of planar graphs – and as Whitney showed (see Theorem 2.13) this class is precisely the set of graphs closed under duality – the Four Colour Theorem tells us that we do have a symmetric situation: if $|A| \geq 4$ then any planar graph has a nowhere-zero A -tension and a nowhere-zero A -flow. It is when we move out of the class of planar graphs that a fundamental difference between the dual notions of flows and tensions arises. Of course within the more general world-view of matroids this asymmetry disappears (there are regular matroids with arbitrarily large flow number, as well as with arbitrarily large chromatic number).

3 The Tutte polynomial

3.1 Deletion-contraction recurrence

The Tutte polynomial of a graph $G = (V, E)$ (and more generally a matroid) may be defined recursively by

$$T(G; x, y) = \begin{cases} T(G/e; x, y) + T(G \setminus e; x, y) & e \text{ ordinary,} \\ xT(G/e; x, y) & e \text{ a bridge,} \\ yT(G \setminus e; x, y) & e \text{ a loop,} \\ 1 & G \text{ has no edges.} \end{cases} \quad (6)$$

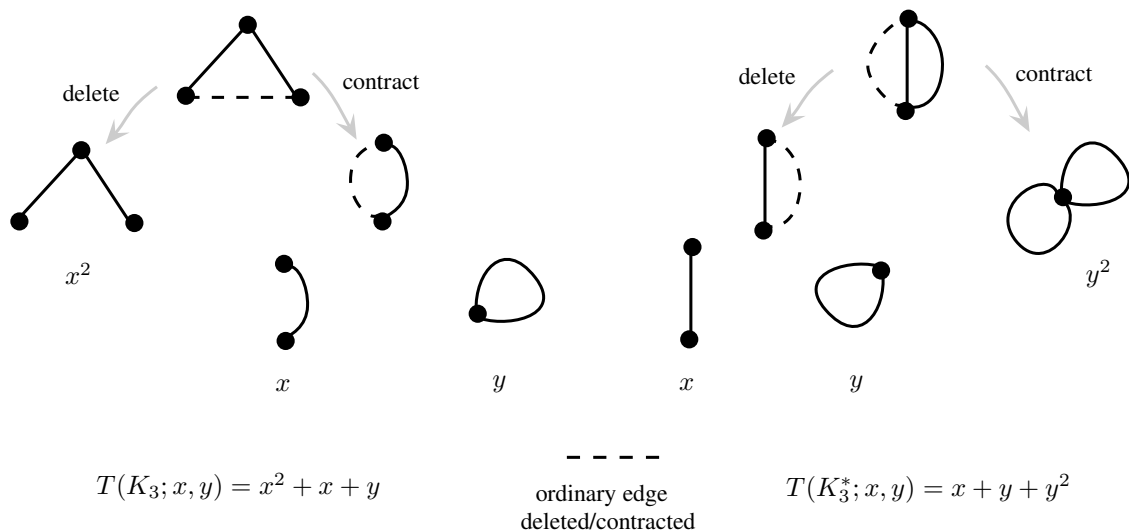


Figure 6: Using deletion-contraction to compute the Tutte polynomial of K_3 and its dual K_3^* .

Alternatively,

$$T(G; x, y) = \begin{cases} T(G/e; x, y) + T(G \setminus e; x, y) & e \text{ ordinary,} \\ x^k y^\ell & G \text{ consists of } k \text{ bridges and } \ell \text{ loops,} \end{cases} \quad (7)$$

It is not immediately clear that it does not matter which order the edges are chosen to calculate $T(G; x, y)$ recursively using (6).

Proposition 3.1. *If e and f are distinct edges of G then the outcome of first applying the recurrence (6) with edge e and then with edge f is the same as with the reverse order, when first taking f and then e .*

Proof. For distinct edges e and f we have the following preservation of edge types under deletion and contraction:

- (a) if e is a bridge in G then e remains a bridge in G/f and $G \setminus f$;
- (a)* if e is a loop in G then e remains a loop in G/f and $G \setminus f$;
- (b) if e is ordinary in G and there is a cutset containing e but not f , then e is ordinary in G/f ;
- (b)* if e is ordinary in G and there is a cycle containing e but not f , then e is ordinary in $G \setminus f$.

For each of the possible combinations of edge types for e and f in (a), (a)*, (b) and (b)*, one verifies that swapping the order of e and f gives the same outcome in the

two-level computation tree going from G to G with edges e and f deleted or contracted. For example, in cases (b) and (b)*, when both edges are ordinary and remain so after contraction or deletion of the other edge, the truth of the statement amounts to the fact that $G/e \setminus f \cong G \setminus f/e$, and similarly for the other three combinations of deletion and contraction.

The remaining cases to consider are the following:

- (c) if e is ordinary in G and any cutset containing e also contains f , then e is a loop in G/f , and in this case f is ordinary in G and a loop in G/e ;
- (c)* if e is ordinary in G and any cycle containing e also contains f , then e is a bridge in $G \setminus f$, and in this case f is ordinary in G and a bridge in $G \setminus e$.

(The hypothesis in (c) is equivalent to saying e is parallel to f .)

In (c), the fact that f must be a loop in G/e , symmetric to the fact that e is a loop in G/f , allow one to complete the argument that the deletion-contraction recurrence applied to e first and then f gives the same result as to f first and then e . Likewise in (c)*, that f is a bridge in $G \setminus e$, symmetric to the fact that e is a bridge in $G \setminus f$, means that the order in which we take e and f does not matter for the resulting value of the polynomial $T(G; x, y)$. \square

A graph G is *2-connected* if and only if has no cut-vertex. A loop on a single vertex (C_1) and a single bridge (K_2) are both 2-connected. For the case of many loops on a single vertex (where one might still consider the vertex not to be a cut-vertex) we refer to the cycle matroid, which is the direct sum of its constituent loops: so this graph is not 2-connected when there is more than one loop.

A *block* of G is a maximal 2-connected induced subgraph of G . If G is not 2-connected then it can be written in the form $G = G_1 \cup G_2$ where $|V(G_1) \cap V(G_2)| \leq 1$. The intersection graph of the blocks of a loopless connected graph is a tree. In particular, if G is loopless and connected and has at least two blocks then there are at least two *endblocks* of G which are blocks containing only one cut-vertex of G .

Proposition 3.2. *The Tutte polynomial of G is multiplicative over the connected components of G and over the blocks of G : if $G = G_1 \cup G_2$ where G_1 and G_2 share at most one vertex then $T(G_1 \cup G_2; x, y) = T(G_1; x, y)T(G_2; x, y)$.*

PROOF. The statement is true when each edge is either a bridge or a loop, since in this case $T(G; x, y) = x^k y^\ell$, where k is the number of bridges and ℓ the number of loops. We argue by induction on the number of ordinary edges of G . Let $G = G_1 \cup G_2$ where $|V(G_1) \cap V(G_2)| = 1$. The endpoints of any edge e must belong to the same block of G ; if e is a bridge or loop then it forms its own block. Suppose $G = G_1 \cup G_2$ where G_1 is a block of G containing an ordinary edge e . Deleting or contracting e can only decrease the

number of ordinary edges of G and since e is ordinary we have, writing $T(G; x, y) = T(G)$,

$$\begin{aligned} T(G) &= T(G/e) + T(G \setminus e) \\ &= T(G_1/e \cup G_2) + T(G_1 \setminus e \cup G_2) \\ &= [T(G_1/e) + T(G_1 \setminus e)]T(G_2) \\ &= T(G_1)T(G_2), \end{aligned}$$

where to obtain the third line we applied the inductive hypothesis. \square

The converse to Proposition 3.2 also holds, although its proof is bit more involved:

Theorem 3.3. [60] *If G is 2-connected graph without loops then $T(G; x, y)$ is irreducible in $\mathbb{Z}[x, y]$.*

The factors of the Tutte polynomial of G therefore correspond precisely to the blocks of G and any loops (each contributing a factor y).

As well as being multiplicative over blocks and connected components, and so unaffected by the operation of identifying vertices in different connected components of G , the Tutte polynomial is also unaffected by Whitney twists (this is what makes the Tutte polynomial of G an invariant of the cycle matroid of G and allows its generalization to a matroid invariant):

Proposition 3.4. *If G and G' are 2-isomorphic then $T(G; x, y) = T(G'; x, y)$. (The Tutte polynomial of G only depends on the cycle matroid of G .)*

Here are some basic properties of the coefficients of $T(G; x, y)$:

Proposition 3.5. *For a graph G with Tutte polynomial $T(G; x, y) = \sum t_{i,j}(G)x^i y^j$,*

- (i) $t_{0,0}(G) = 0$ if $|E(G)| > 0$;
- (ii) if G has no loops then $t_{1,0}(G) \neq 0$ if and only if G is 2-connected;
- (iii) x^k divides $T(G; x, y)$ if and only if G has at least k bridges, and y^ℓ divides $T(G; x, y)$ if and only if G has at least ℓ loops;
- (iv) given G has k bridges and ℓ loops, if $i \geq r(G)$ or $j \geq n(G)$ then $t_{i,j}(G) = 0$ except when $i = r(G)$ and $j = \ell$, or $i = k$ and $j = n(G)$, where we have $t_{r(G),\ell}(G) = 1 = t_{k,n(G)}(G)$.

Proof. For (ii), we use the property that if G is 2-connected, then at least one of G/e and $G \setminus e$ is also 2-connected. A basis for induction is that $T(K_2; x, y) = x$. Given a loopless graph G , if e is not parallel to another edge then both G/e and $G \setminus e$ have no loops, and the equation $t_{1,0}(G) = t_{1,0}(G/e) + t_{1,0}(G \setminus e)$ provides the inductive step. If e is parallel to another edge then G/e has a loop and $t_{1,0}(G) = t_{1,0}(G \setminus e)$; by deleting all but one edge in a parallel class we can thus assume G is simple. For the converse, if G is not 2-connected then by Proposition 3.2 its Tutte polynomial is the product of at least two polynomial

factors, each corresponding to a block of G ; by what we have just proved $t_{1,0}(B) = 1$ for each such block B , and this implies $t_{1,0}(G) = 0$.

For (iv), we shall use induction on the number of ordinary edges to prove that $t_{i,j}(G) = 0$ when $i \geq r(G)$ or $j \geq n(G)$, except for $t_{r(G),\ell}(G) = 1 = t_{k,n(G)}(G)$. The base case is when G has no ordinary edges, consisting of k bridges and ℓ loops. Here $r(G) = k$ and $n(G) = \ell$, and $t_{k,\ell}(G) = 1$, while $t_{i,j}(G) = 0$ for all other values of i, j . Hence the statement is true in this case.

Consider the recurrence formula $t_{i,j}(G) = t_{i,j}(G/e) + t_{i,j}(G \setminus e)$ for ordinary edge e . We have by inductive hypothesis that $t_{i,j}(G/e) = 0$ for $i \geq r(G/e) = r(G) - 1$ except $t_{r(G)-1,\ell}(G/e) = 1$, and for $j \geq n(G/e) = n(G)$ except $t_{k,n(G)}(G/e) = 1$. This gives $t_{i,j}(G) = 0$ for $j \geq n(G)$ except $t_{k,n(G)}(G) = 1$.

Also $t_{i,j}(G \setminus e) = 0$ for $i \geq r(G \setminus e) = r(G)$ except $t_{r(G),\ell}(G \setminus e) = 1$, and for $j \geq n(G \setminus e) = n(G) - 1$ except $t_{k,n(G)-1}(G \setminus e) = 1$. This gives $t_{r(G),\ell}(G) = 0$ for $i \geq r(G)$ except $t_{r(G),\ell}(G) = 1$. \square

Theorem 3.6. “Recipe Theorem” *Let \mathcal{G} be a minor-closed class of graphs. There is a unique graph invariant $f : \mathcal{G} \rightarrow \mathbb{Z}[x, y, \alpha, \beta, \gamma]$ such that for graph $G = (V, E)$*

$$f(G) = \begin{cases} \alpha f(G/e) + \beta f(G \setminus e) & e \text{ ordinary edge of } G, \\ xf(G/e) & e \text{ a bridge in } G, \\ yf(G \setminus e) & e \text{ a loop in } G, \\ \gamma^{|V|} & G \text{ has no edges.} \end{cases} \quad (8)$$

The graph invariant f is equal to the following specialization of the Tutte polynomial:

$$f(G) = \gamma^{c(G)} \alpha^{r(G)} \beta^{n(G)} T(G; \frac{x}{\alpha}, \frac{y}{\beta}). \quad (9)$$

NOTE. (i) If instead of contracting a bridge we require that $f(G) = xf(G \setminus e)$ when e is a bridge, the Tutte polynomial is evaluated at the point $(\gamma x/\alpha, y/\beta)$ instead of $(x/\alpha, y/\beta)$. In particular, when $\gamma = 1$ it does not matter whether bridges are deleted or contracted.

(ii) If either α or β is zero then we interpret (9) as the result of substituting values of the parameters after expanding the expression on the right-hand side as a polynomial in $\mathbb{Z}[\alpha, \beta, \gamma, x, y]$. Given a graph G with k bridges and ℓ loops, if $\alpha = 0$ then $f(G) = \gamma^{c(G)} \beta^{n(G)-\ell} x^{r(G)} y^\ell$, and if $\beta = 0$ then $f(G) = \gamma^{c(G)} \alpha^{r(G)-k} x^k y^{n(G)}$. If both α and β are zero then $f(G) = 0$ if G has an ordinary edge, while $f(G) = \gamma^{c(G)} x^k y^\ell$ if $E(G)$ consists of just k bridges and ℓ loops.

PROOF. Uniqueness of $f(G)$ follows by induction on the number of edges and application of the recurrence (8).

Formula (9) is certainly true for cliques \overline{K}_n . If G consists just of k bridges and ℓ loops and has c connected components, then $f(G) = \gamma^c x^k y^\ell$ and since $r(G) = k$ and $n(G) = \ell$ we have $T(G; \frac{x}{\alpha}, \frac{y}{\beta}) = (\frac{x}{\alpha})^k (\frac{y}{\beta})^\ell$, so (9) is satisfied. Let e be an ordinary edge, and note that

$c(G) = c(G/e) = c(G \setminus e)$, so that $r(G/e) = r(G) - 1$, $r(G \setminus e) = r(G)$ and $n(G/e) = n(G)$, $n(G \setminus e) = n(G) - 1$. By induction on the number of ordinary edges,

$$\begin{aligned} f(G) &= \alpha f(G/e) + \beta f(G \setminus e) \\ &= \alpha \cdot \gamma^{c(G)} \alpha^{r(G)-1} \beta^{n(G)} T(G/e; \frac{x}{\alpha}, \frac{y}{\beta}) + \beta \cdot \gamma^{c(G)} \alpha^{r(G)} \beta^{n(G)-1} T(G \setminus e; \frac{x}{\alpha}, \frac{y}{\beta}) \\ &= \gamma^{c(G)} \alpha^{r(G)} \beta^{n(G)} T(G; \frac{x}{\alpha}, \frac{y}{\beta}). \end{aligned}$$

□

Proposition 3.7. *The chromatic polynomial is given by*

$$P(G; z) = (-1)^{r(G)} z^{c(G)} T(G; 1 - z, 0).$$

More generally, the monochrome polynomial,

$$B(G; k, y) = \sum_{f: V(G) \rightarrow [k]} y^{\#\{uv \in E(G): f(u)=f(v)\}},$$

is the following specialization of the Tutte polynomial:

$$B(G; k, y) = k^{c(G)} (y - 1)^{r(G)} T(G; \frac{y - 1 + k}{y - 1}, y).$$

PROOF. For the chromatic polynomial we have $P(G; z) = (z - 1)P(G/e; z)$ when e is a bridge, for we have $P(G \setminus e; z) = zP(G/e; z)$. A direct argument for $P(G \setminus e; k) = kP(G/e; k)$ when $e = uv$ is a bridge is as follows. Suppose $G \setminus e = G_1 \cup G_2$ with $u \in V(G_1)$ and $v \in V(G_2)$. Then G/e is obtained from $G_1 \cup G_2$ by identifying the vertices u and v to make a cut-vertex w . Given a fixed colour $\ell \in [k]$, there are $P(G_1; k)/k$ proper colourings $f_1 : V(G_1) \rightarrow [k]$ of G_1 with $f_1(w) = \ell$, and $P(G_2; k)/k$ proper colourings $f_2 : V(G_2) \rightarrow [k]$ of G_2 with $f_2(w) = \ell$. Since there are no edges between G_1 and G_2 , there are $P(G_1; k)P(G_2; k)/k^2$ proper colourings of G/e with $f(w) = \ell$. This number is independent of ℓ , so there are $P(G_1; k)P(G_2; k)/k$ proper colourings of G/e . On the other hand, there are $P(G_1; k)P(G_2; k)$ proper colourings of $G \setminus e$. Hence $P(G \setminus e; k) = kP(G/e; k)$ when e is a bridge of G .

A similar argument to the recurrence for the chromatic polynomial gives

$$B(G; k, y) = (y - 1)B(G/e; k, y) + B(G \setminus e; k, y), \quad (10)$$

valid for all edges e . When e is a bridge we have $B(G \setminus e; k, y) = kB(G/e; k, y)$, by a similar argument to the chromatic polynomial, by conditioning on the colour of the cut-vertex w of G/e obtained by identifying the endpoints of e . Instead of proper colourings, consider colourings with exactly m_1 monochrome edges in G_1 and exactly m_2 monochrome edges in G_2 . Then the number of such colourings for $G \setminus e$ (the disjoint union of G_1 and G_2) is k times the number for G/e (the gluing of G_1 and G_2 at a vertex). Collecting together all colourings

for which $m_1 + m_2 = m$, this implies that the coefficient of y^m in $B(G \setminus e; k, y)$ is equal to k times the corresponding coefficient in $B(G/e; k, y)$. Since this holds for each m , it follows that $B(G \setminus e; k, y) = kB(G/e; k, y)$ when e is a bridge, and so $B(G; k, y) = (y-1+k)B(G/e)$ by the recurrence formula (10). When e is a loop $B(G; k, y) = yB(G \setminus e; k, y)$ since a loop is always monochromatic (or by looking at the recurrence formula (10) with $G/e \cong G \setminus e$ when e is a loop).

The result now follows by Theorem 3.6. \square

Remark. The monochrome polynomial is the partition function for the q -state Potts model in disguise (see Section 4.2).

We have seen (Theorem 1.4) that $T(G; 2, 0) = (-1)^{|V(G)|}P(G; -1)$ counts acyclic orientations of G . An acyclic orientation of G has at least one source (all edges outgoing) and at least one sink (all edges incoming).

Theorem 3.8. [Greene and Zavslasky [35]] *Suppose G is a connected graph and $u \in V(G)$. Then the number of acyclic orientations of G with unique source at u is equal to $T(G; 1, 0)$. In particular, this number is independent of the choice of u .*

Note that $T(G; 1, 0) = P'(G; 0)$, the coefficient of z in $P(G; z)$, when G is connected.

Proof. Fix a vertex u of G and let $Q_u(G)$ denote the number of acyclic orientations with a unique source at u .

Suppose G is connected and with at least one edge. Choose an edge $e = uv$ with one endpoint the source vertex u . (Since G is connected there has to be at least one edge incident with u .)

If e is the only edge of G , then $Q_u(G) = 1$ when e is a bridge, and $Q_u(G) = 0$ when e is a loop. Suppose there are other edges.

If e is a loop then $Q_u(G) = 0$.

If e is a bridge then $Q_u(G) = Q_u(G/e)$. For consider an acyclic orientation \mathcal{O} of G with unique source u . Then in the component of $G \setminus e$ containing v , the only source of \mathcal{O} restricted to this component has to be v , otherwise there would be a source other than u in \mathcal{O} . Therefore, acyclic orientations of G with unique source at u are in one-to-one correspondence with acyclic orientations of G/e with unique source at u (which in G/e has been identified with the vertex v).

If e is ordinary then partition acyclic orientations with u as a unique source into two sets: those for which uv is the only edge directed into v (so deleting uv does not give an acyclic orientation of $G \setminus e$ with a unique source) and those for which uv is not the only edge directed into v (here deleting uv gives an acyclic orientation of $G \setminus e$ with unique source at u). The first set is in one-to-one correspondence with acyclic orientations of G/e with unique source at u (in G/e vertex v is identified with vertex u), while the second set is in one-to-one correspondence with acyclic orientations of $G \setminus e$ with unique source at u . Hence when e is ordinary we have $Q_u(G) = Q_u(G/e) + Q_u(G \setminus e)$.

By Proposition 3.6 it follows that $Q_u(G) = T(G; 1, 0)$. \square

Consider a connected graph $G = (V, E)$ in which each edge is deleted independently at random with probability $1 - p$ (e remains with probability p). The probability that G remains connected is known as the (*all-terminal*) *reliability* $R(G; p)$ and is given by

$$R(G; p) = \sum_A p^{|A|} (1 - p)^{|E \setminus A|},$$

where the sum is over all spanning connected subgraphs (V, A) .

Proposition 3.9. *If $G = (V, E)$ is a connected graph then*

$$R(G; p) = (1 - p)^{|E| - |V| + 1} p^{|V| - 1} T(G; 1, \frac{1}{1 - p}).$$

Proof. Establish the recurrence

$$R(G; p) = pR(G/e; p) + (1 - p)R(G \setminus e),$$

by conditioning on the events that e is or is not deleted. By Theorem 3.6 the result follows. \square

When G is not connected the appropriate event to consider is whether G still has the same number of connected components after independently deleting edges at random with probability $1 - p$, i.e., whether its rank of G is preserved. The probability of this event is $(1 - p)^{n(G)} p^{r(G)} T(G; 1, \frac{1}{1 - p})$, by multiplicativity of this invariant over connected components.

3.2 Sugraph expansion of the Tutte polynomial

First let's recap some notation. Let $G = (V, E)$ be a graph and $A \subseteq E$. Identify A with the spanning subgraph $G_A = (V, A)$. The rank of A is defined by $r_G(A) = |V(G)| - c(G_A)$ (this is the matroid rank function on the cycle matroid of G). The nullity of A is defined by $n_G(A) = |A| - r_G(A)$. Thus $r_G(E) = r(G)$ and $n_G(E) = n(G)$ in the notation already introduced for the rank and nullity of the graph G . When context makes it clear what graph G is, we drop the subscript and write $r(A)$ for $r_G(A)$ and $n(A)$ for $n_G(A)$.

It is easy to see that $0 \leq r(A) \leq |A|$ with $r(A) = 0$ if and only if A is empty or a set of loops, and $r(A) = |A|$ if and only if G_A is a forest (set of bridges). Also, $A \subseteq B$ implies $r(A) \leq r(B)$ and $r(A) = r(E)$ if and only if $c(G_A) = c(G)$.

Proposition 3.10. *The Tutte polynomial of a graph $G = (V, E)$ has subgraph expansion*

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{n(A)}. \quad (11)$$

PROOF. Set

$$R(G; u, v) = \sum_{A \subseteq E} u^{r(E) - r(A)} v^{|A| - r(A)},$$

(the Whitney rank-nullity generating function for G). We wish to prove that $T(G; x, y) = R(G; x - 1, y - 1)$ and shall do this by verifying that $R(G; u, v)$ satisfies the recurrence: (i) $R(G; u, v) = 1$ if $E = \emptyset$, (ii) $R(G; u, v) = (u + 1)R(G \setminus e; u, v)$ when e is a bridge, (iii) $R(G; u, v) = (v + 1)R(G \setminus e; u, v)$ when e is a loop, and (iv) $R(G; u, v) = R(G/e; u, v) + R(G \setminus e; u, v)$ when e is ordinary.

When $E = \emptyset$ we have $R(G; u, v) = 1$.

If $e \notin A$ then

$$r_G(A) = r_{G \setminus e}(A). \quad (12)$$

If $e \in A$ then

$$r_{G \setminus e}(A \setminus e) = \begin{cases} r_G(A) - 1 & \text{if } e \text{ is a bridge,} \\ r_G(A) & \text{if } e \text{ is a loop,} \end{cases} \quad (13)$$

and

$$r_{G/e}(A \setminus e) = r_G(A) - 1 \quad \text{if } e \text{ is ordinary or a bridge.} \quad (14)$$

Suppose e is a bridge. Then by (12) and (13),

$$\begin{aligned} R(G; u, v) &= \sum_{A \subseteq E \setminus e} u^{r_G(E) - r_G(A)} v^{|A| - r_G(A)} + \sum_{e \in A \subseteq E} u^{r_G(E) - r_G(A)} v^{|A| - r_G(A)} \\ &= u \sum_{A \subseteq E \setminus e} u^{r_{G \setminus e}(E \setminus e) - r_{G \setminus e}(A)} v^{|A| - r_{G \setminus e}(A)} \\ &\quad + \sum_{B = A \setminus e} u^{r_{G \setminus e}(E \setminus e) + 1 - (r_{G \setminus e}(B) + 1)} v^{|B| + 1 - (r_{G \setminus e}(B) + 1)} \\ &= (u + 1)R(G \setminus e; u, v). \end{aligned}$$

The case when e is a loop is similarly argued.

When e is ordinary, by (12) and (14),

$$\begin{aligned} R(G; u, v) &= \sum_{A \subseteq E \setminus e} u^{r_G(E) - r_G(A)} v^{|A| - r_G(A)} + \sum_{e \in A \subseteq E} u^{r_G(E) - r_G(A)} v^{|A| - r_G(A)} \\ &= \sum_{A \subseteq E \setminus e} u^{r_{G \setminus e}(E \setminus e) - r_{G \setminus e}(A)} v^{|A| - r_{G \setminus e}(A)} \\ &\quad + \sum_{B = A \setminus e} u^{r_{G/e}(E \setminus e) + 1 - (r_{G/e}(B) + 1)} v^{|B| + 1 - (r_{G/e}(B) + 1)} \\ &= R(G \setminus e; u, v) + R(G/e; u, v). \end{aligned}$$

□

It is common to *define* the Tutte polynomial by its subgraph expansion (11), having over the deletion–contraction formulation (6) the advantage of being transparently well-defined. On the other hand, it is not apparent from (11) that the coefficients of the Tutte polynomial are non-negative integers, and often it is easier to derive a combinatorial interpretation for an evaluation of the Tutte polynomial by using the deletion–contraction

recurrence. Nonetheless, it is easy to read off some evaluations of the Tutte polynomial from its subgraph expansion.

Question 24 Let $G = (V, E)$ be a connected graph. Using the subgraph expansion for $T(G; x, y)$ show the following:

- (i) $T(G; 1, 1) = \#\text{spanning trees}$,
 $T(G; 2, 1) = \#\text{spanning forests}$,
 $T(G; 1, 2) = \#\text{connected spanning subgraphs}$,
and $T(G; 2, 2) = 2^{|E|} = \#\text{spanning subgraphs}$.
- (ii) If $(x - 1)(y - 1) = 1$ then $T(G; x, y) = (x - 1)^{r(E)} y^{|E|}$.
- (iii) The generating function for spanning forests of G by number of connected components is given by

$$xT(G; x + 1, 1) = \sum_{\substack{A \subseteq E \\ n(A)=0}} x^{c(G_A)}.$$

- (iv) The generating function for connected spanning subgraphs of G by size is given by

$$y^{|V|-1} T(G; 1, y + 1) = \sum_{\substack{A \subseteq E \\ c(G_A)=c(G)}} y^{|A|}.$$

Along the hyperbola $(x - 1)(y - 1) = z$ we have, for graph $G = (V, E)$,

$$\begin{aligned} T(G; x, y) &= (y - 1)^{-|V|} \sum_{A \subseteq E} \left(\frac{z}{y - 1} \right)^{c(G_A) - c(G)} (y - 1)^{|A| + c(G_A)} \\ &= (y - 1)^{-r(G)} z^{-c(G)} \sum_{A \subseteq E} z^{c(G_A)} (y - 1)^{|A|}. \end{aligned}$$

When $y = 0$ this is the subgraph expansion for the chromatic polynomial obtained by an inclusion–exclusion argument. The polynomial $\sum_{A \subseteq E} z^{c(G_A)} w^{|A|}$ is the partition function for the Fortuin–Kasteleyn random cluster model in statistical physics (the normalizing constant for a probability space on subgraphs of G , the probability of $G_A = (V, A)$ depending on both $|A|$ and $c(A)$). This model generalizes the k -state Potts model, which is the case $z = k \in \mathbb{Z}_+$, and whose partition function we have already met in the form of the monochrome polynomial $B(G; k, y)$.

3.3 Coefficients. Spanning tree expansion.

A graph invariant is called a *Tutte invariant* if it can be found as some function of the coefficients of $T(G; x, y)$. Thus the property of having at least one edge is a Tutte invariant since $t_{0,0}(G) = 0$ if and only if G has an edge. In fact $|E|$ is itself a Tutte invariant since $r(G) = \max\{i : t_{i,j}(G) \neq 0\}$ and $n(G) = \max\{j : t_{i,j}(G) \neq 0\}$ are Tutte invariants and $r(G) + n(G) = |E|$. For another example, from Proposition 3.5 (ii), a loopless graph G is 2-connected if and only if $t_{1,0}(G) \neq 0$.

Examples of graph invariants that are not Tutte invariants include the degree sequence of G and whether G is planar. A tree on n vertices has Tutte polynomial x^{n-1} , and for $n \geq 3$ there are two trees on n vertices with different degree sequences. Less trivially, there are non-2-isomorphic graphs G and G' which have different degree sequences. Likewise, there is a planar graph G and non-planar graph G' with $T(G; x, y) = T(G'; x, y)$. (See [61, Appendix] for examples.)

In this section we shall give Tutte's 1954 inductive proof that, for a connected graph G , the coefficients $t_{i,j}(G)$ count a certain subset of the spanning trees of G . The interpretation of $t_{i,j}(G)$ when G is not necessarily connected follows as an easy consequence of multiplicativity of $T(G; x, y)$ over disjoint unions. A subgraph $G_A = (V, A)$ has $r(A) = r(E)$ and $n(A) = 0$ if and only if G_A is a *maximal spanning forest*, in the sense that no edge can be added to G_A without creating a cycle, i.e., G_A consists of a spanning tree of each connected component of G .

Let $G = (V, E)$ be a connected graph and T a spanning tree of G . Then

- (i) for each $e \in E \setminus T$ there is a unique cycle in G contained in $T \cup \{e\}$, which we shall denote by $\text{cyc}(T, e)$, and
- (ii) for each $e \in T$ there is a unique cut contained in $E \setminus T \cup \{e\}$, which we shall denote by $\text{cut}(T, e)$.

Put a linear order $<$ on E . Say $E = \{e_1, e_2, \dots, e_m\}$, where $e_1 < e_2 < \dots < e_m$.

Definition 3.11. *Given a spanning tree T of a connected graph G with an ordering of its edges, an edge $e \in T$ is internally active with respect to T if e is the least edge in $\text{cut}(T, e)$. An edge $e \in E \setminus T$ is externally active with respect to T if e is the least edge in $\text{cyc}(T, e)$. A spanning tree T has internal activity i and external activity j when there are precisely i internally active edges with respect to T and j externally active edges with respect to T .*

Tutte was led to his spanning tree expansion of the Tutte polynomial of a connected graph by observing that in the recursive definition of $T(G; x, y)$, if one applies deletion and contraction to edges of E in reverse order $e_m, e_{m-1}, \dots, e_2, e_1$, the result will be an expression for $T(G; x, y)$ as a sum in which each summand is obtained by contracting the elements in some spanning tree T of G and deleting the elements of $E \setminus T$. Moreover, in the process of obtaining this summand the edges contracted as bridges will be precisely the internally active edges with respect to T , and the elements of E deleted as loops will be precisely the externally active edges with respect to T .

Theorem 3.12 (Tutte Tutte54). *Let G be a connected graph with an order on its edges and for each $0 \leq i \leq |V| - 1, 0 \leq j \leq |E| - |V| + 1$ let $t_{i,j}(G)$ denote the number of spanning trees of G of internal activity i and external activity j . Then the Tutte polynomial of G is equal to*

$$T(G; x, y) = \sum t_{i,j}(G) x^i y^j.$$

In particular, $t_{i,j}(G)$ is a graph invariant, independent of the ordering of the edges of G .

Proof. We proceed by induction on the number of edges of G .

When there are no edges in G , i.e., $G \cong K_1$, we have $t_{0,0}(G) = 1$ and $t_{i,j}(G) = 0$ for $i + j > 0$.

Let $G = (V, E)$, $E = \{e_1 < e_2 < \dots < e_m\}$, $m \geq 1$, and assume the assertion holds for connected graphs with at most $m - 1$ edges.

The graphs G/e_m and $G \setminus e_m$ are both connected when e_m is ordinary or a loop, while only G/e_m is connected when e_m is a bridge, but this is fine because we only contract bridges in the recurrence for $T(G; x, y)$. We take $E(G/e_m) = E(G \setminus e_m) = \{e_1 < e_2 < \dots < e_{m-1}\}$.

(i) Suppose e_m is a bridge. Then e_m is in every spanning tree of G , and a subgraph T is a spanning tree if and only if $e_m \in T$ and T/e_m is a spanning tree of G/e_m . Also, e_m is internally active in every spanning tree T of G , since $\text{cut}(T, e_m) = \{e_m\}$, so $t_{0,j}(G) = 0$ for each j . Clearly, for $1 \leq k \leq m - 1$ the edge e_k is internally (externally) active in G with respect to T if and only if it is internally (externally) active in G/e_m with respect to T/e_m . Hence $t_{i,j}(G) = t_{i-1,j}(G/e_m)$ for $i \geq 1$. Applying the inductive hypothesis, we obtain

$$\begin{aligned} T(G; x, y) &= \sum t_{i-1,j}(G/e_m) x^i y^j \\ &= x \sum t_{i-1,j}(G/e_m) x^{i-1} y^j \\ &= xT(G/e_m; x, y) = T(G; x, y). \end{aligned}$$

(ii) Suppose e_m is a loop. Then e_m is in no spanning tree of G , and a subgraph T of G is a spanning tree of G if and only if it is a spanning tree of $G \setminus e_m$. Also e_m is externally active with respect to every spanning tree T of G since $\text{cyc}(T, e_m) = \{e_m\}$. For $1 \leq k \leq m - 1$ the edge e_k is internally (externally) active in G with respect to T if and only if it is internally (externally) active in $G \setminus e_m$ with respect to the same spanning tree T . Hence $t_{i,j}(G) = t_{i,j-1}(G \setminus e_m)$, so

$$\begin{aligned} \sum_{i,j} t_{i,j}(G) x^i y^j &= y \sum_{i,j} t_{i,j-1}(G \setminus e_m) x^i y^{j-1} \\ &= yT(G \setminus e_m; x, y) = T(G; x, y). \end{aligned}$$

(iii) Suppose e_m is ordinary.

A subset T is a spanning tree of $G \setminus e_m$ if and only if it is a spanning tree of G not containing e_m . If T is a spanning tree of $G \setminus e_m$ with internal activity i and external activity j then it has the same activities as a spanning tree of G , since every other edge precedes e_m and $\text{cyc}(T, e_m)$ contains an edge other than e_m .

Similarly, T is a spanning tree of G/e_m if and only if $T \cup \{e_m\}$ is a spanning tree of G (no cycles in $T \cup \{e_m\}$ can be created by e_m that would not already be in T in the contraction G/e_m). If T is a spanning tree of G/e_m with internal activity i and external activity j then it has the same activities as a spanning tree of G , since every other edge precedes e_m and $\text{cut}(T, e_m)$ contains an edge other than e_m since e_m is not a bridge.

It follows that $t_{i,j}(G) = t_{i,j}(G/e_m) + t_{i,j}(G \setminus e_m)$ when e_m is ordinary, and this makes the induction step go through for ordinary edges too. \square

A more constructive proof that $t_{i,j}(G)$ is equal to the number of spanning trees of G of internal activity i and external activity j was given by Crapo in 1969. See for example [7, ch. 13], and also [10, X.5].

The definition of internal and external activity extends in the obvious way from spanning trees of connected graphs to maximal spanning forests of graphs more generally.

Corollary 3.13. *Let G be a graph with Tutte polynomial $T(G; x, y) = \sum t_{i,j}(G)x^i y^j$. Then $t_{i,j}(G)$ is equal to the number of maximal spanning forests of G of internal activity i and external activity j .*

Proposition 3.14. *If $|E(G)| > 0$ then $t_{0,0}(G) = 0$. If $|E(G)| > 1$ then $t_{1,0}(G) = t_{0,1}(G)$.*

Proof. If $E = \{e_1, \dots, e_m\}$ is non-empty with order $e_1 < \dots < e_m$, then e_1 is active with respect to any maximal spanning forest F , internally if $e_1 \in F$, externally if $e_1 \notin F$. In particular, $t_{0,0}(G) = 0$.

Note that $t_{1,0}(K_2) = 1, t_{0,1}(K_2) = 0$. Assume $m \geq 2$. If G has at least two blocks containing at least one edge then we can choose an order on E such that e_1 and e_2 belong to different blocks of G . Then e_1 and e_2 are both active with respect to every maximal spanning forest, and so $t_{1,0}(G) = 0 = t_{0,1}(G)$ in this case.

Suppose then that G is 2-connected. (If there are isolated vertices we can ignore them as the Tutte polynomial is unaffected by their presence or absence.) Let T be a spanning tree of internal activity 1 and external activity 0.

The edge e_1 is active with respect to every spanning tree, and so $e_1 \in T$. This implies $e_2 \notin T$, for otherwise e_2 would also be internally active for T ($\text{cut}(T, e_2)$ cannot contain e_1 , which belongs to T). So $e_1 \in \text{cyc}(T, e_2)$, otherwise e_2 would be externally active.

The subgraph $T' = T - \{e_1\} \cup \{e_2\}$ is also a spanning tree of G , and has internal activity 0 and external activity 1 (the edge e_1).

Reversing the argument shows that the map $T \mapsto T'$ is a bijection between trees contributing to $t_{1,0}(G)$ and trees contributing to $t_{0,1}(G)$: if T' is a spanning tree contributing to $t_{0,1}(G)$ then $e_1 \notin T'$ but $e_2 \in T'$, and interchanging e_1 and e_2 yields a spanning tree T contributing to $t_{1,0}(G)$. \square

It is an easy exercise to prove Proposition 3.14 beginning with the fact that $t_{1,0}(G) = 0$ if G is not 2-connected and then inductively by deletion/contraction of an ordinary edge. However, the proof given gives more insight into why the identity holds.

The identities of Proposition 3.14 are the first of a series of identities proved by Brylawski [14]. If $|E(G)| > k$ then

$$\sum_{i=0}^k \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} t_{i,j}(G) = 0.$$

Thus if $|E(G)| > 2$ then $t_{2,0}(G) - t_{1,1}(G) + t_{0,2}(G) = t_{1,0}(G)$.

The fact that $T(G; x, y)$ has degree $r(G)$ as a polynomial in x and degree $n(G)$ as a polynomial in y is immediate from the fact that $t_{i,j}(G)$ is the number of maximal spanning forests of internal activity i and external activity j . Choose the edge order $e_1 < e_2 < \dots < e_m$ so that $e_1, \dots, e_{r(G)}$ are the edges of a maximal spanning forest: all are internally active, and no edges are externally active when G has no loops. Or, when choosing the edge order so that $e_1, \dots, e_{n(G)}$ are the edges in the complement of a maximal spanning forest of G , the latter having internal activity 0 provided there are no bridges, and external activity $n(G)$.

Proposition 3.15. *Let $G = (V, E)$ be a 2-connected loopless graph with Tutte polynomial $T(G; x, y) = \sum t_{i,j}(G) x^i y^j$. Then $t_{i,0}(G) > 0$ for $1 \leq i \leq |V| - 1$ and $t_{0,j}(G) > 0$ for $1 \leq j \leq |E| - |V| + 1$.*

Proof. See [10, ch. X.5]. □

3.4 The Tutte polynomial of a planar graph

Let $G = (V, E, F)$ be a connected plane graph, with set of faces F , and let $G^* = (V^*, E^*, F^*)$ be its geometric dual. To construct G^* , put a vertex in the interior of each face of G , and connect two such vertices of G^* by edges that correspond to common boundary edges between the corresponding faces of G . If there are several common boundary edges the result is a multiple edge of G^* .

We identify V^* with F , E^* with E , and F^* with V .

Proposition 3.16. *If G is a connected planar graph with dual G^* then $T(G^*; x, y) = T(G; y, x)$.*

PROOF. A bridge in G is a loop in G^* , a loop in G is a bridge in G^* , and deleting (contracting) an edge in G corresponds to contracting (deleting) an edge in G^* . In other words, $(G/e)^* \cong G^* \setminus e$ and $(G \setminus e)^* \cong G^*/e$. From these properties, that $T(G^*; x, y) = T(G; y, x)$ follows from the deletion-contraction recurrence for the Tutte polynomial. □

We indicate too how to derive $T(G^*; x, y) = T(G; y, x)$ by way of the spanning tree expansion for the Tutte polynomial.

For a spanning tree T of G , let T^e denote its set of externally active edges and T^i its set of internally active edges.

Proposition 3.17. *There is a bijection $T \mapsto T^*$ between spanning trees of G and spanning trees of G^* which switches internal and external activities. Specifically, $T^* = E \setminus T$, and $t_{i,j}(G^*) = t_{j,i}(G)$.*

Proof. The set of edges T^* in the dual G^* corresponding to the set of edges $E \setminus T$ in G together connect all the faces of G , since T has no cycles. (A cycle of edges would be required to separate one set of faces from another, their edges forming a simple closed curve partitioning the plane into inside and outside. If there are no cycles the plane remains in one piece.) Also, T^* does not contain a cycle, for otherwise it would separate some vertices in G inside the cycle from vertices outside, and this is impossible because T is spanning and its edges are disjoint from T^* .³

This shows that T^* is a spanning tree of G^* .

Given an edge $e \in T$ we have $\text{cut}(T, e) = \text{cyc}(T^*, e)$. Dually, given an edge $e \in E \setminus T$ we have $\text{cyc}(T, e) = \text{cut}(T^*, e)$. Consequently $T^\vee = (T^*)^\epsilon$ and $T^\epsilon = (T^*)^\vee$, from which it follows that $t_{i,j}(G^*) = t_{j,i}(G)$. \square

Corollary 3.18. *If G is a connected planar graph with dual G^* then $T(G^*; x, y) = T(G; y, x)$.*

Note that a bridge in G is a loop in G^* , a loop in G is a bridge in G^* , and that deleting (contracting) an edge in G corresponds to contracting (deleting) an edge in G^* . In other words, $(G/e)^* \cong G^* \setminus e$ and $(G \setminus e)^* \cong G^*/e$. From these properties, that $T(G^*; x, y) = T(G; y, x)$ also follows from the deletion-contraction recurrence for the Tutte polynomial.

More generally, a subgraph of G on edges $A \subseteq E$ has no cycles (i.e., is a forest) if and only if the subgraph in the dual G^* on edges $E \setminus A$ is connected. If there is a cycle in A then its edges form the boundary of a simple closed curve in the plane, inside which lies at least one vertex of G^* (corresponding to a face enclosed by the cycle) and outside of which lies another vertex of G^* . Likewise, the edges of A form a connected subgraph of G if and only if the edges of $E \setminus A$ form a forest of G^* : any cycle in G^* has to cross an edge of a connected subgraph A .

The rank and nullity functions of a planar graph and its dual are related by

$$r_{G^*}(A) = n_G(E) - n_G(E \setminus A) = |A| - r_G(E) + r_G(E \setminus A),$$

and

$$n_{G^*}(A) = r_G(E) - r_G(E \setminus A) = |A| - n_G(E) + n_G(E \setminus A).$$

Note then that $r_{G^*}(E) - r_{G^*}(A) = |E \setminus A| - r_G(E \setminus A) = n_G(E \setminus A)$.⁴ Thus

$$T(G; x, y) = \sum_{E \setminus A \subseteq E} (x - 1)^{n_{G^*}(E \setminus A)} (y - 1)^{r_{G^*}(E) - r_{G^*}(E \setminus A)} = T(G^*; y, x).$$

³Note that Euler's formula $|V| - |E| + |F| = 2$ follows from $|V(T)| = |V|$, $|V(T^*)| = |F|$, $|E(T)| + |E(T^*)| = |E|$ and $|E(T)| = |V(T)| - 1 = |V| - 1$, $|E(T^*)| = |V(T^*)| - 1 = |F| - 1$.

⁴In the terminology of the next section, an edge $e \in E \setminus A$ is independent of A in G if and only if it is a dependent edge of $E \setminus A$ in G^* . (And the dual statement holds: an edge $e \in A$ is a dependent edge of G if and only if it is an independent edge of $E \setminus A$.) The maximum number k of edges e_1, \dots, e_k such that e_i is independent of $A \cup \{e_1, \dots, e_{i-1}\}$ for each $1 \leq i \leq k$ is equal to $r_G(E) - r_G(A)$, which is therefore equal to the maximum number k of edges e_1, \dots, e_k so that e_i is dependent on $E \setminus (A \cup \{e_1, \dots, e_i\})$ for each $1 \leq i \leq k$, i.e., $n_{G^*}(A)$.

A subgraph of G on edges $A \subseteq E$ has no cycles (i.e., is a forest) if and only if the subgraph in the dual G^* on edges $E \setminus A$ is connected. If there is a cycle in A then its edges form the boundary of a simple closed curve in the plane, inside which lies at least one vertex of G^* (corresponding to a face enclosed by the cycle) and outside of which lies another vertex of G^* . Likewise, the edges of A form a connected subgraph of G if and only if the edges of $E \setminus A$ form a forest of G^* : any cycle in G^* has to cross an edge of a connected subgraph A .

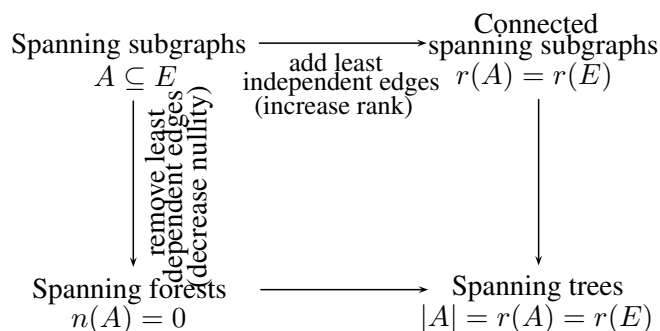
3.5 The spanning tree partition of subgraphs.

The remarks in this section rely on many facts given without proof (for which see e.g. [7, ch. 13]).

Let $G = (V, E)$ be a connected graph with a given order on its edges. For each spanning tree T of G , we have a set of externally active edges, T^e , and a set of internally active edges, T^i . The Boolean lattice of subgraphs $2^E = \{A : A \subseteq E\}$ is partitioned into Boolean intervals $[T \setminus T^i, T \cup T^e] = \{A : T \setminus T^i \subseteq A \subseteq T \cup T^e\}$ indexed by spanning trees. Given $A \subseteq E$, we have $n(A) = 0$ (i.e., $r(A) = |A|$) if and only if (V, A) is a forest, and $r(A) = r(E)$ if and only if (V, A) is a connected spanning subgraph. An edge e is *independent* of A if $r(A \cup e) = r(A) + 1$, otherwise e is dependent, and $n(A \cup e) = n(A) + 1$. Use the order on E to successively add to A the least edges $e_1, e_2, \dots, e_{r(E)-r(A)}$ that are independent of A . This creates a connected spanning subgraph $A \cup \{e_1, \dots, e_{r(E)-r(A)}\}$ containing A .

Similarly, given $A \subseteq E$, by removing edges dependent on A we decrease its nullity, and if $e_1, \dots, e_{n(A)}$ are chosen to be the least such dependent edges then we obtain a unique subgraph $A \setminus \{e_1, \dots, e_{n(A)}\}$ of nullity zero, i.e., a spanning forest of G .

If we first add least independent edges to A to make a connected spanning subgraph, and then remove least dependent edges of A we obtain a spanning tree T of G . Likewise, if we first remove the least dependent edges to make a spanning forest and then add the least independent edges we obtain (the same) spanning tree T .



This procedure locates which interval $[T \setminus T^i, T \cup T^e]$ the subset A belongs to. Call A an *internal subgraph* if we only need add independent edges to A in order to place it in its interval $[T \setminus T^i, T \cup T^e]$. In particular, (V, A) is a spanning forest and contains no externally active edges of T , i.e. $A \in [T \setminus T^i, T]$. (Note that A is internal in this sense if

and only if it contains no broken cycle: the least edge in a cycle contributes to the external activity of the tree T containing A .)

Similarly, call A an *external subgraph* if we need only remove dependent edges from A in order to place it in $[T, T \cup T^\epsilon]$. Then (V, A) is a connected spanning subgraph containing no internally active edges of T . (If A is external, then $E \setminus A$ contains no “broken cuts”.)

From the expansion $T(G; x, y) = \sum_{i,j} t_{i,j}(G) x^i y^j$ we see that $T(G; 2, 0)$ is the number of internal subgraphs (this also follows from Whitney’s Broken Cycle Theorem) and $T(G; 0, 2)$ is the number of external subgraphs. Moreover, $T(G; 1, 0)$ counts the number of internal trees, and $T(G; 0, 1)$ the number of external trees.

	General	Connected	External
General	$T(G; 2, 2) = 2^{ E }$	$T(G; 1, 2)$	$T(G; 0, 2)$
Forest	$T(G; 2, 1)$	$T(G; 1, 1)$	$T(G; 0, 1)$
Internal	$T(G; 2, 0)$	$T(G; 1, 0)$	$T(G; 0, 0) = 0$

(We have already seen that $T(G; 2, 0)$ counts acyclic orientations, and for a connected graph $T(G; 1, 0)$ counts acyclic orientations with unique prescribed source. See e.g. [6, Fig. 20] for an interpretation of $T(G; x, y)$ for other values of $x, y \in \{0, 1, 2\}$ in terms of orientations of G . In fact, Las Vergnas [53] gives an interpretation for $2^{i+j} t_{i,j}(G)$ in terms of orientations of G and an order on E , quoted as Theorem 25 in [23].)

Given the spanning tree partition $2^E = \bigcup_T [T \setminus T^\nu, T \cup T^\epsilon]$ of all subgraphs of G , the subgraph expansion of the Tutte polynomial may be rewritten as follows:

$$\begin{aligned}
T(G; x, y) &= \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{n(A)} \\
&= \sum_T \sum_{A \in [T \setminus T^\nu, T \cup T^\epsilon]} (x-1)^{|A \cap T^\nu|} (y-1)^{|A \cap T^\epsilon|} \\
&= \sum_T \sum_{k, \ell} \binom{|T^\nu|}{k} (x-1)^k \binom{|T^\epsilon|}{\ell} (y-1)^\ell \\
&= \sum_T x^{|T^\nu|} y^{|T^\epsilon|},
\end{aligned}$$

which gives Tutte’s spanning tree expansion by internal and external activities.

3.6 The beta invariant.

The coefficient $t_{1,0}(G)$ is known as Crapo’s *beta invariant*, or also the *chromatic invariant*, with $t_{1,0}(G) = (-1)^{|V(G)|} P'(G; 1)$.

We know from the corresponding property of the chromatic polynomial that the beta invariant is unaffected by the addition or removal of parallel edges. A direct proof can be given by a deletion/contraction of a parallel edge, noting $t_{1,0}(G) = 0$ if G has a loop.

By Propositions 3.17 and 3.14, $t_{1,0}(G) = t_{1,0}(G^*)$ when G is a connected planar graph.

Two graphs are *homeomorphic* if they can both be obtained from the same graph by subdividing its edges (inserting vertices of degree 2).

Proposition 3.19. *If G and G' are homeomorphic connected graphs with at least two edges then $t_{1,0}(G) = t_{1,0}(G')$.*

NOTE. The condition on the number of edges is necessary: $t_{1,0}(K_2) = 1$ but for any path P_n on $n \geq 3$ vertices, which is homeomorphic to K_2 , we have $t_{1,0}(P_n) = 0$.

Proof. Homeomorphic graphs have each some subdivision that makes them isomorphic. Hence it suffices to prove that if G' is obtained from G by subdividing an edge e into two edges e_1 and e_2 then $t_{1,0}(G) = t_{1,0}(G')$.

If e is a bridge of G then since G has another edge it is not 2-connected, so $t_{1,0}(G) = t_{1,0}(G') = 0$.

If e is not a bridge of G then e_1 is neither a bridge nor a loop of G' , so $t_{1,0}(G') = t_{1,0}(G'/e_1) + t_{1,0}(G' \setminus e_1)$. As e_2 is a block of $G' \setminus e_1$ and there is another edge of G' we have $t_{1,0}(G' \setminus e_1) = 0$. Since $G'/e_1 \cong G$ this yields the desired result that $t_{1,0}(G') = t_{1,0}(G)$. \square

Definition 3.20. *A series-parallel graph is a graph constructed from C_2 (two vertices joined by two parallel edges) by a sequence of the following two operations:*

- (i) *subdividing an edge (introducing a vertex of degree 2),*
- (ii) *placing an edge parallel to an existing edge.*

Series-parallel graphs are 2-connected, have no loops, and are planar.

Theorem 3.21. *Let G be a 2-connected graph with at least one edge. Then $t_{1,0}(G) \geq 1$ with equality if and only if G is series-parallel.*

Proof. If G is not 2-connected then $t_{1,0}(G) = 0$.

We prove the statement by induction on the number of edges. The base case C_2 has $T(C_2; x, y) = x + y$.

Suppose G is 2-connected with $m \geq 3$ edges and assume the truth of the assertion for 2-connected graphs with less than m edges. If G has an edge e that has been introduced in series (one of its endpoints has degree 2), then G/e is 2-connected while $G \setminus e$ is not. Hence $t_{1,0}(G \setminus e) = 0$ while by inductive hypothesis $t_{1,0}(G/e) = 1$.

On the other hand, if e is parallel to another edge of G then G/e has a loop and at least one other edge and hence is not 2-connected, while $G \setminus e$ is 2-connected. By inductive hypothesis we have $t_{1,0}(G \setminus e) = 1$, so that $t_{1,0}(G) = 0 + t_{1,0}(G \setminus e) = 1$.

For the converse we use the fact that a 2-connected graph G is series-parallel if and only if it contains no K_4 minor (Dirac, 1952), and that $t_{i,j}(H) \leq t_{i,j}(G)$ whenever H is a minor of a 2-connected graph G (Brylawski, [14, Corollary 6.9]). It follows in particular that $t_{1,0}(K_4) = 2 \leq t_{1,0}(G)$ whenever a 2-connected graph G is not series-parallel. \square

Exercise 3.22. Let W_n be the wheel on $n + 1$ vertices (an n -cycle all of whose vertices are joined to a new central vertex). By first calculating the chromatic polynomial of W_n , find $t_{1,0}(W_n)$.

By using $P(K_n; z) = z^n$, show that $t_{1,0}(K_n) = (n - 2)!$.

Proposition 3.23. If $G = G_1 \cup G_2$ where $|V(G_1) \cap V(G_2)| = s \geq 2$ and the induced subgraph on $V(G_1) \cap V(G_2)$ is a clique K_s , then

$$t_{1,0}(G) = t_{1,0}(G_1)t_{1,0}(G_2)/(s - 2)!.$$

Note that if G has a 1-separation then it is not 2-connected and $t_{1,0}(G) = 0$.

Proof. This follows from the expression for the chromatic polynomial of a quasi-separation given in Proposition 1.8, written as

$$P(G; 1 - z)P(K_s; 1 - z) = P(G_1; 1 - z)P(G_2; 1 - z),$$

where, for connected G ,

$$P(G; 1 - z) = (-1)^{|V|-1}(1 - z) \sum_{1 \leq i \leq |V|-1} t_{i,0}(G)z^i,$$

and the fact that $t_{1,0}(K_s) = (s - 2)!$. Comparing coefficients of z^2 gives the result. \square

In particular, edge-gluing a series-parallel graph to G does not change its beta invariant.

The only 3-connected graph G with beta invariant $t_{1,0}(G) = 2$ is K_4 , and a similar classification of 3-connected graphs with beta invariant up to 9 has been made (see references given in [23, §7.1]). An *outerplanar* graph is a planar graph with an embedding in the plane with the property that all vertices of G lie on the outer face. A graph is outerplanar if and only if it has no K_4 minor (so it is series-parallel) or $K_{2,3}$ minor.

Theorem 3.24. [33] *If G is a simple 2-connected series-parallel graph then $t_{2,0}(G) \geq t_{0,2}(G) + 1$ with equality if and only if G is outerplanar.*

It turns out that the beta invariant $t_{1,0}(G)$ counts a certain subset of those acyclic orientations counted by $T(G; 1, 0)$ (Theorem 3.8 above).

Theorem 3.25. [Greene and Zaslavsky, 1983; Las Vergnas, 1984]⁵ *Let G be a connected graph and $uv \in E(G)$. The number of acyclic orientations of G with u as unique source and v as unique sink is equal to $t_{1,0}(G)$.*

⁵The original proofs of Greene and Zaslavsky of this result and Theorem 3.8 use hyperplane arrangements. A contraction–deletion proof was given by Gebhard and Sagan [30]. Las Vergnas proved a stronger theorem in [53], giving an orientation expansion for the Tutte polynomial.

Proof. Let $Q_{uv}(G)$ denote the number of acyclic orientations of G with u as unique source and v as unique sink.

Recall that $t_{1,0}(G) = 0$ if G is not 2-connected. We know that $t_{1,0}(G) = t_{1,0}(G/e) + t_{1,0}(G \setminus e)$ for an ordinary edge e , and if G has more than one edge and e is a bridge of G then $t_{1,0}(G) = 0$ (since G is not 2-connected). Also $t_{1,0}(K_2) = 1$. Finally, $t_{1,0}(G) = 0$ if G has a loop e .

When G is not 2-connected it is impossible to have an acyclic orientation of G with unique source u and unique sink v . First, if G is not connected then there are not even any acyclic orientations with unique source u , since each component has a source. Second, if G is connected with 1-separation $G_1 \cup G_2$ having $|V(G_1) \cap V(G_2)| = 1$, then an acyclic orientation restricted to G_1 has at least one source and sink, at least one of which survives as a source or sink in G . Similarly for G_2 . But then there is either a source or sink in G_1 and in G_2 , and these are not connected by an edge. Hence u and v are not unique as source and sink.

Clearly $Q_{uv}(K_2) = 1$ and $Q_{uv}(G) = 0$ if G has a loop.

If G has at least two edges, is 2-connected and has no loops, then G has no bridges. It remains to prove that in this case $Q_{uv}(G) = Q_{uv}(G/e) + Q_{uv}(G \setminus e)$, where e is an ordinary edge. We can choose $e = uv$ with $w \neq u, v$. In an acyclic orientation of G with unique sink v the edge uv is directed from w to v . Since u is the unique source there is at least one edge directed into w . If there is also at least one other edge directed out of w , then deleting e gives an acyclic orientation of $G \setminus e$ with unique source u and unique sink v . On the other hand, if e is the only edge directed out of w then contracting the edge e gives an acyclic orientation of G/e with unique source u and unique sink v (which is identified with w in the graph G/e). Thus partitioning acyclic orientations of G with unique source u and unique sink v according to whether or not $G \setminus uv$ is also an acyclic orientation with this property, we find that $Q_{uv}(G) = Q_{uv}(G/uv) + Q_{uv}(G \setminus uv)$. \square

3.7 Computational complexity

The Tutte polynomial can be computed in polynomial time at some particular points. Specifically, these points are: $(0, 0)$ (whether there are any edges), $(1, 1)$ (number of spanning trees – see Section 3.8 below), $(2, 2)$ (number of subgraphs), $(-1, 0)$ (whether bipartite or not), $(0, -1)$ (whether Eulerian or not), $(-1, -1)$ (up to easily determined sign equal to number of bicycles), and also in the last section interpretations for evaluations at $(e^{2\pi i/3}, e^{-2\pi i/3})$ and $(i, -i)$, the former involving the dimension the space spanned by vectors that are simultaneously \mathbb{Z}_3 -flows and \mathbb{Z}_3 -tensions.

Recall also that $T(G; x, y) = (x - 1)^{r(G)} y^{|E(G)|}$ when $(x - 1)(y - 1) = 1$, so that the Tutte polynomial is also polynomial time computable at points on this hyperbola (the points $(0, 0)$ and $(2, 2)$ were already mentioned in the previous paragraph).

Theorem 3.26 below says that we have in fact now encountered all such “easy points”.

A computational (enumeration) problem can be regarded as a function mapping inputs to solutions (graphs to the number of their proper vertex 3-colourings, for example). A problem is *polynomial time computable* if there is an algorithm which computes the output

in length of time (number of steps) bounded by a polynomial in the size of the problem instance. The class of such problems is denoted by P . If A and B are two problems, we say that A is *polynomial time reducible* to B , written $A \propto B$, if it is possible with the help of a subroutine for problem B to solve problem A in polynomial time.

The class $\#\mathsf{P}$ can be roughly described as the class of all enumeration problems in which the structures being counted can be recognized in polynomial time (i.e., instances of an NP problem). For example, counting Hamiltonian paths in a graph is in $\#\mathsf{P}$ because it is easy to check whether a given set of edges is a Hamiltonian path.

The class $\#\mathsf{P}$ has a class of “hardest” problems called the $\#\mathsf{P}$ -complete problems. A problem A belonging to $\#\mathsf{P}$ is $\#\mathsf{P}$ -complete if for any other problem B in $\#\mathsf{P}$ we have $B \propto A$. A prototypical example of a $\#\mathsf{P}$ -complete problem is $\#\mathsf{SAT}$, the problem of counting the number of satisfying assignments of a Boolean function. Many of the thousands of problems known to be $\#\mathsf{P}$ complete have been shown to be so by reduction to $\#\mathsf{SAT}$. Counting Hamiltonian paths is an example of a $\#\mathsf{P}$ -complete problem (even when restricted to planar graphs with maximum degree 3).

A problem is $\#\mathsf{P}$ -hard if any problem in $\#\mathsf{P}$ is polynomial time reducible to it. In other words, A is $\#\mathsf{P}$ -hard if the existence of a polynomial time algorithm for A would imply the existence of a polynomial time algorithm for any problem in $\#\mathsf{P}$. (A $\#\mathsf{P}$ -hard problem is $\#\mathsf{P}$ -complete if it belongs to the class $\#\mathsf{P}$ itself.)

We have found that many evaluations of the Tutte polynomial count structures associated with a graph. Sometimes though it is not apparent what an evaluation of the Tutte polynomial at a particular point (a, b) might count. However, we can still speak of whether the problem of computing $T(G; a, b)$ can be done in polynomial time or if it is a $\#\mathsf{P}$ -hard problem (being able to evaluate it for any graph in polynomial time would imply that every problem in $\#\mathsf{P}$ could be computed in polynomial time).

Theorem 3.26 ([46]). *Evaluating the Tutte polynomial of a graph at a particular point of the complex plane is $\#\mathsf{P}$ -hard except when either*

(i) *the point lies on the hyperbola $(x - 1)(y - 1) = 1$,*

(ii) *the point is one of the special points $(1, 1)$, $(-1, 0)$, $(0, -1)$, $(-1, -1)$, $(i, -i)$, $(-i, i)$, $(e^{2\pi i/3}, e^{-2\pi i/3})$, $(e^{-2\pi i/3}, e^{2\pi i/3})$.*

In the special cases (i) and (ii) evaluation can be carried out in polynomial time.

In [85] Vertigan and Welsh show that the same statement in Theorem 3.26 holds even when restricting the problem to computing the Tutte polynomial for bipartite graphs.

Around the same time as [85], but only much later published, Vertigan showed that restricting the problem of evaluating the Tutte polynomial to planar graphs only yields extra “easy points” on the hyperbola $(x - 1)(y - 1) = 2$ (corresponding to the partition function of the Ising model, which in the planar case is polynomial time computable due to Kasteleyn’s expression for the partition function of the Ising model as the Pfaffian of an associated matrix).

Theorem 3.27 ([87]). *The problem of computing the Tutte polynomial of a planar graph at a particular point of the complex plane is #P-hard except when either*

- (i) *the point lies on one the hyperbolae $(x - 1)(y - 1) = 1$ or $(x - 1)(y - 1) = 2$,*
- (ii) *the point is one of the special points $(1, 1)$, $(-1, 0)$, $(0, -1)$, $(-1, -1)$, $(e^{2\pi i/3}, e^{-2\pi i/3})$, $(e^{-2\pi i/3}, e^{2\pi i/3})$.*

In the special cases (i) and (ii) evaluation can be carried out in polynomial time.

See e.g. [90] for a more detailed account of the complexity of counting problems, with special emphasis on those related to the Tutte polynomial.

3.8 The Laplacian and the number of spanning trees

Proposition 3.28. *Let D be the incidence matrix (with respect to some orientation) of a graph G , and let A be the adjacency matrix of G (whose (u, v) -entry is the number of edges joining u to v). Then*

$$Q = DD^\top = \Delta - A,$$

where Δ is the diagonal matrix whose (v, v) -entry is the degree of the vertex v (a loop on v contributing 2 to its degree). Consequently, Q is independent of the orientation given to G .

The matrix Q is called the *Laplacian matrix* of G .

Let $Q[u]$ denote the matrix obtained by deleting the row and column indexed by u , and $Q[u, v]$ the matrix obtained by further deleting the row and column indexed by v .

Write $Q = Q(G)$ when D is the incidence matrix of G . Note that if e is a loop then $Q(G) = Q(G \setminus e)$, since the column of the incidence matrix D of G indexed by e is zero and contributes nothing to DD^\top .

Theorem 3.29. *Let G be a connected graph with Laplacian matrix Q . If u is an arbitrary vertex of G then $\det Q[u]$ is equal to the number of spanning trees of G .*

Proof. We show that $Q(G)[u]$ satisfies the same deletion–contraction recurrence as the number of spanning trees of G , $\tau(G)$, which satisfies the deletion–contraction recurrence $\tau(G) = \tau(G \setminus e) + \tau(G/e)$ when e is not a loop, and $\tau(G) = \tau(G \setminus e)$ when e is a loop. (Note that when e is a bridge, $\tau(G) = \tau(G/e)$ because $G \setminus e$ is disconnected so that $\tau(G \setminus e) = 0$.)

When e is a loop on u , $Q(G)[u] = Q(G \setminus e)[u]$.

Choose an ordinary edge $e = uv$, and let R be the $V \times V$ diagonal matrix with $R_{v,v} = 1$, and all other entries equal to 0. Then

$$Q(G)[u] = Q(G \setminus e)[u] + R,$$

from which

$$\det Q[u] = \det Q(G \setminus e)[u] + \det Q(G \setminus e)[u, v]. \quad (15)$$

Note that $Q(G \setminus e)[u, v] = Q[u, v]$. Assume in forming G/e we contract u onto v , so that $V(G/e) = V \setminus \{u\}$. Then $Q(G/e)[v]$ has rows and columns indexed by $V \setminus \{u, v\}$ with (x, y) -entry equal to $Q_{x,y}$, and so we also have that $Q(G/e)[v] = Q[u, v]$. Thus we can rewrite (15) as

$$\det Q[u] = \det Q(G \setminus e)[u] + \det Q(G/e)[v].$$

By induction $\det Q(G \setminus e)[u] = \tau(G \setminus e)$ and $\det Q(G/e)[v] = \tau(G/e)$. By the recurrence for $\tau(G)$ the result follows. □

Since $\tau(G) = T(G; 1, 1)$ when G is connected, and the Tutte polynomial of an arbitrary graph is multiplicative over its connected components, Theorem 3.29 provides a polynomial-time algorithm for computing $T(G; 1, 1)$.

Other points (x, y) at which we already know that $T(G; x, y)$ can be computed in polynomial time in the size of G include points on the hyperbola $\{(x, y) : (x-1)(y-1) = 1\}$, where $T(G; x, y) = (x-1)^{r(E)}y^{|E|}$, and the point $(-1, 0)$ (since the number of proper 2-colourings amounts to testing for bipartiteness). We shall shortly see that $T(G; 0, -1)$ is computable in polynomial time (what does it count?), and later that $T(G; a, \bar{a})$ is also polynomial-time computable when a is a second, third or fourth root of unity, and \bar{a} its conjugate.

3.9 Hamming weight enumerator for tensions and flows

Let $G = (V, E)$ be a graph, A a commutative ring of k elements, and \mathcal{Z} the module of A -flows of G and its orthogonal complement \mathcal{Z}^\perp the module of A -tensions of G .

The monochrome polynomial $B(G; k, y)$ of G was defined in Proposition 1.15 in terms of vertex k -colourings, but we can write it in terms of A -tensions as follows:

$$k^{-c(G)} B(G; k, y) = \sum_{\mathbf{z} \in \mathcal{Z}^\perp} y^{|E| - |\text{supp}(\mathbf{z})|}. \quad (16)$$

In coding theory $|\text{supp}(\mathbf{z})|$ is called the *Hamming weight* of the vector \mathbf{z} and the polynomial on the right-hand side of (16) is known as the (Hamming) *weight enumerator* of the code \mathcal{Z}^\perp .

By deletion-contraction and the Recipe Theorem we have seen that

$$B(G; k, y) = k^{c(G)} (y-1)^{r(G)} T(G; \frac{y-1+k}{y-1}, y). \quad (17)$$

A code over a field \mathbb{F} is a special type of matroid, namely one that is representable over \mathbb{F} . The point $(\frac{y-1+k}{y-1}, y)$ lies on the hyperbola $(x-1)(y-1) = k$. Greene [34] was first to make the connection between the Tutte polynomial and linear codes over a field of k elements, proving that the Tutte polynomial of the matroid of a code specializes on the hyperbola $(x-1)(y-1) = k$ to the weight enumerator of the code (effectively, identity (17) generalized to codes/representable matroids).

The dual version of the monochrome polynomial (the weight enumerator for A -tensions (16)) is the weight enumerator for A -flows:

$$C(G; k, x) = \sum_{\mathbf{z} \in \mathcal{Z}} x^{|\mathcal{E}| - |\text{supp}(\mathbf{z})|} = (x - 1)^{n(G)} T(G; x, \frac{x - 1 + k}{x - 1}). \quad (18)$$

(This identity can be proved by an inductive deletion-contraction argument, as for the monochrome polynomial.) Thus by identities (17) and (18) we have

$$B(G; k, y) = k^{|V(G)| - |E(G)|} (y - 1)^{|E(G)|} C(G; k, \frac{y - 1 + k}{y - 1}), \quad (19)$$

which amounts to MacWilliams identity in coding theory.

3.10 Bicycles

We describe combinatorial interpretations for the special points with complex coordinates given in Theorem 3.26 at which the Tutte polynomial is easy to compute. To do this we return to A -flows and A -tensions, for $A = \mathbb{Z}_2$ and, in the next section, $A = \mathbb{Z}_3$. Our notation differs slightly from earlier, in that we use conventional boldface vector notation rather than Greek lettered maps for tensions and flows.

The incidence mapping $D : \mathbb{Z}_2^E \rightarrow \mathbb{Z}_2^V$ for a graph $G = (V, E)$ has kernel $\ker D = \mathcal{Z}$ and $\mathcal{Z}^\perp = \text{im } D^\top = \mathcal{K}$. Vectors in \mathcal{Z} are indicator vectors of *Eulerian subgraphs* of G (sometimes just called cycles – although we shall reserve the term cycle for a connected 2-regular subgraph – or *even subgraphs* of G). Vectors in $\mathcal{Z}^\top = \mathcal{K}$ are indicator vectors of (edge) *cuts* of G . (A 2-tension has support a cut, equal for some $V_0 \subseteq V, V_1 = V \setminus V_0$ to the set of edges with one endpoint in V_0 and the other in V_1 .)

An Eulerian subgraph meets a cut in an even number of edges (by orthogonality of flows and tensions, and by definition when considering cuts comprising edges from $\{v\}$ to $V \setminus \{v\}$, these vertex-cuts together spanning all cuts).

We identify a subset of edges of G with its indicator vector.

A vector \mathbf{x} in the intersection $\mathcal{Z} \cap \mathcal{K}$ is called a *bicycle* of G , and is self-orthogonal, i.e., $\mathbf{x}^\top \mathbf{x} = 0$. So a bicycle has an even number of edges.

A bicycle is an Eulerian subgraph that meets every other Eulerian subgraph in an even number of edges (as well as every cut in an even number of edges). Alternatively, a bicycle is a cut that meets every other cut in an even number of edges (as well as meeting every Eulerian subgraph in an even number of edges).

In short, a bicycle is a cutset that is also an Eulerian subgraph of G . In particular, if G is itself a bipartite Eulerian graph then E (the all-one vector) is a bicycle.

For more about bicycles see Sections 14.15-16 and 15.7 in [32] (from which the material in this section is adapted), and for the usefulness of bicycles in relation to knots see Chapter 17 of the same reference.

Theorem 3.30. *Let e be the edge of a graph G . Then precisely one of the following holds:*

- (i) e belongs to a bicycle,
- (ii) e belongs to a cut B such that $B \setminus \{e\}$ is Eulerian,
- (iii) e belongs to an Eulerian subgraph C such that $C \setminus \{e\}$ is a cut.

Proof. Suppose $e \in E(G)$ and \mathbf{e} is its indicator vector in \mathbb{Z}_2^E . If e belongs to a bicycle with indicator vector \mathbf{x} then $\mathbf{x}^\top \mathbf{e} \neq 0$ and therefore $e \notin (\mathcal{Z} \cap \mathcal{Z}^\perp)^\perp = \mathcal{Z}^\perp + \mathcal{Z}$. If e does not belong to a bicycle then \mathbf{e} is orthogonal to all vectors in $\mathcal{Z} \cap \mathcal{Z}^\perp$ and so $\mathbf{e} \in \mathcal{Z} + \mathcal{Z}^\perp$. In other words, e is either contained in a bicycle or e is the symmetric difference of an Eulerian subgraph and a cutset.

In any representation of e as the symmetric difference of an Eulerian subgraph and a cut, either e will always belong to the Eulerian subgraph, or e will always belong to the cut. For suppose that $\mathbf{e} = \mathbf{z} + \mathbf{y} = \mathbf{z}' + \mathbf{y}'$ where $\mathbf{z}, \mathbf{z}' \in \mathcal{Z}$ and $\mathbf{y}, \mathbf{y}' \in \mathcal{Z}^\perp$. Then $\mathbf{z} + \mathbf{z}' \in \mathcal{Z}$ and $\mathbf{y} + \mathbf{y}' \in \mathcal{Z}^\perp$ so $\mathbf{z} + \mathbf{z}' = \mathbf{y} + \mathbf{y}'$ is a bicycle. Since e does not belong to a bicycle, it must belong to both or neither of \mathbf{z} and \mathbf{z}' , and to neither or both of \mathbf{y} and \mathbf{y}' , respectively (since $\mathbf{e} = \mathbf{z} + \mathbf{y}$). \square

An edge e of G is of *bicycle-type*, *cut-type* or *flow-type* according as (i), (ii) or (iii) holds in the statement of Theorem 3.30, respectively. This is known as the principal tripartition of the edges of G .

A bridge is an edge of cut-type [take cut $B = \{e\}$ in (ii)] and a loop is an edge of flow-type [take Eulerian subgraph $\{e\}$ in (iii)].

If G is planar then edges of bicycle-type in G remain of bicycle-type in G^* . By flow-tension duality, edges of cut-type in G are edges of flow-type in G^* , and similarly edges of flow-type in G^* are edges of cut-type in G^* .

See [32, Theorem 14.16.2] for a simple polynomial-time algorithm, involving the Laplacian matrix DD^\top , to decide what type an edge has in the principal tripartition.

Lemma 3.31. *Let G be a graph with bicycle space of dimension d , and e an edge of G . The following table gives the dimension of the bicycle space of G/e and $G \setminus e$.*

Type of e	G/e	$G \setminus e$
Bridge or loop	d	d
Bicycle-type	$d - 1$	$d - 1$
Cut-type, not bridge	d	$d + 1$
Flow-type, not loop	$d + 1$	d

Proof. A bridge belongs to no cycle and hence to no Eulerian subgraph, and therefore to no bicycle. So any bicycle of G is a bicycle of $G \setminus e$. Conversely, a bicycle of $G \setminus e$ is also a bicycle of G . Likewise, bicycles of G/e correspond to bicycles of G .

Similarly, a loop belongs to no cut and hence to no bicycle, so bicycles of G are bicycles of $G \setminus e$, and conversely. For a loop we have $G/e \cong G \setminus e$.

For an ordinary edge e we shall find the following two observations useful:

- (i) If e is not a loop and belongs an Eulerian subgraph C , then $C \setminus \{e\}$ is neither an Eulerian subgraph of G nor of $G \setminus e$. On the other hand, $C \setminus \{e\}$ is an Eulerian subgraph of G/e .
- (ii) Dually, if e is not a bridge and belongs to a cut B , then $B \setminus \{e\}$ is neither a cut of G nor of G/e . On the other hand, $B \setminus \{e\}$ is a cut of $G \setminus e$.

Suppose then that e is an ordinary edge. We distinguish the three cases of the principal tripartition:

- (a) e belongs to a bicycle A .

By (i) and (ii), $A \setminus \{e\}$ is not a bicycle of G , $G \setminus e$ or G/e . On the other hand, any bicycle of G which does not contain e remains a bicycle of $G \setminus e$ and G/e . Hence the bicycle spaces of $G \setminus e$ and of G/e both correspond to the subspace of bicycles of G that do not contain e , and their dimensions are therefore 1 less than the bicycle dimension of G .

- (b) e belongs to a cut B , such that $B \setminus \{e\}$ is an Eulerian subgraph of G .

By (ii), the set $B \setminus \{e\}$ is a cut of $G \setminus e$, but not of G or G/e . Hence $B \setminus \{e\}$ is a bicycle of $G \setminus e$, but not of G or G/e . The effect is to increase the dimension of the bicycle space of $G \setminus e$ by 1. All bicycles of G are bicycles of $G \setminus e$ since e is of cut-type, and so bicycles of $G \setminus e$ are bicycles of G together with symmetric difference of bicycles of G with the fixed set $B \setminus \{e\}$. On the other hand, the dimension of the bicycle space of G/e coincides with that of G , all bicycles of G being bicycles of G/e , and no others.

- (c) e belongs to an Eulerian subgraph C such that $C \setminus \{e\}$ is a cut.

By (i), the set $C \setminus \{e\}$ is an Eulerian subgraph of G/e , but not of G or $G \setminus e$. Hence $C \setminus \{e\}$ is a bicycle of G/e , but not of G or $G \setminus e$. Similarly to case (b), this implies the dimension of the bicycle space of G/e is 1 more than that of G , while $G \setminus e$ has the same bicycle dimension as G .

□

Lemma 3.32. *Let $G = (V, E)$ be a graph with bicycle space of dimension $b(G)$, and let e be an edge of G . Then the graph invariant*

$$f(G) = (-1)^{|E|} (-2)^{b(G)}$$

satisfies

$$f(G) = \begin{cases} (-1)f(G/e) & e \text{ a bridge,} \\ (-1)f(G \setminus e) & e \text{ a loop,} \\ f(G/e) + f(G \setminus e) & e \text{ ordinary.} \end{cases}$$

Proof. We use Lemma 3.31.

If e is a bridge or loop then the bicycle spaces of G/e , $G \setminus e$ and G are all of the same dimension, and this implies the first two cases.

Suppose e is ordinary. If e is of cut-type then

$$\begin{aligned} f(G/e) + f(G \setminus e) &= (-1)^{|E|-1}(-2)^{b(G)} + (-1)^{|E|-1}(-2)^{b(G)+1} \\ &= (-1)^{|E|}(-2)^{b(G)}. \end{aligned}$$

If e is of flow-type then

$$\begin{aligned} f(G/e) + f(G \setminus e) &= (-1)^{|E|-1}(-2)^{b(G)+1} + (-1)^{|E|-1}(-2)^{b(G)} \\ &= (-1)^{|E|}(-2)^{b(G)}. \end{aligned}$$

If e belongs to a bicycle then

$$f(G/e) + f(G \setminus e) = 2(-1)^{|E|-1}(-2)^{b(G)-1} = (-1)^{|E|}(-2)^{b(G)}.$$

□

By the Recipe Theorem (Theorem 3.6) we obtain:

Theorem 3.33 ([71]). *Let $G = (V, E)$ be a graph and let $b(G)$ denote the dimension of its bicycle space. Then $(-1)^{|E|}(-2)^{b(G)} = T(G; -1, -1)$.*

Corollary 3.34. *A connected graph G has no non-trivial bicycles if and only if G has an odd number of spanning trees.*

Proof. We have $T(G; -1, -1) \equiv T(G; 1, 1) \pmod{2}$. □

3.11 \mathbb{Z}_3 -tension-flows

In this section we take $A = \mathbb{Z}_3$ and consider the intersection of the space of \mathbb{Z}_3 -flows and the space of \mathbb{Z}_3 -tensions. If $D : \mathbb{Z}_3^E \rightarrow \mathbb{Z}_3^V$ is the incidence mapping, and we let $\mathcal{Z} = \ker D$, so that $\mathcal{Z}^\perp = \text{im } D^\top$, then we shall call a vector in $\mathcal{Z} \cap \mathcal{Z}^\perp$ a \mathbb{Z}_3 -tension-flow. In other words, a \mathbb{Z}_3 -tension-flow is both a \mathbb{Z}_3 -tension and a \mathbb{Z}_3 -flow, and is self-orthogonal in \mathbb{Z}_3^E . (In this terminology we could have called bicycles \mathbb{Z}_2 -tension-flows.)

Let $\omega = e^{2\pi i/3}$ be a primitive cube root of unity. In [45] Jaeger proved by a deletion-contraction argument that $T(G; \omega, \omega^2) = \pm \omega^{|E| + \dim \mathcal{Z}} (i\sqrt{3})^{\dim(\mathcal{Z} \cap \mathcal{Z}^\perp)}$, using the *principal quadripartition* of the edges of a graph (a generalization to flows and tensions over finite fields of characteristic $\neq 2$ of the principal tripartition). Gioan and Las Vergnas [31] provide a linear algebra proof that has the benefit of determining the sign. It is this latter proof that we shall present here.

Recall that we say vectors \mathbf{y} and \mathbf{z} are orthogonal if $\mathbf{y}^\top \mathbf{z} = 0$. A *self-orthogonal* vector (also called an *isotropic* vector) is a vector \mathbf{z} with $\mathbf{z}^\top \mathbf{z} = 0$.

Lemma 3.35. *Let \mathcal{Z} be a finite-dimensional vector space over a field of characteristic not equal to 2. Then \mathcal{Z} has an orthogonal basis.*

Proof. Let $\{\mathbf{z}_1, \dots, \mathbf{z}_d\}$ be a basis for \mathcal{Z} . If there is an index $1 \leq i \leq d$ such that \mathbf{z}_i is not self-orthogonal then reindex in such a way that $i = 1$ and set $\mathbf{z}'_1 = \mathbf{z}_1$. Otherwise, if there is an index $2 \leq i \leq d$ such that $\mathbf{z}_1 + \mathbf{z}_i$ is not self-orthogonal then set $\mathbf{z}'_1 = \mathbf{z}_1 + \mathbf{z}_i$. In both cases update \mathbf{z}_j as $\mathbf{z}_j - \frac{\mathbf{z}'_1{}^\top \mathbf{z}_j}{\mathbf{z}'_1{}^\top \mathbf{z}'_1} \mathbf{z}'_1$ for $2 \leq j \leq d$. Now \mathbf{z}'_1 and \mathbf{z}_j are orthogonal for $2 \leq j \leq d$.

Otherwise the vectors \mathbf{z}_j are self-orthogonal for $1 \leq j \leq d$, and $\mathbf{z}_1 + \mathbf{z}_j$ is self-orthogonal for $2 \leq j \leq d$. The latter implies $\mathbf{z}'_1{}^\top \mathbf{z}_1 + 2\mathbf{z}_1{}^\top \mathbf{z}_j + \mathbf{z}_j{}^\top \mathbf{z}_j = 2\mathbf{z}'_1{}^\top \mathbf{z}_j = 0$. Hence $\mathbf{z}'_1{}^\top \mathbf{z}_j = 0$ in characteristic $\neq 2$. Set $\mathbf{z}'_1 = \mathbf{z}_1$.

In all three cases $\mathbf{z}'_1, \mathbf{z}_2, \dots, \mathbf{z}_d$ comprise a basis of \mathcal{Z} such that \mathbf{z}'_1 is orthogonal to the space generated by the remaining vectors $\mathbf{z}_2, \dots, \mathbf{z}_d$.

The result now follows by induction. \square

Lemma 3.36. *The self-orthogonal vectors of an orthogonal basis of \mathcal{Z} form a basis for $\mathcal{Z} \cap \mathcal{Z}^\perp$.*

Proof. Let $\mathbf{z}_1, \dots, \mathbf{z}_d$ form an orthogonal basis for \mathcal{Z} , and $\mathbf{z} = \sum_{1 \leq j \leq d} a_j \mathbf{z}_j \in \mathcal{Z} \cap \mathcal{Z}^\perp$. For $1 \leq i \leq d$ we have $0 = \mathbf{z}^\top \mathbf{z}_i = \sum_{1 \leq j \leq d} a_j \mathbf{z}_j{}^\top \mathbf{z}_i = a_i \mathbf{z}_i{}^\top \mathbf{z}_i$. Hence if $\mathbf{z}_i{}^\top \mathbf{z}_i \neq 0$ then $a_i = 0$. It follows that \mathbf{z} is generated by the self-orthogonal vectors of the basis, which, being independent, therefore form a basis of $\mathcal{Z} \cap \mathcal{Z}^\perp$. \square

Proposition 3.37. *Let \mathcal{Z} be a subspace of \mathbb{Z}_3^E . Then*

$$\sum_{\mathbf{z} \in \mathcal{Z}} \omega^{|\text{supp}(\mathbf{z})|} = (-1)^{d+d_1} (i\sqrt{3})^{d+d_0},$$

where $d = \dim \mathcal{Z}$, $d_0 = \dim(\mathcal{Z} \cap \mathcal{Z}^\perp)$, and d_1 is the number of basis vectors of support size congruent to 1 modulo 3 in any orthogonal basis of \mathcal{Z} .

Proof. Observe that for $\mathbf{z} \in \mathbb{Z}_3^E$ we have $|\text{supp}(\mathbf{z})| \equiv \mathbf{z}^\top \mathbf{z} \pmod{3}$. It follows that $\omega^{|\text{supp}(\mathbf{z})|} = \omega^{\mathbf{z}^\top \mathbf{z}}$.

By Lemma 3.35 there is an orthogonal basis $\{\mathbf{z}_1, \dots, \mathbf{z}_d\}$ of \mathcal{Z} . In particular, the inner product of $\mathbf{z} = \sum_{1 \leq j \leq d} a_j \mathbf{z}_j$ with itself is equal to $\sum_{1 \leq j \leq d} a_j^2 \mathbf{z}_j{}^\top \mathbf{z}_j$. So we find that

$$\begin{aligned} \sum_{\mathbf{z} \in \mathcal{Z}} \omega^{\mathbf{z}^\top \mathbf{z}} &= \sum_{(a_1, \dots, a_d) \in \mathbb{Z}_3^d} \omega^{\sum_{1 \leq j \leq d} a_j^2 \mathbf{z}_j{}^\top \mathbf{z}_j} \\ &= \sum_{(a_1, \dots, a_d) \in \mathbb{Z}_3^d} \prod_{1 \leq j \leq d} \omega^{a_j^2 \mathbf{z}_j{}^\top \mathbf{z}_j} \\ &= \prod_{1 \leq j \leq d} \sum_{a_j \in \mathbb{Z}_3} \omega^{a_j^2 \mathbf{z}_j{}^\top \mathbf{z}_j} \\ &= \prod_{1 \leq j \leq d} (1 + 2\omega^{\mathbf{z}_j{}^\top \mathbf{z}_j}) \\ &= 3^{d_0} (1 + 2\omega)^{d_1} (1 + 2\omega^2)^{d-d_0-d_1}, \end{aligned}$$

where d_0 (resp. d_1) is the number of vectors $\mathbf{z}_j, 1 \leq j \leq d$, such that $\mathbf{z}_j^\top \mathbf{z}_j = 0$ (resp. $= 1$). With $1 + 2\omega = i\sqrt{3}$, $1 + 2\omega^2 = -i\sqrt{3}$, and $d_0 = \dim(\mathcal{Z} \cap \mathcal{Z}^\perp)$ by Lemma 3.36, the statement of the proposition now follows. \square

As Gioan and Las Vergnas [31] observe in their Corollary 2, it is not obvious that the parity of the number of vectors in an orthogonal basis for \mathcal{Z} with support size congruent to 1 modulo 3 is independent of the choice of basis, a fact implied by Proposition 3.37.

We reach another polynomial time computable evaluation of the Tutte polynomial (bases for finite-dimensional vector spaces being easy to find by Gaussian elimination, and Lemma 3.35 providing a polynomial time algorithm for constructing an orthogonal basis):

Theorem 3.38. *Let $G = (V, E)$ be a graph and $\omega = e^{2\pi i/3}$. We have*

$$T(G; \omega, \omega^2) = (-1)^{d_2} \omega^{|E|+d} (i\sqrt{3})^{d_0},$$

where d_0 is the dimension of the space of \mathbb{Z}_3 -tension-flows of G , d the dimension of the space of \mathbb{Z}_3 -flows, and d_2 is the number of vectors with support size congruent to 2 modulo 3 in any orthogonal basis for the space of \mathbb{Z}_3 -flows.

Proof. Setting $k = 3$ and $x = \omega^2 = \omega^{-1}$ in equation (18) we have

$$\sum_{\mathbf{z} \in \mathcal{Z}} \omega^{-|E|+|\text{supp}(\mathbf{z})|} = (\omega^2 - 1)^d T(G; \omega^2, \omega),$$

where $d = \dim \mathcal{Z} = n(G)$ is the dimension of the space of \mathbb{Z}_3 -flows. Then by Proposition 3.37 and $\omega^2 - 1 = i\sqrt{3}\omega$ we obtain

$$\omega^{-|E|} (-1)^{d+d_1} (i\sqrt{3})^{d+d_0} = (i\sqrt{3}\omega)^d T(G; \omega^2, \omega).$$

Since $T(G; \omega^2, \omega)$ is the complex conjugate of $T(G; \omega, \omega^2)$ the result follows. \square

In Section 3.10 we saw that $T(G; -1, -1) = (-1)^{|E(G)|} (-2)^{b(G)}$, where $b(G)$ is the bicycle dimension of G , i.e., the dimension of the the subspace of \mathbb{Z}_2 -tension-flows. The point $(-1, -1)$ lies on the hyperbola $(x-1)(y-1) = 4$, so that by identity (18)

$$T(G; -1, -1) = (-2)^{-n(G)} \sum_{\mathbb{Z}_2 \times \mathbb{Z}_2\text{-flows } \mathbf{z}} (-1)^{|E|-|\text{supp}(\mathbf{z})|}.$$

This might lead one to expect rather an expression for $T(G; -1, -1)$ in terms of the space of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -tension-flows in \mathbb{F}_4^E . Indeed, the dimension of the space of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -tension-flows is equal to the bicycle dimension $b(G)$. A $\mathbb{Z}_2 \times \mathbb{Z}_2$ -tension-flow decomposes by projection into a pair of \mathbb{Z}_2 -tension-flows, and conversely such a pair of \mathbb{Z}_2 -tension-flows can be pieced together to make a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -tension-flow. Hence there are precisely $(2^{b(G)})^2$ vectors that are $\mathbb{Z}_2 \times \mathbb{Z}_2$ -tension-flows, i.e., they comprise a space of dimension $b(G)$ over \mathbb{F}_4 . Hence we could also have written that $T(G; -1, -1) = (-1)^{|E|} (-2)^{d_0}$, where d_0 is the dimension of the space of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -tension-flows.

Question 25 Are there in general as many \mathbb{Z}_4 -tension-flows as $\mathbb{Z}_2 \times \mathbb{Z}_2$ -tension-flows?

Vertigan proved that the Tutte polynomial evaluated at the point $(i, -i)$ on the hyperbola $(x - 1)(y - 1) = 2$ has the following interpretation:

Theorem 3.39 ([86]). *Let G be a graph with bicycle dimension $b(G)$. Then*

$$|T(G; i, -i)| = \begin{cases} \sqrt{2}^{b(G)} & \text{if every bicycle has size a multiple of 4,} \\ 0 & \text{otherwise.} \end{cases}$$

For example, $T(C_4; i, -i) = i^3 + i^2 + i - i = -i - 1 = -\sqrt{2} \frac{1+i}{\sqrt{2}}$, where $\frac{1+i}{\sqrt{2}}$ is a primitive eighth root of unity. Recall also that every bicycle has even size, so that the bicycles of size a multiple of 4 either comprise all bicycles, or exactly half of them. Theorem 3.39 implies a polynomial time algorithm for evaluating $T(G; i, -i)$.

4 The Tutte polynomial in statistical physics

4.1 Colourings and flows in the ice model

Square ice consists of an $n \times n$ lattice arrangement of oxygen atoms. Between any two adjacent O-atoms lies one hydrogen atom, and there are also H-atoms at the left and right boundaries. The problem is to count all possible configurations in which every O-atom is attached to exactly two of its surrounding H-atoms, forming H_2O .

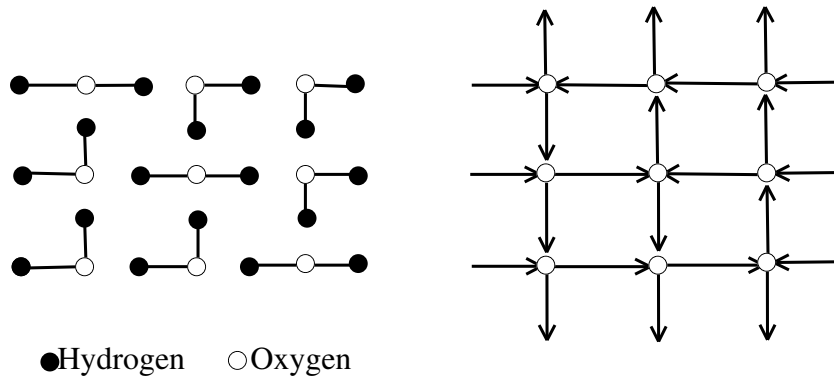


Figure 7: A configuration in the 3×3 square ice model, and its associated orientation.

There is a bijection between $n \times n$ ice configurations and Eulerian orientations on the lattice graph of O-atoms, with boundary conditions. Let u and v be two adjacent O-atoms. Orient the edge $u \rightarrow v$ if the H-atom between u and v is attached to v . On the left and

right boundaries all edges are incoming (each H-atom on the boundary is attached to an O-atom horizontally). On the top and bottom boundaries all edges are outgoing. See Figure 7.

In this way we get an Eulerian orientation of the $n \times n$ lattice graph with hanging boundary edges (each missing one endpoint).

The number of ice configurations is the number of Eulerian orientations of the $n \times n$ lattice graph with boundary conditions (incoming edges left and right, outgoing edges top and bottom). Each O-atom has six possible attachments to neighbouring H-atoms, corresponding to the six possible orientations at a vertex of degree 4 with two incoming and two outgoing edges. (This gives the alternative name of “six-vertex model” for the ice model.)

The $n \times n$ lattice graph of O-atoms with directed edges added as described gives an $(n+1) \times (n+1)$ array of square cells, where each O-vertex is incident with four cells. The cells can be \mathbb{Z}_3 -coloured by the following rule. Colour the top left corner 0. Suppose a and b are neighbouring cells such that the edge that separates them has orientation having a to the left and b to the right, and that a and b have colours $c(a)$ and $c(b)$ respectively. Then $c(b) = c(a) + 1$. In other words add one modulo 3 going from left to right across a directed edge. The boundary colours appear in sequence $0, 1, 2, 0, \dots$, with the bottom right corner coloured 0 like the top left. (The sequence along the top is the mirror image of that along the bottom, and likewise for left and right boundaries.)

This gives a bijection between $n \times n$ ice configurations and proper \mathbb{Z}_3 -colourings of the $(n+1) \times (n+1)$ -array of cells, observing the boundary conditions.

An alternative way to see this 3-colouring procedure is to first add edges to the $n \times n$ lattice graph $L_{n,n}$ to make it a 4-regular graph as follows. Given $L_{n,n}$ on vertex set $[n] \times [n]$, add edges between $(i, 1)$ and $(1, i)$ for each $i \in [n]$ and edges between (i, n) and (n, i) for each $i \in [n]$. This yields a 4-regular planar graph $\tilde{L}_{n,n}$ (with loops at the two corners $(1, 1)$ and (n, n)). An Eulerian orientation of $\tilde{L}_{n,n}$ is obtained by the same rule of directing O-atom u towards O-atom v when v is attached to the H-atom between u and v , the orientation of edges joining boundary O-atoms being determined by always directing edge into those vertices on the left or right boundaries. By tension-flow duality, each nowhere-zero \mathbb{Z}_3 -flow (Eulerian orientation) of $\tilde{L}_{n,n}$ corresponds to a nowhere-zero \mathbb{Z}_3 tension of the dual graph $\tilde{L}_{n,n}^*$, i.e. to three proper \mathbb{Z}_3 -colourings of the faces of $\tilde{L}_{n,n}$. Fixing the colour of either of the loop faces to be 0, it is easy to see that this corresponds to the cell-colouring described above. See Figure 8.

This 3-coloured version of the square ice problem is the starting point for the proof of the remarkable formula obtained by Zeilberger and Kuperberg in 1996: the number of $n \times n$ ice configurations is equal to

$$\frac{(3n-2)!(3n-5)! \cdots 4!1!}{(2n-1)!(2n-2)! \cdots (n+1)!n!}$$

See [2, Chapter 10] and [13].

In the general case, an ice model concerns the number of ways of orienting a 4-regular

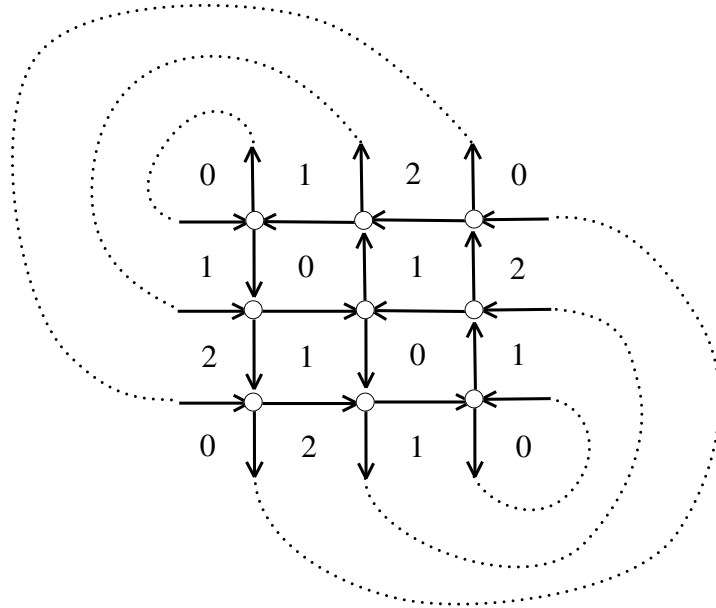


Figure 8: Eulerian orientation of 4-regular graph corresponds to a nowhere-zero \mathbb{Z}_3 -flow, whose dual is a nowhere-zero \mathbb{Z}_3 -tension, from which we get a proper face 3-colouring.

graph G such that each vertex has 2 incoming edges and 2 outgoing edges, i.e., an Eulerian orientation of G .

In Proposition 2.20 we saw that Eulerian orientations of a 4-regular graph correspond to nowhere-zero \mathbb{Z}_3 -flows of G , so that there are $F(G; 3)$ ice configurations on G .

Although finding an Eulerian orientation can be done polynomial time, in general computing the number of them is $\#\text{P}$ -complete, as proved by Mihail and Winkler [63]. In other words, computing $F(G; 3)$ is $\#\text{P}$ -complete even on the class of 4-regular graphs.

Proposition 4.1. *Let $G = (V, E)$ be a 4-regular graph. Then $F(G; 3) \geq \left(\frac{3}{2}\right)^{|V|}$.*

Proof. Use induction on the number of vertices of G . The case of a single vertex with two loops has $F(G; 3) = 4 \geq \frac{3}{2}$.

For a graph on n vertices, choose one, say v , and partition Eulerian orientations of G according to which of the six possible configurations is at v . Fix an Eulerian orientation of G . Let a, b, c, d be the neighbours of v and suppose that $a \rightarrow v, b \rightarrow v, v \rightarrow c, v \rightarrow d$.

Define a 2-in 2-out digraph G_1 on vertex set $V \setminus \{v\}$ as follows. Take the same edge orientations as G for edges not incident with v , together with directed edges $a \rightarrow c, b \rightarrow d$ to replace the four edges of G incident with v . Similarly, define the 2-in 2-out digraph G_2 by in a similar way except taking directed edges $a \rightarrow d$ and $b \rightarrow c$.

According to the configuration of oriented edges incident with v the resulting digraphs G_1 and G_2 have each one of the three types of “transition” at v , as illustrated in Figure 9 below.

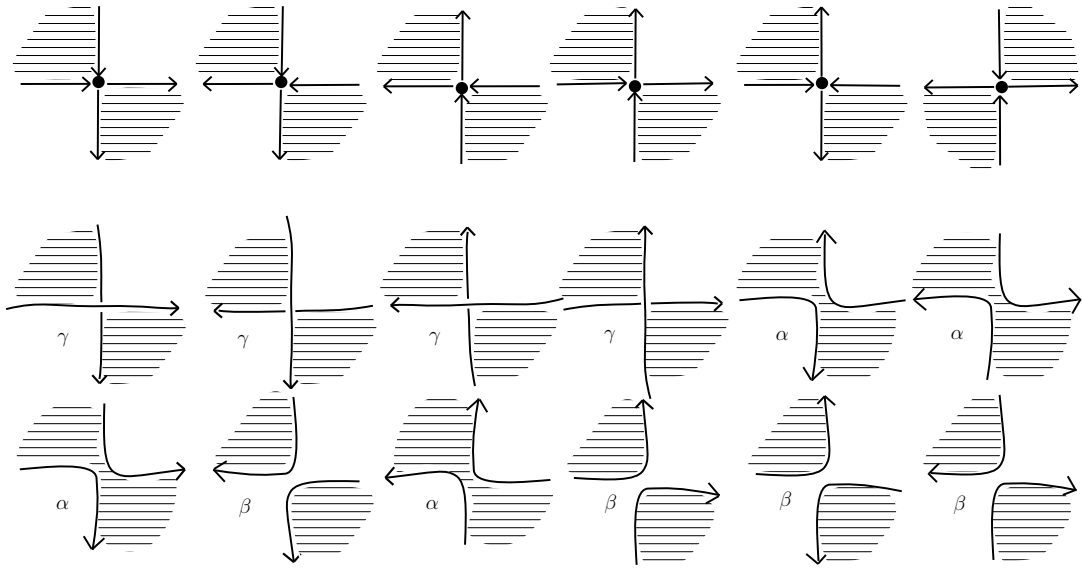


Figure 9: Two possible transitions at a vertex for each of the six configurations of orientations of its four incident edges (given in the top row). Three types of transition: white (α), black (β) and crossing (γ).

Depending on which of the six possible configurations of directed edges is at v , the digraphs G_1 and G_2 are Eulerian orientations of two of three possible 4-regular graphs $G_\alpha, G_\beta, G_\gamma$, according as the transition type at v is white (α), black (β) or crossing (γ). See Figure 10 below.

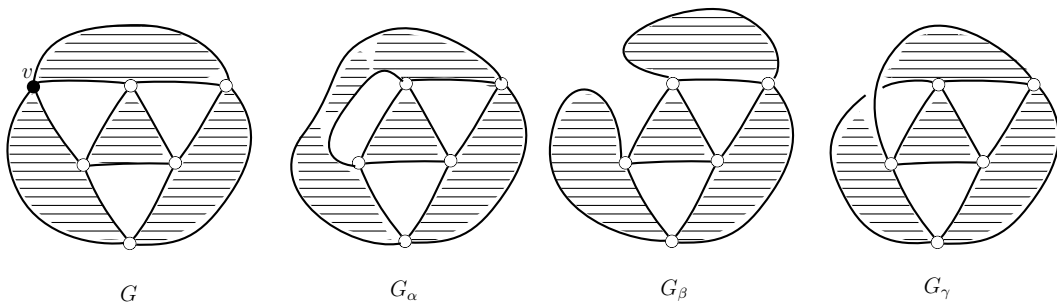


Figure 10: The three possible types of transition (black, white, crossing) at a vertex v of a 4-regular graph G . Which transition occurs depends on how the oriented edges incident with v are “tied together” when eliminating v from G to obtain either of the two 4-regular graphs G_1 or G_2 .

In Figure 10 the transition type α occurs four times, with all four possible configurations of orientations of the two edges. A similar observation holds for the transition types β

and γ .

Therefore, by considering the two possible ways to “tie together” two edges with matching directions in all six configurations of orientations of edges incident with v , we find that

$$F(G_\alpha; 3) + F(G_\beta; 3) + F(G_\gamma; 3) \leq 2F(G; 3),$$

and by induction hypothesis

$$3 \cdot \left(\frac{3}{2}\right)^{n-1} \leq 2F(G; 3),$$

yielding the desired lower bound. \square

In the square ice model we take $G \cong \tilde{L}_{n,n}$ the $n \times n$ grid with edges added between $(i, 1)$ and $(1, i)$ and edges between (i, n) and (n, i) , for each $i \in [n]$.

Lieb proved in 1967 that for the square lattice

$$\lim_{n \rightarrow \infty} F(\tilde{L}_{n,n}; 3)^{\frac{1}{n^2}} = \left(\frac{4}{3}\right)^{\frac{3}{2}} \approx 1.5396.$$

This is quite close to the lower bound of $\frac{3}{2}$ given by Proposition 4.1.

Suppose for a moment that G is the medial graph of a cubic planar graph H . Then $P(G; 3)$ is the number of proper edge 3-colourings of H , so if we had a positive lower bound for $F^*(G; 3)$ rather than $F(G; 3)$ we would have a quantitative version of the Four Colour Theorem: bounding the number of proper edge 3-colourings of a cubic planar graph H from below positively would yield a lower bound on the number of proper face 4-colourings of H . Needless to say such a lower bound on $F(G^*; 3)$ is not forthcoming.

4.2 The Potts model

The q -state Potts model on a graph $G = (V, E)$ is a generalization of the Ising model in which there are q possible states at a vertex rather than the two up/down states. In this model introduced by Askin and Teller (1943) and Potts (1952) the energy between two adjacent spins at vertices i and j is taken to be zero if the spins are the same and equal to a constant J_{ij} if they are different. For a state σ the Hamiltonian is defined by

$$H(\sigma) = \sum_{ij \in E} J_{ij}(1 - \delta(\sigma_i, \sigma_j)),$$

where δ is the Kronecker delta function ($\delta(a, b) = 1$ if $a = b$ and $\delta(a, b) = 0$ if $a \neq b$). We shall assume there is no external magnetic field. The Hamiltonian $H(\sigma)$ represents the energy of the state σ . The partition function of the q -state Potts model is defined by

$$Z(G) = \sum_{\sigma} e^{-\beta H(\sigma)},$$

where the sum is over all $q^{|V|}$ possible states σ and β is the inverse temperature $\beta = \frac{1}{kT}$ as for the Ising model.

Just as for the Ising model, we have

$$\Pr(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z(G)},$$

the Boltzmann maximum entropy distribution on the state space subject to a given expected value of $H(\sigma)$. (This expected value is the internal energy of the system, which is constant when the system is isolated/ in equilibrium with its environment. This is the First Law of Thermodynamics, expressing the principle of conservation of energy.)

If we replace J_{ij} by $-2J_{ij}$ then the partition function of the 2-state Potts model is the same as that of the Ising model scaled by $e^{-\beta \sum_{ij \in E} J_{ij}}$.

Returning to the q -state Potts model, if $J_{ij} = J$ is constant over all edges and we write $K = \beta J$ then the partition function can be written in the following ways:

$$\begin{aligned} Z(G) &= \sum_{\sigma \in [q]^V} e^{-K(|E| - \#\{ij \in E: \sigma_i = \sigma_j\})} \\ &= e^{-K|E|} B(G; q, e^K) \\ &= q^{|V| - |E|} (1 - e^{-K})^{|E|} C(G; q, \frac{e^K - 1 + q}{e^K - 1}) \\ &= q^{c(G)} (e^K - 1)^{r(G)} e^{-K|E|} T(G; \frac{e^K - 1 + q}{e^K - 1}, e^K). \end{aligned}$$

The point $(\frac{e^K - 1 + q}{e^K - 1}, e^K)$ lies on the hyperbola $(x - 1)(y - 1) = q$.

Here is a summary of correspondences between the Potts model and the Tutte plane (taken from [89]):

Potts model on G	Tutte polynomial $T(G; x, y)$
Ferromagnetism	Positive $(x, y > 1)$ branch of $(x - 1)(y - 1) = q$
Antiferromagnetism	Negative $(x < 0)$ branch of $(x - 1)(y - 1) = q$ with $y > 0$
High temperature	Asymptote of $(x - 1)(y - 1) = q$ to $y = 1$
Low temp. ferromagnetic	Positive branch of $(x - 1)(y - 1) = q$ asymptotic to $x = 1$
Zero temp. antiferromagnetic	Proper vertex q -colourings, $x = 1 - q, y = 0$.

4.3 The Fortuin-Kasteleyn random cluster model

The random cluster model on a connected graph $G = (V, E)$ with parameters p and q is a probability space on all spanning subgraphs of G . The probability measure of a subgraph $A \subseteq E$ is

$$\mu(A) = \frac{1}{Z(G)} p^{|A|} (1 - p)^{|E \setminus A|} q^{c(A)},$$

where as usual $c(A)$ denotes the number of connected components of the subgraph (V, A) , and $Z(G)$ is the normalizing constant

$$Z(G) = \sum_{A \subseteq E} p^{|A|} (1-p)^{|E \setminus A|} q^{c(A)}.$$

When $q = 1$ this is the *bond percolation* model on G , where an edge is open with probability p and otherwise closed. This model is used for such processes as molecules penetrating a porous solid, diffusion, and the spread of infection through a community (passage/contagion is possible along open edges).

Letting $q \rightarrow 0$, a subgraph has non-zero probability if and only if it is connected and in this case the partition function is the reliability polynomial:

$$\begin{aligned} Z(G) &= \sum_{A \subseteq E} p^{|A|} (1-p)^{|E \setminus A|} \\ &= (1-p)^{|E| - |V| + 1} p^{|V| - 1} T(G; 1, \frac{1}{1-p}). \end{aligned}$$

When q is a positive integer the random cluster model is equivalent to the q -state Potts model with $p = 1 - e^{-K}$. Using the subgraph expansion of the Tutte polynomial we have the following:

Proposition 4.2. *The partition function of the random cluster model on a connected graph $G = (V, E)$ with parameters $0 \leq p \leq 1$ and $q > 0$ is given by*

$$Z(G) = q(1-p)^{|E| - |V| + 1} p^{|V| - 1} T(G; 1 + \frac{(1-p)q}{p}, \frac{1}{1-p}),$$

and the probability measure of the subgraph A is given by

$$\mu(A) = \frac{\left(\frac{p}{1-p}\right)^{|A|} q^{c(A) - 1}}{\left(\frac{p}{1-p}\right)^{|V| - 1} T(G; \frac{p+q-pq}{p}, \frac{1}{1-p})}.$$

When $q > 1$ there is a bias towards edges joining vertices in an existing component than edges uniting two old components, since a larger number of components are favoured. More precisely, given $B \subseteq E$ and $e \in E \setminus B$, under the probability distribution μ we have

$$\begin{aligned} \Pr(e \in A \mid A \setminus \{e\} = B) &= \frac{\Pr(A = B \cup \{e\})}{\Pr(A - \{e\} = B)} = \frac{\mu(B \cup \{e\})}{\mu(B \cup \{e\}) + \mu(B)} \\ &= \begin{cases} p & \text{if } c(B \cup \{e\}) = c(B), \\ \frac{p}{p+q(1-p)} & \text{if } c(B \cup \{e\}) = c(B) - 1, \end{cases} \end{aligned}$$

where, for $0 < p < 1$,

$$\frac{p}{p+q(1-p)} \begin{cases} < p & \text{if } q > 1 \\ > p & \text{if } 0 < q < 1. \end{cases}$$

Percolation in the random cluster model (the existence of an infinite component of open edges) is intimately related to *two-point correlation* (long-distance correlation between vertex colours) in the q -state Potts model. Given fixed vertices i and j , in the Ising model the two-point correlation between i and j is defined to be the expected value of $\sigma_i \sigma_j$ over all states σ . For the Potts model the two-point correlation is the expected value of $\delta(\sigma_i, \sigma_j)$, i.e., the probability that σ_i equals σ_j .

A key result of Fortuin and Kasteleyn (1969) is the following (see e.g. [36, Theorem 2.1]):

Theorem 4.3. *For any pair of vertices i and j and positive integer q , the probability that σ_i equals σ_j in the q -state Potts model is given by*

$$\frac{1}{q} + \left(1 - \frac{1}{q}\right) \mu\{i \rightsquigarrow j\},$$

where μ is the random cluster probability measure on G obtained by taking $p = 1 - e^{-K}$ and $\{i \rightsquigarrow j\}$ is the event that there is an open path from i to j , i.e.,

$$\{i \rightsquigarrow j\} = \bigcup \{A \subseteq E : i \text{ and } j \text{ belong to the same component of } (V, A)\}.$$

The expression on the right-hand side in Theorem 4.3 can be regarded as being made up of two parts. The first term $1/q$ is the probability that under a uniformly random colouring of the vertices of G the vertices i and j have the same colour. The second term measures the probability of long-range interaction. So Theorem 4.3 expresses an equivalence between long-range spin correlations and percolatory behaviour.

Phase transition (in the infinite system) occurs at the onset of an infinite cluster (connected component) in the random cluster model and corresponds to spins on the vertices of the Potts model having long-range two-point correlation.

See [90, Chapter 4] for further discussion of percolation in the random cluster model, as well as the detailed account of [37] from the point of view of probability theory.

5 Graph homomorphisms

Many generalizations of the Tutte polynomial have been studied that have been motivated by applications in statistical physics (see e.g. [76] for the multivariate Tutte polynomial, equivalent to the partition function of the general Potts model where edge interactions vary from edge to edge), and by knot theory (see e.g. [65] for the U -polynomial), as well as the V -functions studied by Tutte himself, these being the most general multivariate polynomials which satisfy a deletion-contraction recurrence whose parameters may depend on which particular edge is being deleted/contracted.

Another perspective is to regard the chromatic polynomial, and more generally, the partition function of the q -state Potts model on a graph G , as arising from counting homomorphisms from G to a graph H (possibly with weights on its edges). For example,

$P(G; k)$ is equal to the number of homomorphisms from G to K_k (think of the vertices of K_k as being colours). More generally, the monochrome polynomial $B(G; k, y)$ is the number of homomorphisms from G to the complete graph on k vertices, each vertex with a loop of weight y attached to it. By the identity $Z(G) = e^{-K|E|}B(G; q, e^K)$ it follows that the partition function $Z(G)$ for the q -state Potts model is the number of homomorphisms from G to the complete graph on q vertices, each vertex with a loop of weight 1 and non-loop edges of weight e^{-K} .

Another example of a homomorphism counting function of interest to statistical physics is the Widom-Rowlinson model (introduced in 1969 as a model for liquid-vapour phase transitions), where the target graph consists of a star $K_{1,k}$ with a loop of weight 1 on each vertex. The number of homomorphisms from G to this graph is equal to the number of partial k -colourings of the vertices of G with the property that no edge has an endpoint of different colours (but it is allowed to have one endpoint a coloured vertex and the other uncoloured).

Amongst all possible weighted graphs H , the number of homomorphism from G to H is an evaluation of the Tutte polynomial for every graph G if and only if H is a Potts model graph [27], [28]. (A Potts model graph is K_q with a constant weight on its edges, together with loops attached, also of constant weight.) In fact, given just that the number of graph homomorphisms from G to H is an evaluation of the Tutte polynomial for G a cycle or path or the dual of a cycle or path, it must be the case that H is a Potts model graph [28].

As we have seen, the partition function of the Potts model is the specialization of the Tutte polynomial to the hyperbola $(x - 1)(y - 1) = q$. In [28] it is shown that any evaluation of the Tutte polynomial can be interpolated from its values on the hyperbolae $(x - 1)(y - 1) = q$ for positive integer q . For a familiar example, the number of acyclic orientations, $T(G; 2, 0)$, the point $(2, 0)$ lying on $(x - 1)(y - 1) = -1$, can be found by interpolation from the values $T(G; 1 - q, 0)$ for $r(G) + 1$ choices of positive integer q , the points $(1 - q, 0)$ lying on the hyperbolae $(x - 1)(y - 1) = q$. In this sense, the partition functions of the q -state Potts model for all positive integers q contain all the information about a graph that the Tutte polynomial does. What about when only finitely many values of q are chosen? Although it seems likely that a finite number of Potts model partition functions will not determine the Tutte polynomial in general, it seems difficult to produce examples of a pair of graphs that have different Tutte polynomials but the same q -state Potts model for even a fixed value of $q \geq 3$. (For $q = 2$ there are small examples of graphs with the same Ising model partition function but different Tutte polynomials.)

5.1 Graph invariants and graph homomorphism profiles

Many graph invariants can be expressed in terms of counting homomorphisms, including the chromatic polynomial (the familiar example, $P(G; k) = \text{hom}(G, K_k)$ for $k \in \mathbb{N}$), the flow polynomial (not so obvious, but we saw how earlier on), the Tutte polynomial (also not so obvious [28]), and other polynomial invariants such as the characteristic polynomial.

One of the fundamental questions about a graph invariant is whether it determines a given graph G up to isomorphism: for example, is G determined by its Tutte polynomial,

or even just by its chromatic polynomial? What about say the chromatic polynomial and characteristic polynomial jointly: do they together determine G ? By using the language of graph homomorphisms we can unify these sorts of question by using *homomorphism profiles*.

Let \mathcal{G} denote the set of all finite (multi)graphs up to isomorphism (i.e., graphs in \mathcal{G} are pairwise non-isomorphic, any given graph is isomorphic to exactly one graph in \mathcal{G}).

Definition 5.1. Let $\mathcal{P} \subseteq \mathcal{G}$ be given in some fixed enumeration $\mathcal{P} = \{P_1, P_2, \dots\}$. The left \mathcal{P} -profile of a graph G is the sequence $(\text{hom}(P, G) : P \in \mathcal{P})$ and the right \mathcal{P} -profile is the sequence $(\text{hom}(G, P) : P \in \mathcal{P})$.

Definition 5.2. A graph invariant is a function $f : \mathcal{G} \rightarrow S$, where S is a set (often with some algebraic or combinatorial structure that “encodes” some of the graphical combinatorial structure).

For example, the Tutte polynomial $T(G; x, y)$ is a graph invariant taking values in the ring $\mathbb{Z}[x, y]$. Multiplication in the ring corresponds to the disjoint union of graphs, $T(G_1 \cup G_2; x, y) = T(G_1; x, y)T(G_2; x, y)$. As we have seen, many combinatorial parameters of a graph G are reflected in properties of the Tutte polynomial $T(G; x, y)$. For example, a graph G with at least two edges is 2-connected if and only if the coefficient of x is non-zero, and G is k -colourable if and only if $T(G; 1 - k, 0) \neq 0$.

The left- (or right-) \mathcal{P} -profile defines an invariant taking values in \mathbb{N}^ω , the set of infinite sequences of natural numbers. Multiplication in the monoid \mathbb{N}^ω corresponds to the disjoint union of graphs for the left-profile, $\text{hom}(G_1 \cup G_2, H) = \text{hom}(G_1, H)\text{hom}(G_2, H)$, and to the direct product of graphs for the right-profile, $\text{hom}(F, G_1 \times G_2) = \text{hom}(F, G_1)\text{hom}(F, G_2)$.

A graph invariant induces a partition of \mathcal{G} on whose subsets the function f is constant, i.e., two graphs G and G' are *f-equivalent* if $f(G) = f(G')$. If on the other hand $f(G) \neq f(G')$ then the graphs G and G' are *distinguished by f*, belonging as they do to different subsets of the partition of \mathcal{G} induced by f .

If f induces the trivial partition consisting entirely of singletons, then f determines graphs up to isomorphism. There is great interest in finding graph invariants with this property, because of the possible implications for the status of the graph isomorphism problem (still of unknown complexity).

A slightly weaker requirement than that f determine all graphs up to isomorphism is that f determine *almost all* graphs up to isomorphism. Letting $\mathcal{G}(n) \subset \mathcal{G}$ denote the set of all graphs on n vertices, this is to say that

$$\frac{\#\{G \in \mathcal{G}(n), G \text{ determined by } f\}}{|\mathcal{G}(n)|} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

If $\mathcal{G}_1 \subset \mathcal{G}$ is a block, or union of blocks, of the partition of \mathcal{G} induced by f then we say the the class \mathcal{G}_1 is *determined by f*. In this situation, knowing the value of $f(G)$ we can determine whether $G \in \mathcal{G}_1$. Another way of phrasing this is to say that the property of a graph belonging to the class \mathcal{G}_1 is an “*f*-invariant”. For example, the property of being 2-connected is a Tutte polynomial invariant. When \mathcal{G}_1 consists of just a single graph G , the graph G itself is determined by f up to isomorphism.

Question 26

- (i) Explain why the property of having no cycles is a chromatic polynomial invariant.
- (ii) Prove that the complete graph K_k and cycle C_k are both determined by their chromatic polynomials.

Conjecture 5.3. [11] *Almost all graphs are determined by their chromatic polynomial.*

Bollobás, Pebody and Riordan also make the weaker conjecture – but still far from being solved – that almost all graphs are determined by their Tutte polynomial.

The Tutte polynomial of any forest on m edges is equal to x^m ; conversely if $T(G; x, y) = x^m$ then G is a forest on m edges. Thus, although forests not individually determined by the Tutte polynomial, the class of all forests on m edges is so determined. (Likewise for the chromatic polynomial, except now one needs to take into account the number of connected components too.)

5.2 Homomorphism profiles determining graph invariants

There are graph invariants that are known to determine each graph G up to isomorphism. An example, trivial by definition, is the equivalence class of the adjacency matrix of G (up to permutation of rows and columns). But despite this triviality one shouldn't overlook the fact that algebraic properties of the adjacency matrix A of a graph G correspond to graphical properties of G in a way that may permit analysis of the latter (for example, the matrix powers of A enumerate walks on G – see below).

The homomorphism \mathcal{G} -profile of G , an infinite sequence of natural numbers, may also seem to be too unwieldy a graph invariant to be useful (even allowing that for given G it is possible to truncate the profile to those graphs with at most as many vertices as G). However, we saw in the final lecture how the correspondence $\text{hom}(G, H_1 \times H_2) = \text{hom}(G, H_1)\text{hom}(G, H_2)$ between the direct product of graphs and multiplication in \mathbb{N} could be used to prove the non-trivial result that $G \times G \cong H \times H$ implies $G \cong H$. This required the fact that these profiles do indeed determine all graphs up to isomorphism:

Theorem 5.4. (Lovász,[57], and [58]) *Let \mathcal{G} be the set of all finite graphs in some enumeration, no two graphs isomorphic.*

Then

- (i) *The left- \mathcal{G} -profile of a (possibly edge-weighted) graph G determines G up to isomorphism.*
- (ii) *The right \mathcal{G} -profile of a graph G determines G up to isomorphism.*

Can we “thin out” the class \mathcal{G} to make a smaller set \mathcal{P} with the property that every graph is still determined by its left- (and right-) \mathcal{P} -profile?

Dvořák [21] has given two examples for left-profiles. A graph H is k -degenerate if each subgraph of H contains a vertex of degree at most k . Every graph with tree-width k is k -degenerated. 1-degenerated graphs are precisely forests, but there are 2-degenerated graphs with arbitrary tree-width; the complete graph with each edge subdivided by two new vertices is 2-degenerate.

Theorem 5.5. (Dvořák [21]) *Every graph is determined by its left \mathcal{P} -profile when*

- (i) \mathcal{P} is the set of all 2-degenerate graphs.
- (ii) \mathcal{P} consists of all graphs homomorphic to a fixed non-bipartite graph (in other words, an down-set in the homomorphism order with minimal element a non-bipartite graph).

We may extend the terminology of right \mathcal{P} -profiles to the case where \mathcal{P} is a collection of edge-weighted graphs.

Question 27 Show the following:

- (i) The right $\{K_k : k = 1, 2, \dots\}$ -profile of G determines $P(G; x)$.
- (ii) The right $\{K_k^{1-k} : k = 1, 2, \dots\}$ -profile of G determines $F(G; x)$, where K_k^y denotes the complete graph on k vertices with a loop of weight y on each vertex (here $y = 1 - k$).
- (iii) The right $\{K_k^y : k, y = 1, 2, \dots\}$ -profile of G determines $T(G; x, y)$. (Here all that matters is that y ranges over some infinite set of values.) [More fiddly as requires bivariate polynomial interpolation. See [28] for details.]
- (iv) The right $\{K_1^1 + \overline{K}_k : k = 1, 2, \dots\}$ -profile determines the independence polynomial $I(G; x) = \sum x^{|U|}$, where the sum is over all stable sets U in G . (The graph K_1^1 is a single vertex with a loop attached; the graph $K_1^1 + \overline{K}_k$ the star $K_{1,k}$ with a loop on its central vertex.)

Conjecture 5.3 thus states that the right $\{K_k : k = 1, 2, \dots\}$ -profile determines almost all graphs (or in its weaker form, that the right $\{K_k^y : k, y = 1, 2, \dots\}$ -profile determines almost all graphs).

How about the right $\{K_1^1 + \overline{K}_k : k = 1, 2, \dots\}$ -profile? Well, as Noy showed [66], using the fact that on average a random graph on n vertices has independence (stability) number $O(\log n)$, almost all graphs are *not* determined by the independence polynomial. So here we have an example of a homomorphism profile by an infinite number of non-isomorphic graphs for which we know it is not true that the profile determines almost all graphs.

5.3 Spectrum and degree sequence by left profiles

A k -walk in a graph is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \dots, v_{k-1}, e_k, v_k$, where $e_{i+1} = v_i v_{i+1}$ for $0 \leq i \leq k-1$. A k -walk is *closed* if $v_0 = v_k$. A 0-walk is just a vertex and is always closed. A 1-walk is a walk from a vertex to an adjacent vertex. A closed 1-walk is a loop.

Lemma 5.6. *Let H be an edge-weighted graph with adjacency matrix A . Then*

$$\text{hom}(C_k, H) = \text{tr}(A^k).$$

Proof. The matrix A^k has (i, j) entry the sum of edge-weighted k -walks from i to j , as can be proved by induction. (The weight of a walk is the product of its edge weights, with multiplicities counted for repeated edges.) A closed k -walk corresponds to a homomorphic image of C_k . The diagonal entries of A^k then together sum to $\text{hom}(C_k, H)$. By diagonalization, $A = B^{-1}DB$ for orthogonal matrix B and diagonal matrix $D = \text{diag}(\lambda_1, \dots, \lambda_n)$, where the λ_i are the eigenvalues of A taken with multiplicity. \square

Corollary 5.7. *Let H be an edge-weighted graph H on n vertices with adjacency matrix A . Then the left $\{C_k : 1 \leq k \leq n\} \cup \{K_1\}$ -profile of H determines the spectrum of A .*

Proof. If A has eigenvalues $\lambda_1, \dots, \lambda_n$ then $\text{tr}(A^k) = \sum_i \lambda_i^k$. In particular, $\text{tr}(A^0) = \text{hom}(K_1, H) = n$ gives the number n of vertices of H , i.e., the size of A . Given the power sums $\sum \lambda_i^k$ for $1 \leq k \leq n$, Newton's relations yield the elementary symmetric polynomials in the λ_i and hence the λ_i are uniquely determined (as the roots of the characteristic polynomial of A). \square

Restricting attention to simple unweighted undirected graphs, graphs determined by their spectrum include K_n , $K_{n,n}$ and C_n . (Curiously, the line graphs $L(K_n)$ of complete graphs are also determined by their spectrum with the exception of the case $n = 8$, where there are three other non-isomorphic graphs with the same spectrum.) Similar to Conjecture 5.3 about the chromatic polynomial, it is conjectured that almost all graphs are determined by their spectrum [16]. On the other hand, almost all trees are not determined by their spectrum, and there are many constructions of cospectral non-isomorphic graphs. The smallest pair of graphs with the same spectrum is $C_4 \cup K_1$ and $K_{1,4}$.

The *characteristic polynomial* of G is defined by $\phi(G; x) = \det(A - xI) = (x - \lambda_1) \cdots (x - \lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A (taken with multiplicity). So in all the above we could have talked about the characteristic polynomial of G (rather than the spectrum) being determined by the left $\{K_{1,k} : k = 0, 1, \dots\}$ -profile of G .

By Corollary 5.7 the conjecture of Van Dam and Haemers [16] is that almost all graphs are determined by their left $\{K_{1,k} : k = 0, 1, \dots\}$ -profile.

Lemma 5.8. *Let H be an edge-weighted graph on n vertices with adjacency matrix A , and let $\mathbf{1}$ denote the $n \times 1$ all-one vector. Then the left $\{K_{1,k} : 1 \leq k \leq n\}$ -profile of H determines the vector $A\mathbf{1}$.*

Proof. The homomorphic image of $K_{1,k}$ is a multiset of k edges incident with a common vertex. If H has vertex set $[n] = \{1, \dots, n\}$ and adjacency matrix $A = (a_{u,v})_{u,v \in [n]}$ then

$$\text{hom}(K_{1,k}, H) = \sum_{v \in [n]} \left(\sum_{u \in [n]} a_{u,v} \right)^k,$$

by taking all possible choices of a multiset of k edges incident with common vertex v as the image of $K_{1,k}$. By taking $k = 1, \dots, n$ we can determine the column sums $\sum_{u \in [n]} a_{u,v}$ of A , i.e., the vector $\mathbf{1}^\top A$. Since A is symmetric this also gives the row sums and the vector $A\mathbf{1}$. \square

When H is an unweighted graph, i.e. its adjacency matrix has entries either 0 or 1, with degree sequence d_1, \dots, d_n (vertex degrees listed in non-increasing order), we have as a particular case of Lemma 5.8 that

$$\text{hom}(K_{1,k}, H) = \sum_i d_i^k.$$

(The case $k = 0$ gives us n : $\text{hom}(K_1, H) = |V(H)|$.) By taking $k = 0, 1, \dots, n$ we obtain the following:

Corollary 5.9. *The left $\{K_{1,k} : k = 0, 1, \dots\}$ -profile of (an unweighted graph) H determines the degree sequence of H .*

Are almost all graphs determined up to isomorphism by their degree sequence? If ab and cd are edges of a graph then the (multi)graph obtained from G by deleting ab and cd and replacing these by edges ac and bd has the same degree sequence: provided ac and bd are not already edges then this gives another simple graph G' with the same degree sequence as G . Of course, it may be that there are no such pair of edges ab and cd for which this exchange both yields a simple graph and one that is non-isomorphic to G' .

6 From graphs to matroids

The Tutte polynomial was defined originally for graphs and extends more generally to matroids. The many natural combinatorial interpretations of its evaluations and coefficients for graphs then translate to not obviously related combinatorial quantities in other matroids. For example, the Tutte polynomial evaluated at $(2, 0)$ gives the number of acyclic orientations of a graph.⁶ Zaslavsky proved that the Tutte polynomial at $(2, 0)$ also counts

⁶This interpretation of the evaluation of the chromatic polynomial of a graph at -1 , due to Stanley [77], is itself surprising and is the tip of the iceberg with regard to the connection between orientations and colourings of graphs.

the number of different arrangements of sets of hyperplanes in n -dimensional Euclidean space (the underlying matroid is of a set of n -dimensional vectors over \mathbb{R}).⁷

For another example of extending the scope of a definition, the Tutte polynomial $T(G; x, y)$ along the hyperbola $xy = 1$ when G is planar specializes to the *Jones polynomial* of the alternating link associated with G (via the medial graph of G). Starting from the knot theory context, analogues of the Tutte polynomial have recently been defined for signed graphs (needed for encoding arbitrary links, not just alternating) and embedded graphs (in other surfaces than the plane).

We shall see the connection of the Tutte polynomial and the Jones polynomial via the *Kauffman bracket* of a link: the deletion-contraction recurrence for the Tutte polynomial of a graph is mirrored in the skein relation for the Jones polynomial (involving local transformations of a knot).

But before this we shall see with the *interlace polynomial* another example of this phenomenon – originally defined meaningfully only for a restricted class of graphs (namely interlace graphs, or *circle graphs*), its recursive definition (analogous to deletion-contraction for the Tutte polynomial) applies to any graph. Interpreting its values for graphs generally remains an open area of research. Its definition has already been extended to matroids too.

Common to the application to knots and to Eulerian tours on 2-in 2-out digraphs is the operation of taking the medial graph of a plane graph and considering the types of transition that may occur at a vertex when following the knot (under or over) or Eulerian tour of the graph (which of the edges to follow out of the vertex). There are three possible types of transition when following edges in and out of at a vertex of degree 4, and this would bring us to Jaeger’s *transition polynomial* (see [2]) which includes the *Penrose polynomial* as a special case, and is also intimately related to the Tutte polynomial. However, we shall only have time to consider the interlace polynomial and Jones polynomial.

6.1 Cuts, circuits and cycles

We start with undirected graphs. A set A of edges in a graph $G = (V, E)$ is an *edge cut*, or *cutset*, if the graph $G - A = (V, E \setminus A)$ has more components than G . When G is a connected graph, a cutset is a set of edges which disconnects G . An inclusion minimal edge cut of a graph G is called a *bond*. A bond is always contained in a single component of G . If A is a bond of a connected graph G then the graph $G - A$ has exactly two components, say with vertex sets V_1 and V_2 . A is the set of all edges between V_1 and V_2 , denoted by $E(V_1, V_2)$. A cut of size one is called a *bridge*.

A *path* we interpret both as a subgraph and as a sequence $v_0, e_1, v_1, \dots, e_t, v_t$ of vertices and edges of the graph, in which the vertices v_i (and hence edges too) are distinct. In a *trail* vertices may be repeated, only edges e_i being distinct, and in a *walk* both vertices and

⁷A hyperplane in n -dimensional Euclidean space is an $(n - 1)$ -dimensional flat subset (congruent to $(n - 1)$ -dimensional space), i.e., affine subspace of dimension $n - 1$. Flats in Euclidean spaces are affine subspaces such as points, lines, planes,... More generally, a *flat* in a matroid is a subset with the property that adding any other element to it increases its rank, and a *hyperplane* in a matroid of rank r is a flat of rank $r - 1$.

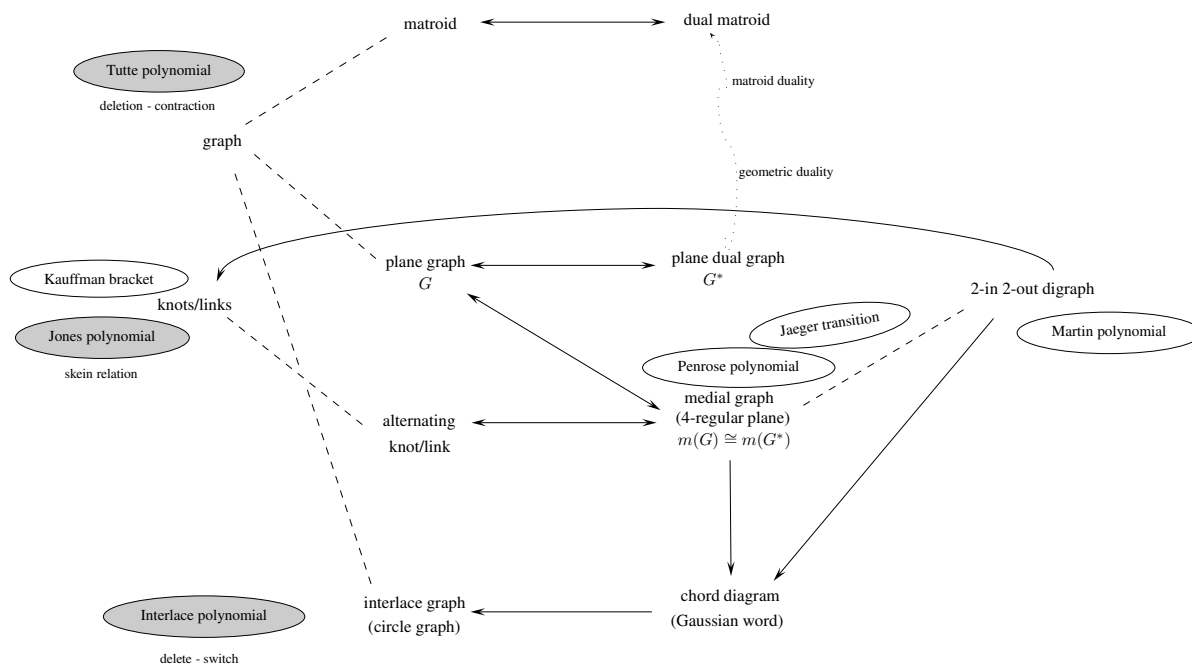


Figure 11: Schematic diagram of interrelationships between various combinatorially defined polynomials. The dashed lines from one class up to another indicate containment. Arrows indicate operations (invertible when double arrow) from one class to another. Once defined for a class of objects, a polynomial may be extended in its scope by lifting up to a more general class of objects – this is what happens for the Tutte polynomial (graphs to matroids) and interlace polynomial (circle graphs to graphs to binary matroids). In the reverse direction, restricting the Jones polynomial to alternating links gives the Tutte polynomial along $xy = 1$. Missing from the diagram are the class of signed plane graphs: these encode all knots/links and by extending to signed graphs generally there is a *signed Tutte polynomial* which satisfies a deletion-contraction recurrence, only with two cases according to the sign of the edge.

edges may be repeated. A *circuit* of length t is formed by a sequence $v_0, e_1, v_1, \dots, e_t, v_t$ in which all vertices are distinct with the exception only of $v_0 = v_t$ (so it can be thought of as a “closed path”). Similarly, a trail $v_0, e_1, v_1, \dots, e_t, v_t$ satisfying $v_0 = v_t$ is called a *closed trail* or a *cycle*.⁸

As all the edges of a closed trail are distinct, a closed trail may be identified with the subset of edges traversed by it. By saying that a set of edges is a cycle is meant that for some ordering it will be form a closed trail. If all the edges of the connected graph G are traced by some cycle then G is an *Eulerian graph*.

Proposition 6.1. *A graph is Eulerian if and only if it is connected and all its vertices have even degree.*

It follows that any Eulerian subgraph is an edge-disjoint union of circuits (this is sometimes called Veblen’s theorem [84]). The following are two algorithms for constructing an Eulerian cycle of a connected graph G all of whose vertices are of even degree:

- (i) **Fleury’s algorithm** [25] (“Don’t burn your bridges”) starts with an arbitrarily chosen vertex v_0 and an arbitrary starting edge e_0 incident with v_0 . The edge e_0 is deleted and its other endpoint is the next vertex v_1 to be chosen. At each stage, at current vertex v_i an edge e_i can be chosen as the next edge in the cycle if it is not a bridge in the remaining graph, unless there is no such edge, in which case the only remaining edge left at the current vertex is chosen. It then moves to the other endpoint v_{i+1} of edge e_i , after which e_i is deleted from the graph. At the end of the algorithm there are no edges left, and the sequence in which the edges were chosen forms an Eulerian cycle. [In this algorithm, the last edges chosen from v_i ($i > 0$) before returning to v_0 form a spanning tree of G – each of them is a bridge by definition of the algorithm.]
- (ii) **Hierholzer’s algorithm** [38] chooses any starting vertex v_0 , and follows a trail of edges from that vertex until it returns to v_0 . It is not possible to get stuck at any vertex other than v_0 , because all vertex degrees being even ensures that when the trail enters a vertex $v \neq v_0$ there must be an unused edge leaving v . The trail formed in this way is closed, but may not cover all the vertices and edges of the initial graph. As long as there exists a vertex v belonging to the current cycle that has incident edges not yet in the cycle, start another trail from v , following unused edges until returning to v , and join the cycle formed in this way to the previous cycle. [This algorithm may be thought of as gluing cycles together to form a Eulerian cycle covering all the edges of G , cf. Veblen’s theorem that an Eulerian cycle can be partitioned into circuits.]

Proposition 6.2. *The intersection of any cutset with any cycle is even.*

⁸Circuits and cycles have been defined this way round in order to have circuits in the graph theory sense coincide with circuits in the matroid theory sense. For us any circuit is a cycle, not vice versa. However, this is sometimes counter to graph theory terminology found elsewhere, where cycles are circuits....

Question 29

- (i) Prove (or recall the proofs of) Propositions 6.1 and 6.2.
- (ii) Prove that the symmetric difference of two bonds contains a bond.
- (iii) Prove that the symmetric difference of two circuits contains a circuit.

6.2 Orthogonality of cycles and cutsets

If \mathbb{F} is a field then the set of all vectors of length m with entries in \mathbb{F} forms a vector space \mathbb{F}^m of dimension m . This vector space is equipped with inner product $\mathbf{x}\mathbf{y}^\top = \sum_{i=1}^m x_i y_i$. If V is a vector subspace of \mathbb{F}^m then the set of all vectors \mathbf{y} which are orthogonal to all vectors $\mathbf{x} \in V$, is again a vector space which is called the *orthogonal complement* of V and denoted by V^\perp . Observe that $V^{\perp\perp} = V$ and that $\dim V^\perp = m - \dim(V)$.

Consider a graph $G = (V, E)$ with m edges and the field $\mathbb{F} = \mathbb{Z}_2$ on two elements. The vector space \mathbb{Z}_2^E formed by all vectors $\mathbf{x} = (x_e : e \in E)$, where $x_e \in \mathbb{Z}_2$, is isomorphic to the vector space \mathbb{Z}_2^m . A vector \mathbf{x} may be thought of as the characteristic (indicator) vector of a subset of edges of G . Now consider the set \mathcal{Z} of all vectors \mathbf{x} which correspond to cycles (i.e., to Eulerian subgraphs). Also, denote by \mathcal{K} the set of all vectors \mathbf{x} which correspond to edge cuts of G . We have the following basic result:

Theorem 6.3. *For any graph G the following hold:*

- (i) *Both \mathcal{Z} and \mathcal{K} are vector subspaces of \mathbb{Z}_2^E .*
- (ii) *\mathcal{Z} and \mathcal{K} are orthogonal complements of each other.*
- (iii) *$\dim(\mathcal{Z}) = m - n + k$, where $n = |V(G)|$ and k is the number of components of G , and $\dim(\mathcal{K}) = n - k$.*

Before proving this result, recall that a *spanning forest* of a graph $G = (V, E)$ is any (edge-) inclusion maximal subgraph (V, A) not containing any cycle. A spanning forest is just a *spanning tree* if G is connected. Of course $|A| = |V| - c(G)$, where $c(G)$ is the number of components of G . A spanning forest (V, A) can be dually defined as an inclusion-minimal subgraph of G which has the same number of connected components as G .

Proof. Both \mathcal{Z} and \mathcal{K} are vector subspaces of \mathbb{Z}_2^E by virtue of Question 29 above. Proposition 6.2 says that $\mathcal{K} \subseteq \mathcal{Z}^\perp$, so that $\dim \mathcal{K} \leq m - \dim \mathcal{Z}$. It therefore suffices to prove $\dim(\mathcal{Z}) \geq m - n + k$ and $\dim(\mathcal{K}) \geq n - k$.

Let A be the edge set of a spanning forest of G (so $|A| = n - k$, $|E \setminus A| = m - n + k$). By maximality of A the graph $(V, A \cup \{e\})$ contains, for every $e \in E \setminus A$, a uniquely determined cycle Z_e containing the edge e . By the minimal connectivity definition of a spanning forest we know that for every $e \in A$ the graph $(V, A \setminus \{e\})$ has more components than G and thus

there exists a unique cutset C_e of G containing the edge e with the same components as the graph $(V, A \setminus \{e\})$. But now both cycle Z_e and cut C_e are the only selected cycles and cuts containing the edge e . Since $e \notin Z_f$ for each $f \in E \setminus (A \cup \{e\})$, and similarly $e \notin C_f$ for each $f \in A \setminus \{e\}$, the cycles $\{Z_e : e \in E \setminus A\}$ are linearly independent, as are the cutsets $\{C_e : e \in A\}$. This proves both $\dim(\mathcal{Z}) \geq m - n + k$ and $\dim(\mathcal{K}) \geq n - k$. \square

Question 30

- (i) Prove that if Z is a cycle in G and A the set of edges of a spanning forest of G then $\sum_{e \in Z \setminus A} Z_e = Z$ (for simplicity we identify in this notation a cycle with its characteristic vector).
- (ii) Prove a similar statement for cuts.

6.3 Graph duality

Henceforth we allow graphs with parallel edges and loops. A *simple graph* is a graph with no parallel edges or loops. When clarity demands it, the term *multigraph* is used for a graph in which there may be parallel edges and loops.

Two simple graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there is a bijection $f : V \rightarrow V'$ such that $uv \in E$ if and only if $f(u)f(v) \in E'$, for all $u, v \in V$.

Definition 6.4. Two multigraphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there are functions $f : V \rightarrow V'$ and $g : E \rightarrow E'$ such that

- (i) if e has endpoints u and v then $g(e)$ has endpoints $f(u)$ and $f(v)$;
- (ii) f and g are bijections.

An isomorphism between multigraphs is an isomorphism between their underlying simple graphs together with the condition that edge multiplicities are the same (including loops). A multigraph $G = (V, E)$ can be represented by its adjacency matrix $A = A(G)$ with (u, v) entry equal to the number of edges joining u and v . Multigraphs G and G' are isomorphic if and only if the matrices $A(G)$ and $A(G')$ are permutation-equivalent.

A graph $G = (V, E)$ is *planar* if it can be drawn in the plane so that in the drawing distinct arcs are openly disjoint and share only end-vertices in the case that corresponding edges are incident. A graph with such a drawing is called a *plane graph*. An edge may lie on the boundary of one face (and this if and only if it is a bridge) or two faces. As we are considering multigraphs, a face may be formed by only two edges.

Let $G = (V, E, F)$ be an (undirected) plane graph with set of faces F . The *geometric dual* of G is the graph $G^* = (V^*, E^*, F^*)$, with $V^* = F$, $E^* = \{e^* : e \in E\}$, where e^* has endpoints the faces of G which have e on their boundary (thus e^* is a loop when e is a bridge). The face set F^* of G^* may be identified with the vertex set of G ; and then the edge set E^* may be identified with the edge set E of G .

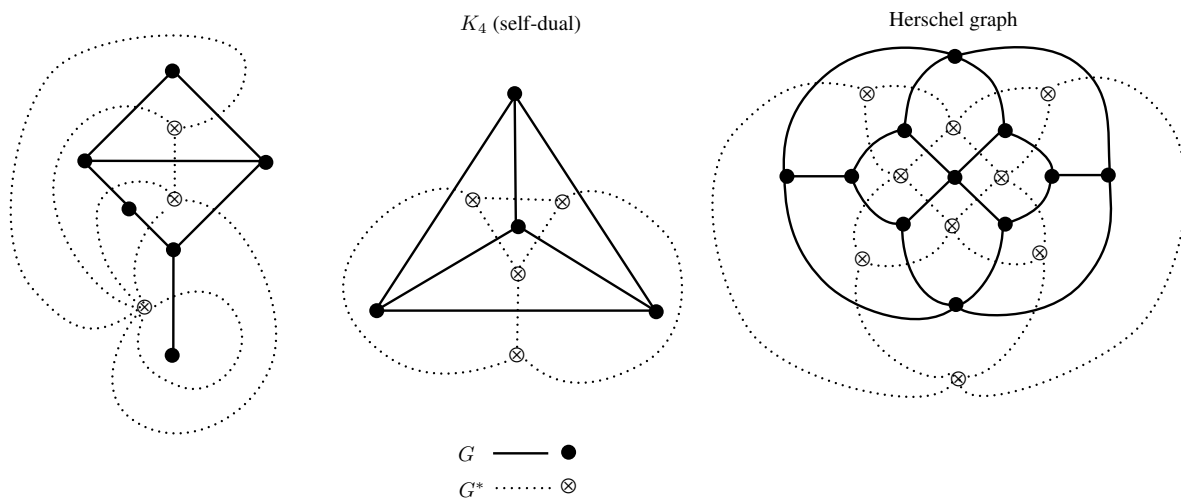


Figure 12: Some plane graphs and their duals. The leftmost example illustrates how edges in series become parallel edges in the dual and bridges become loops in the dual.

The dual graph G^* is again planar and $(G^*)^* \cong G$.

A simple graph may have a graph with parallel edges and loops as its dual. For example, if T is a tree with n vertices then the dual of any of its plane drawings is a single vertex graph with $n - 1$ loops. As another example consider any 2-connected plane graph G (so in particular there are no bridges) and let G' be the graph which arises from G by subdividing every edge of G by one vertex. Then the dual graph G'^* arises from G^* by replacing every edge by two “parallel” edges.

Further examples of dual pairs are shown in Figure 12.

Question 31

- (i) Can you find some more examples of self-dual plane graphs?
- (ii) The *medial graph* $m(G)$ of a plane graph G is defined by placing vertices on the midpoints of edges of G and joining vertices by an edge when they lie on consecutive edges of a face (by a double edge if consecutive in two different faces). Prove that $m(G) \cong m(G^*)$. (See Figure 13.)

Theorem 6.1. Let $G = (V, E)$ be a plane graph. Then the correspondence $e \mapsto e^*$ has the following properties:

- (i) $A \subseteq E$ is a cycle if and only if $A^* = \{e^* : e \in A\}$ is a cut in G^* .
- (ii) $A \subseteq E$ is a cut if and only if $A^* = \{e^* : e \in A\}$ is a cycle in G^* .

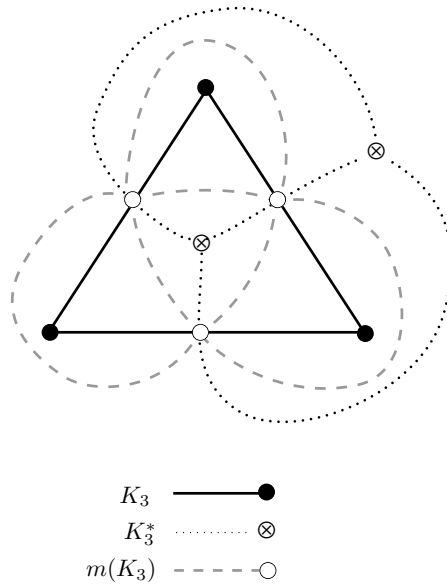


Figure 13: The medial graph (dashed grey lines) of K_3 (solid black lines) is isomorphic to the medial graph of the dual graph K_3^* (the graph K_3^* is shown with dotted black lines).

To prove this rigorously requires appeal to the Jordan Curve Theorem.

Question 32

- (i) Give an example of a 2-connected planar graph G with two inequivalent embeddings, i.e., find G with two embeddings that have non-isomorphic geometric duals G^* .
- (ii) For any given k , describe a 2-connected planar graph with at least k pairwise inequivalent embeddings.

6.4 Matroids

Whitney [93] introduced matroids in 1935 as an abstraction of both linear independence and the properties of cycles in graphs. Matroids present a natural “self-dual” notion which captures both cycles and cutsets.

There are many “cryptomorphic” ways to define a matroid axiomatically. First we start with *independence* as the primitive notion – for graphs a set of edges is independent if it spans a forest, i.e., contains no cycles.

Definition 6.5. (Independent sets) *Let E be a finite set. A matroid M on E is a pair (E, \mathcal{I}) where \mathcal{I} is a non-empty collection of subsets of E with the properties:*

- (i) If $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$ then $I_2 \in \mathcal{I}$ (\mathcal{I} is an ideal),
- (ii) (Exchange Property) If $I_1, I_2 \in \mathcal{I}$, $|I_2| < |I_1|$, then there exists $e \in I_1$ such that $I_2 \cup \{e\} \in \mathcal{I}$.

If E is a family of vectors in a vector space V , and \mathcal{I} is the set of linearly independent subsets of E , then (E, \mathcal{I}) is a matroid (called a *vector matroid*).

A *basis* is a maximal independent set with respect to inclusion, i.e., a subset of edges that is independent with the property that adding any other edges destroys the property of independence.

The *rank* of $A \subseteq E$ is defined by

$$\rho(A) = \max\{|I| : I \in \mathcal{I}, I \subseteq A\}.$$

A *circuit* is a minimal non-independent set of edges with respect to inclusion, i.e., a subset of edges that is not independent but with the property that any proper subset is independent. Equivalently, a circuit is a minimal subset of edges contained in no basis. (A circuit in a graphic matroid corresponds to a spanning subgraph in which all vertices have degree 2 or 0.)

A matroid on E may be alternatively defined using any one of these three notions just defined as primitive:

Definition 6.6. (Circuits) *Let E be a finite set. A matroid M on E is a pair (E, \mathcal{C}) where \mathcal{C} is a non-empty collection of subsets of E with the properties:*

- (i) No member of \mathcal{C} contains another (\mathcal{C} is an antichain),
- (ii) If $C_1, C_2 \in \mathcal{C}$ are distinct and $e \in C_1 \cap C_2$ there exists $C_3 \in \mathcal{C}$ such that $C_3 \subset C_1 \cup C_2$ and $e \notin C_3$.

Definition 6.7. (Rank function) *Let E be a finite set. A matroid M on E is a pair (E, ρ) where ρ is a function defined on subsets of E with the following properties:*

- (i) $\rho(A) \leq |A|$ is a non-negative integer;
- (ii) if $A \subseteq B \subseteq E$ then $\rho(A) \leq \rho(B)$ (ρ is monotone), and $\rho(\{x\}) \leq 1$;
- (iii) (Semimodularity) For any $A, B \subseteq E$,

$$\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B).$$

Definition 6.8. (Bases) *Let E be a finite set. A matroid M on E is a pair (E, \mathcal{B}) where \mathcal{B} is a non-empty collection of subsets of E with the properties:*

- (i) No member of \mathcal{B} contains another (\mathcal{B} is an antichain);

- (ii) (Steinitz-MacLane Exchange Lemma) If $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$ then there is $f \in B_2 \setminus B_1$ such that $B_1 \setminus \{e\} \cup \{f\} \in \mathcal{B}$.

For more about matroids see for example Peter Cameron's notes at www.maths.qmul.ac.uk/~pjc/comb/matroid.pdf (see in particular Section 3, in which the connection between the success of the greedy algorithm and bases of a matroid is explained), and for even more the books [88] and [68].

Question 33

- (i) Prove that for any graph $G = (V, E)$ the collection \mathcal{C} of all subsets of E that form a circuit in G forms a matroid on E .

What are the independent sets of this matroid? What is the corresponding rank function? And what are the bases?

Matroids which are defined in this way are called *graphic matroids*, usually denoted by $M(G)$. If $M(G)$ is defined by the cycles of G then we also speak about the *cycle matroid* of G .

- (ii) Let E be a finite (multi)set of vectors (in a vector space). Subsets of E will be independent if they are linearly independent. Bases and rank function (defined as the dimension of the space generated by the set) have a clear meaning from linear algebra.

What are the circuits?

Matroids which are defined in this way are called *linear* or *representable matroids*. If the vector space is over field \mathbb{F} then the matroid is \mathbb{F} -*representable*, and a *binary matroid* is one that is \mathbb{Z}_2 -representable.

- (iii) Prove that the lines of the Fano plane (see Figure 14) form the circuits of a matroid (called the *Fano matroid* and denoted by *Fano*). Alternatively, declare a set of points to be independent if it does not contain a line and prove that this defines a matroid (the same one of course). Is *Fano* a binary matroid?

- (iv) Let D be a digraph and S and T subsets of its vertices (not necessarily disjoint). Show that M defined on T is a matroid when it is stipulated that $I \subseteq T$ is independent if there exists a set of vertex-disjoint paths each starting in S and whose endpoints are exactly I . (This is a *gammoid*; a *strict gammoid* is the case where $S = T$ comprises all vertices of D .)

When are two matroids on sets E and E' isomorphic? One obvious thing to demand is a bijection $\iota : E \rightarrow E'$ such that, for every $A \subseteq E$, (i) $\iota(A)$ is a circuit iff A is a circuit,

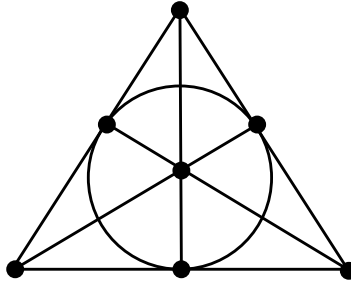


Figure 14: The Fano plane, consisting of 7 points and 7 lines, each containing 3 points. One line is represented as a circle.

(ii) $\iota(A)$ is independent iff A is independent, (iii) $\iota(A)$ is a basis iff A is a basis, and (iv) $\rho(A) = \rho(\iota(A))$.

But when do two graphs have isomorphic cycle matroids? This is more interesting and it leads to the following notion:

Definition 6.9. *Two graphs G and G' are 2-isomorphic if G can be transformed into G' by means of the following two operations and their inverses:*

- (i) *Identify two vertices in different connected components of G ;*
- (ii) *Suppose G is obtained from disjoint graphs G_1 and G_2 by identifying the vertices u_1 of G_1 and u_2 of G_2 , and identifying v_1 of G_1 and v_2 of G_2 . The Whitney twist of G is the graph obtained by identifying u_1 with v_2 and u_2 with v_1 .*

The first operation joins two components in a 1-cut (its inverse separating a graph at a 1-cut). The Whitney twist acts by flipping the graph G about one of its 2-cuts, and is illustrated in Figure 15.

Question 34 Suppose that the graphs G_1 and G_2 are connected planar graphs in Figure 15. Let G be the graph obtained by identifying u_1 with v_1 and u_2 with v_2 , and G' the Whitney twist of G . Describe how $(G')^*$ is related to G^* in terms of the graphs G_1^* and G_2^* .

Proposition 6.10. *If two graphs are 2-isomorphic then their cycle matroids are isomorphic*

Proof. Clearly the edge set of cycles are unchanged when identifying two vertices in different components. Suppose G' is obtained from G by a Whitney twist about a given 2-cut of G . A cycle that does not pass through either vertex of the 2-cut remains unchanged. A cycle of G passing through one of the vertices of the 2-cut must pass through the other. If traversing this cycle we encounter the edges $e_1, e_2, \dots, e_i, f_1, f_2, \dots, f_j$, where the e -edges belong to G_1 and the f -edges to G_2 , then in the Whitney twist corresponds the cycle in

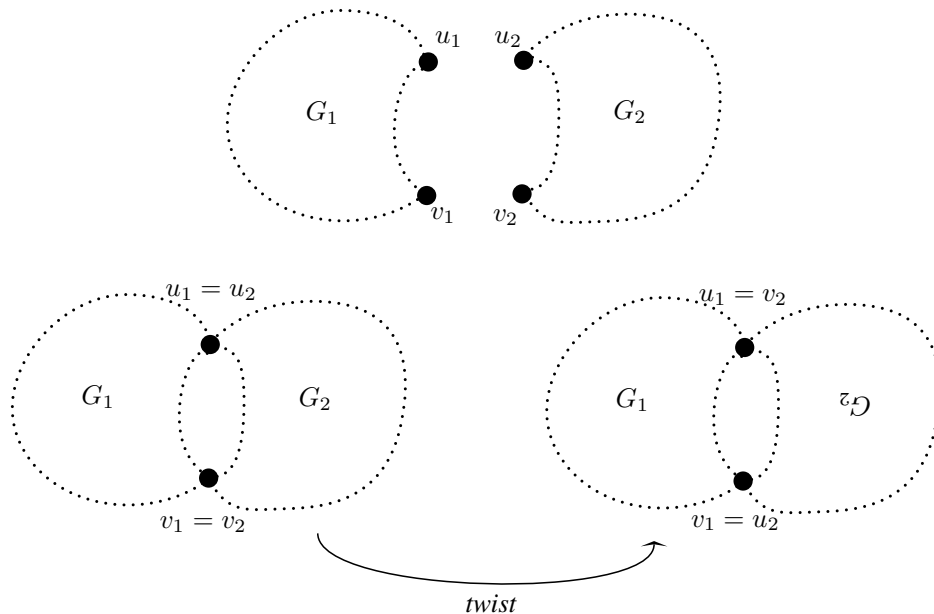


Figure 15: The Whitney twist of a graph about a 2-cut.

whose traversal we meet the edges in the order $e_1, \dots, e_i, f_j, f_{j-1}, \dots, f_1$. Thus the edge sets of cycles are the same in both graphs. \square

Theorem 6.2. Whitney [92] *The cycle matroids of G and G' are isomorphic if and only if G and G' are 2-isomorphic. In particular, if G is 3-connected and G has isomorphic cycle matroid to G' then G and G' are isomorphic.*

Geometric duals of different embeddings of a plane graph G are 2-isomorphic, although they may not be isomorphic when G is not 3-connected.

6.5 Dual matroids

Consider a graph $G = (V, E)$ and its cycle matroid $M(G)$. The independent sets of $M(G)$ correspond to the edge sets of spanning forests of G , the bases to maximal spanning forests of G . Thus the rank of the set E is $|V| - c(G)$ where $c(G)$ is the number of components of G and any basis of $M(G)$ has rank $|V| - c(G)$.

A maximal set of edges not containing any circuit (basis of G) is a maximal spanning forest of G . Circuits and cutsets are in dual correspondence for planar graphs. The dual notion of a basis for G is a maximal set of edges not containing any cutset of G , which is precisely the complement of a maximal spanning forest of G . Each of these sets has size $|E| - |V| + c(G)$.

Definition 6.11. *Let $M = (E, \mathcal{B})$ be matroid given by its bases. Then the dual matroid M^* is given by (E, \mathcal{B}^*) , where $\mathcal{B}^* = \{E \setminus B; B \in \mathcal{B}\}$.*

Question 35 Prove that M^* is indeed a matroid and that $M^{**} = M$.

Lemma 6.12. Let M be a matroid on E with rank function ρ and M^* the dual of M , with rank function ρ^* . For $A \subseteq E$, let $A^* = E \setminus A$. Then,

$$|A^*| - \rho^*(A^*) = \rho(E) - \rho(A),$$

and

$$\rho^*(E) - \rho^*(A^*) = |A| - \rho(A).$$

Proof. Let I be a maximal independent subset of A in M , and $I \cup J$ a basis ($J \subseteq A^*$ by maximality). Set $K = A^* \setminus J = E \setminus (A \cup J) \subseteq E \setminus (I \cup J)$. The set K is an independent subset of A^* (since $E \setminus (I \cup J)$ is a basis of M^*). We then have

$$\rho^*(A^*) \geq |K| = |A^*| - |J| = |A^*| - \rho(E) + \rho(A),$$

with $\rho(E) - \rho(A)$ independent elements in J . Dually,

$$\rho(A) \geq |A| - \rho^*(E) + \rho^*(A^*).$$

But $|A| + |A^*| = |E| = \rho(E) + \rho^*(E)$, so the two inequalities are in fact equalities. \square

The rank function ρ^* of the dual matroid is thus given in terms of the rank function ρ for M by

$$\rho^*(A) = |A| - \rho(E) + \rho(E \setminus A).$$

The dual of a graphic matroid need not be a graphic matroid.

Question 36

- (i) Let the rank function be given by $\rho(A) = \min(|A|, r)$ (r a fixed positive integer). Determine the bases and circuits of the corresponding matroid. (This matroid called *uniform* matroid and it is denoted by U_m^r , where $m = |E|$.) What is its dual?
- (ii) A matching in a graph is a subset of pairwise disjoint edges. Do matchings form a matroid on a set of edges?

6.6 Deletion and contraction

A *loop* of a matroid M is an element e such that $\{e\}$ is not independent (i.e., $\rho(\{e\}) = 0$), equivalently e which lies in no independent set, or in no maximal independent set (basis).

Dually, a *coloop* is an element e contained in every basis of M . A coloop in a connected graph is an edge whose removal disconnects the graph. (Such an edge is commonly called a *bridge* or *isthmus*.)

Let M be a matroid on a set E given by its set of circuits \mathcal{C} . For e not a loop, denote by \mathcal{C}' those sets in \mathcal{C} not containing e and for e not a coloop by \mathcal{C}'' sets of the form $C \setminus \{e\}$ where the circuit C contains e . It is easy to see that both sets $\mathcal{C}', \mathcal{C}''$ satisfy the axioms for the circuits of a matroid.

This matroid defined by \mathcal{C}' is the matroid obtained by *deletion* of e (or *restriction* to $E \setminus \{e\}$), denoted by $M \setminus e$. For \mathcal{C}'' the matroid is that obtained by *contraction* of e , denoted by M/e .

If M is the cycle matroid of a graph $G = (V, E)$ then, for an edge $e \in E$, $M \setminus e$ and $M \setminus A$ are the cycle matroid of the graph $G \setminus e$. The matroid M/e is the cycle matroid of the graph G/e obtained by contraction of edge e .

It is intuitively clear (but involves the Jordan Curve Theorem) that if G is a plane graph then contraction of an edge e in G corresponds to deletion of edge e^* in the dual graph G^* and that the deletion of an edge e in G corresponds to contraction of edge e^* in G^* . This duality holds in general for matroids and their duals:

Proposition 6.13. *(i) e is a loop in M if and only if e is a coloop in M^* , and vice versa.*

(ii) If e is not a loop then $(M/e)^ \cong M^* \setminus e$.*

(iii) If e is not a coloop then $(M \setminus e)^ \cong M^*/e$.*

Proof. The element e lies in every basis of M (i.e., e is a coloop of M) if and only if it lies in no basis of M^* (i.e., e is a loop of M^*), and dually.

Suppose that e is not a loop of M . The bases of M/e are the bases of M containing e with e removed. The complement of such a basis in $E \setminus \{e\}$ is a basis of M^* not containing e , which is to say a basis of $M^* \setminus e$. So $(M/e)^* \cong M^* \setminus e$. Statement (iii) is proved dually. \square

Definition 6.14. *A matroid M' is a minor of matroid M if M' can be obtained from M by a sequence of contractions and deletions, which is denoted by $M' \prec M$.*

Using the above-mentioned facts we see that M' is a minor of M if and only if M'^* is a minor of M^* . Matroid minors feature in beautiful characterization theorems such as the following:

Theorem 6.3. *A matroid M can be represented by linear independence over the field \mathbb{Z}_2 (M is a binary matroid) if and only if U_4^2 fails to be a minor of M .*

Theorem 6.4. *A matroid M is the cycle matroid of a graph if and only if U_4^2 , Fano and Fano* fail to be minors of M and the cycle matroids of $K_{3,3}$ and K_5 fail to be minors of the dual matroid M^* .*

Theorem 6.5. *A matroid M is representable by linear independence of vectors over every finite field (such matroids are called regular) if and only if U_4^2 , Fano and Fano* fail to be minors of M .*

(The dual of the Fano matroid ought not be confused with its geometric dual when represented by a drawing in the plane.)

It follows that every graphic matroid (cycle matroid) is regular, but this can be seen much more easily directly. Another equivalent condition for regular matroids is that they can be represented by linear independence of columns of a totally unimodular matrix (a matrix whose square submatrices all have determinants equal to 0, 1, or -1). *Rota's conjecture* is that matroids representable over a fixed finite field are characterized by finitely many forbidden minors. In contrast with this, matroids in general and even matroids representable by independence of real vectors are not characterized by finitely many forbidden minors.

Planar graphs have a matroid characterization. In fact the connection between planar graphs (and thus to the Four Colour Conjecture) and linear algebra was a motivation for the concept of a matroid in the 1930s (H. Whitney, B. L. Van der Waerden).

Theorem 6.6. *A graph G is planar if and only if the dual matroid $M^*(G)$ of the cycle matroid $M(G)$ is graphic.*

Put otherwise, G is planar if and only if it has a dual graph. For duality of general graphs we need matroids. (Here the matroid *dual* is distinct from the *geometric dual*, defined for a 2-cell embedding of a graph in surfaces of arbitrary genus – genus 0 is the case of plane embedding, or, equivalently, embedding in the 2-dimensional sphere.)

7 Connections to knot theory

7.1 The medial of a plane graph

To form the medial graph $m(G)$ of a connected plane graph G that has at least one edge first place a vertex v_e into the interior of each edge e of G . Then, for each face F of G , join v_e and v_f by an edge lying in F if and only if the edges e and f are consecutive on the boundary of F . The medial graph $m(G)$ is 4-regular, as each face creates two adjacencies for each edge on its boundary. The faces of $m(G)$ divide naturally into two types: those that contain vertices of G (vertex-faces), and those corresponding to faces of G (face-faces). Vertex-faces will be coloured black and face-faces coloured white. See Figure 16.

If G^* is the planar dual of G then $m(G^*) \cong m(G)$ (if $e \mapsto e^*$ is the duality mapping between edges of G and edges of G^* then e and f are consecutive edges of a face in G if and only if e^* and f^* are consecutive edges in a face of G^*).

The plane graph G is the *black face graph* of $m(G)$, i.e., the graph whose vertices are the black faces of $m(G)$ and whose edges join two black faces of $m(G)$ that share a vertex. The plane graph G^* is the *white face graph* of $m(G)$. The embedding of $m(G^*)$ differs from

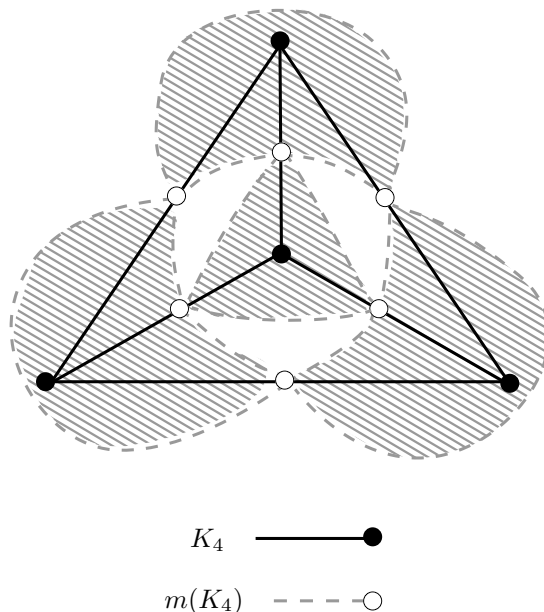


Figure 16: K_4 and its medial graph, with faces containing vertices of G shaded black.

that of $m(G)$ in having a different outer face: black faces in one (vertices of G) become white faces in the other (faces of G).

Forming the black face graph is inverse to the medial construction. A 4-regular connected plane graph H has bipartite dual graph so we can always 2-colour the faces of H properly with colours black and white, making the exterior face white. If $G(H)$ is the black face graph of H then $m(G(H)) = G$.

Question 37 Describe how to construct the graph $m(G)^*$. What type of graph is it?

7.2 Eulerian tours of digraphs

Definition 7.1. Let D be a digraph. An Euler cycle of D is a closed trail in D , i.e., a closed walk in which each edge of D is traversed at most once. An Euler tour⁹ of D is a closed trail that traverses all the edges of D , i.e., an Euler cycle in which each edge of D is traversed exactly once. In other words, a walk $v_1, e_1, v_2, e_2, \dots, v_m, e_m, v_1$ where e_1, \dots, e_m

⁹In [2] the term *Euler cycle* is used, for both Euler tour and Euler cycle. In [4] the term *Euler circuit* is used both for *Euler tour* and *Euler cycle*, the distinction not being made between tour of the graph and tour of a subgraph. The use of the word *cycle* for digraphs has been chosen here to correspond with that of its use for graphs (in which a circuit is minimal dependent set of edges, i.e., a closed path): the underlying graph of an Euler cycle of D is a cycle (Eulerian subgraph) of the underlying undirected graph of D .

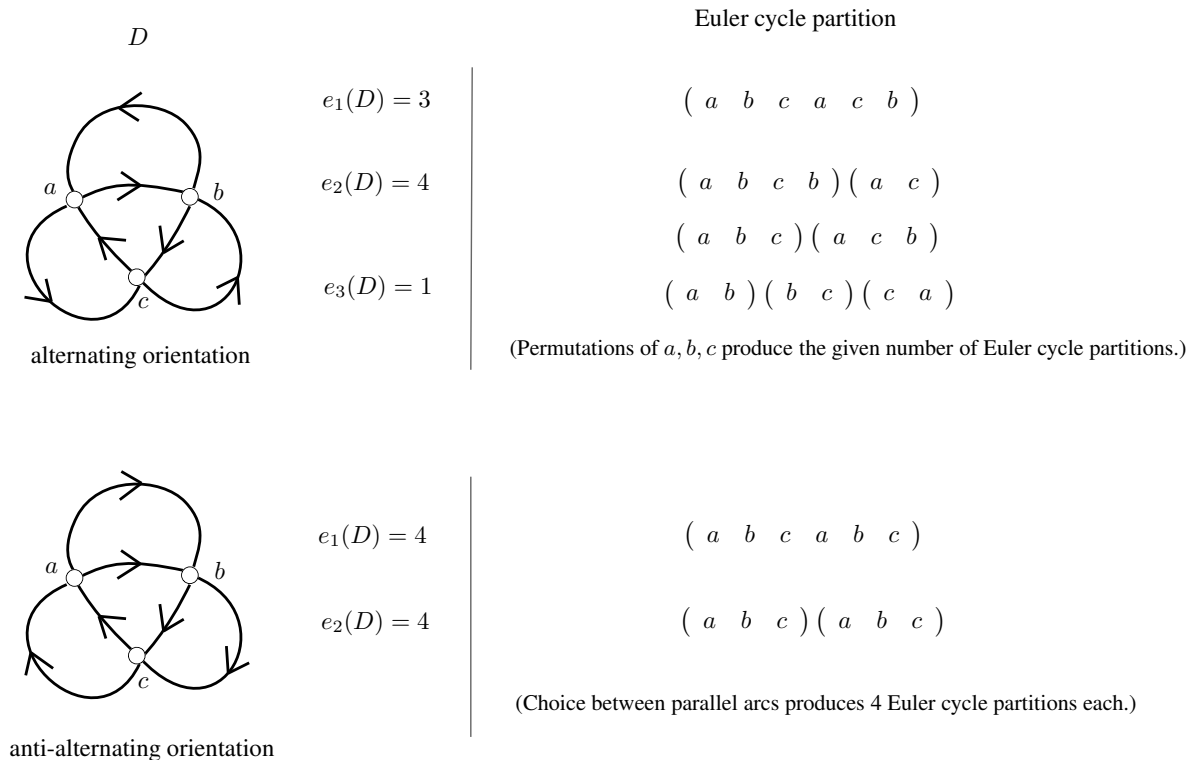


Figure 17: Two orientations of $m(K_3)$ and their Euler cycle partitions, in which the Euler cycles are presented as cyclic vertex words. An Eulerian orientation of 4-regular graph is *alternating* if at each vertex incoming edges alternate with outgoing edges, and *anti-alternating* if at each vertex incoming edges do not alternate with outgoing edges.

comprises a list of all the edges of D with no repetitions, and e_i is the edge directed from v_i to v_{i+1} (in which $v_{m+1} = v_1$).

Definition 7.2. An Euler cycle k -partition of D is a partition of the edges of D into k non-empty Euler cycles C_1, \dots, C_k . The number of Euler cycle k -partitions of D is denoted by $e_k(D)$. (Thus $e_1(D)$ is the number of Euler tours of D .)

For an Euler tour C , let $\mathbf{ve}(C) = (v_1 e_1 v_2 e_2 \dots v_m e_m)$ denote the cyclic word comprising vertices and edges visited when traversing C (tours and cycles are considered equivalent up to starting point). The word $\mathbf{ve}(C)$ has the property that vertex v appears exactly $d^+(v)$ times and each edge appears exactly once.

Similarly, we denote by $\mathbf{v}(C) = (v_1 v_2 \dots v_m)$ the cyclic word comprising vertices in the order visited when traversing C . The word $\mathbf{v}(C)$ has the property that vertex v appears exactly $d^+(v)$ times and each edge appears exactly once.

If there are no parallel directed edges in D then $\mathbf{v}(C)$ determines $\mathbf{ve}(C)$ uniquely.

Given an Euler tour C , the digraph D is uniquely determined by C . The vertex-edge word $\mathbf{ve}(C)$ contains not only the vertex-edge incidences and edge directions that determine

D , but also the fact that this is an Euler tour, so C in fact contains more information than the adjacency matrix for D . For given C , we thus write $D = D(C)$ to emphasize that the digraph D is determined by C .

A digraph D has an Euler tour if and only if each vertex has the same indegree as outdegree, $d^+(v) = d^-(v)$, one of the originating results of graph theory, due to Euler [24]. The vertex v is encountered exactly $d^+(v)$ times in traversing an Euler tour of D .

The algorithms of Fleury and Hierholzer for constructing Euler tours of a graph G , described in Section 6.1 above, suggest two ways to think of an Euler tour of a digraph D .

The first (corresponding to Hierholzer's algorithm) is to construct C by gluing together Euler cycles C_1, \dots, C_k of D , where C_1 is arbitrary, and C_{i+1} is a cycle that uses a vertex appearing in at least one of the cycles C_1, \dots, C_i . In other words, the cycles of an Euler k -partition of D can be glued together to form an Euler tour of D . Given two cyclic vertex words $(v \ x)$ and $(v \ y)$, where x and y are sequences of vertices, representing Euler cycles C and C' , the cyclic vertex word $(v \ x \ v \ y)$ represents the cycle obtained by gluing C and C' together, i.e., first traversing C starting at v and then, upon returning to the vertex v , following the cycle C' . By iterating this gluing procedure, all k cycles in an Euler k -partition of D glue together to form an Euler tour of D .

The second (corresponding to Fleury's algorithm) arises by fixing a starting point u for a given Euler tour and marking the last out-edge traversed from vertex $v \neq u$ before returning to u for the last time. These edges together form a spanning tree of D in which every vertex $v \neq u$ is connected to u by a directed path, in other words, a *spanning arborescence* of D rooted at u . Conversely, given a spanning arborescence T of D rooted at u , an Euler tour can be traversed by first freely choosing one of $d^+(u)$ out-edges of u , and then at each vertex v freely choosing any out-edge not on T , as long as there are any, and only when the edge on T remains taking it. This gives, for each spanning arborescence T rooted at u ,

$$d^+(u)! \prod_{v \neq u} (d^+(v) - 1)!$$

Euler tours starting at a given outedge from u , i.e.,

$$\prod_v (d^+(v) - 1)!$$

Euler tours. Moreover, an Euler tour corresponding to spanning arborescence T rooted at u cannot equal an Euler tour corresponding to a different spanning arborescence T' rooted at u .

By the Matrix Tree Theorem for digraphs, the number of spanning arborescences of D rooted at u is given by

$$t_u(D) = \det L_{V \setminus \{u\}},$$

where L is the Laplacian matrix $L = \Delta - A$, in which A is the adjacency matrix of D , with (v, w) entry equal to the number of directed edges from v to w , and Δ is the diagonal matrix with (v, v) entry equal to $d^+(v)$, and $L_{V \setminus \{u\}}$ is the matrix L with row and column indexed by u removed. (Alternatively, $t_u(D) = \frac{1}{n} \lambda_1 \cdots \lambda_{n-1}$, where $\lambda_1, \dots, \lambda_{n-1}$ are the non-zero

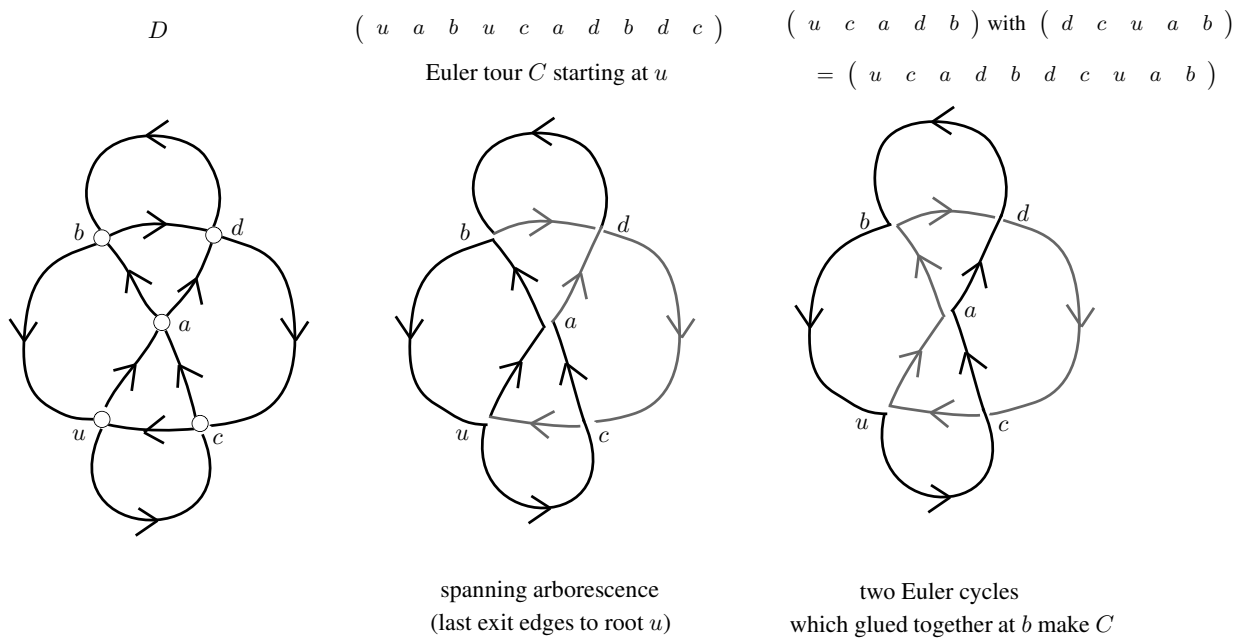


Figure 18: Euler tour C of 2-in 2-out digraph D starting from u corresponds to spanning arborescence rooted at u , whose edges are the second out-edge taken from $v \neq u$ when traversing C . The tour C may also be constructed by gluing together Euler cycles (illustrated here is one possibility, gluing the two given cycles at vertex b , which produces a shifted version of C).

eigenvalues of L .) It is perhaps surprising that the number of spanning arborescences of D does not depend on the root u .¹⁰

Theorem 7.3. (“BEST Theorem”, [1, 69]) *The number of Euler tours of digraph $D = (V, E)$ is given by*

$$e_1(D) = t_u(D) \prod_{v \in V} (d^+(v) - 1)!,$$

where u is an arbitrary vertex of D and $t_u(D)$ the number of spanning arborescences of D rooted at u .

7.3 2-in 2-out digraphs

From now on D will be a *2-in 2-out digraph*, i.e., each vertex has indegree 2 and outdegree 2. If D has n vertices then it has $2n$ edges.

For an Euler tour C of 2-in 2-out digraph D , the word $\text{ve}(C)$ has the property that each vertex occurs exactly twice and each edge exactly once. For an Euler cycle the corresponding cyclic vertex-edge word has the property that each vertex occurs at most twice and each edge at most once. Similarly, the cyclic vertex word $\mathbf{v}(C)$ has the property that each vertex occurs exactly twice and for an Euler cycle the corresponding cyclic vertex word has the property that each vertex occurs at most twice.

For a 2-in 2-out digraph D there is a one-one correspondence between Euler tours of D and spanning arborescences of D rooted at a fixed vertex u . See Figure 18. However, we shall concentrate on the viewpoint of Euler cycle k -partitions (whose cycles glued together form an Euler tour).

Given an undirected graph $G = (V, E)$, an Eulerian orientation of G is an orientation of the edges with the property that each vertex has as many incoming edges as outgoing edges: $d^-(v) = d^+(v)$ for each $v \in V$. When orienting edges according to a traversal of an Euler tour of G the result is an Eulerian orientation. Counting the number of Eulerian orientations of G is $\#\mathcal{P}$ -complete [62], even for plane 4-regular graphs [29]. (The situation is quite different for a given digraph D , where by Theorem 7.3 counting Eulerian tours of D can be done in polynomial time.)

Proposition 7.4. *If G is a connected 4-regular graph on n vertices then G has $(-1)^{n-1}T(G; 0, -2)$ Eulerian orientations.*

Note that this does not extend to graphs G generally: the number of Eulerian orientations of G is only given by this evaluation of the Tutte polynomial when all its vertex degrees belong to $\{0, 1, 2, 4\}$.

PROOF. The easiest way to prove this relies on knowing what a nowhere-zero \mathbb{Z}_3 -flow is and establishing that $(-1)^{n-1}T(G; 0, -2) = F(G; 3)$ is the number of nowhere-zero \mathbb{Z}_3 -flows of

¹⁰A similar phenomenon occurs for acyclic orientations of a connected graph G with unique sink at vertex u : this turns out to be independent of u and is in fact given by the Tutte polynomial evaluation $T(G; 1, 0)$.

connected 4-regular G . (Making an inductive deletion-contraction argument and applying the Recipe Theorem for evaluations of the Tutte polynomial is thwarted by the fact that the property of being 4-regular is destroyed by edge deletion and contraction.) \square

Definition 7.5. Let C be an Euler cycle of D with cyclic vertex-edge word

$$\mathbf{ve}(C) = (x \ a \ y \ b \ x' \ a \ y' \ b),$$

in which the vertices a, b are interlaced, and x, y, x', y' are vertex-edge sequences (possibly empty). The transposition of C along a and b is the Euler cycle C^{ab} of D defined by the cyclic vertex-edge word

$$\mathbf{ve}(C^{ab}) = (x \ a \ y' \ b \ x' \ a \ y \ b).$$

Note that transposition along an interlaced pair in an Euler cycle C produced another Euler cycle of the same size; in particular, an Euler tour upon transposition becomes another Euler tour.

If the cyclic vertex word for C is $\mathbf{v}(C) = (x \ a \ y \ b \ x' \ a \ y' \ b)$, then that for C^{ab} is $\mathbf{v}(C^{ab}) = (x \ a \ y' \ b \ x' \ a \ y \ b)$. If D has no parallel edges then transposition can be defined in terms of vertex words rather than vertex-edge words, because in this case each vertex word $\mathbf{v}(C)$ uniquely determines the vertex-edge word $\mathbf{ve}(C)$. For the sake of simplicity we shall work with vertex words rather than vertex-edge words, with the understanding that only minor modifications need to be made in order to incorporate the case of parallel directed edges.

Definition 7.6. Let a be a vertex of a 2-in 2-out digraph D such that $(u, a), (a, v), (u', a), (a, v')$ are the directed edges of D incident with a . (Possibly $u = v$ or $u' = v'$, corresponding to loops on a .) A transition at a is one of the two possible pairings $\{u, v\}, \{u', v'\}$ or $\{u, v'\}, \{u', v\}$. (For the first pairing, the pair of edges $(u, a), (a, v)$ and the pair of edges $(u', a), (a, v')$ both form directed paths, and similarly for the second pairing.)

Remark then that a 2-in 2-out digraph D on n vertices has 2^n Euler cycle partitions, corresponding to the independent choice of two possible transitions at each vertex of D .

Let C be an Euler cycle. When $\mathbf{v}(C) = (\dots \ u \ a \ v \ \dots \ u' \ a \ v' \ \dots)$ the transition of C at a is $\{u, v\}, \{u', v'\}$.

Given $\mathbf{v}(C) = (\dots \ u \ a \ v \ \dots \ b \ \dots \ u' \ a \ v' \ \dots \ b)$, the transposition along interlaced a and b is given by

$$\mathbf{v}(C^{ab}) = (\dots \ u \ a \ v' \ \dots \ b \ \dots \ u' \ a \ v \ \dots \ b),$$

in which the transition at a has been switched from $\{u, v\}, \{u', v'\}$ to $\{u, v'\}, \{u', v\}$.

See Figure 19.

Definition 7.7. Suppose C is an Euler cycle with $\mathbf{v}(C) = (\dots \ u \ a \ v \ \dots \ u' \ a \ v' \ \dots)$. The contraction of C by a is the Euler cycle of $D \setminus \{a\}$ denoted by $C - a$ and given by

$$\mathbf{v}(C - a) = (\dots \ u \ v \ \dots \ u' \ v' \ \dots).$$

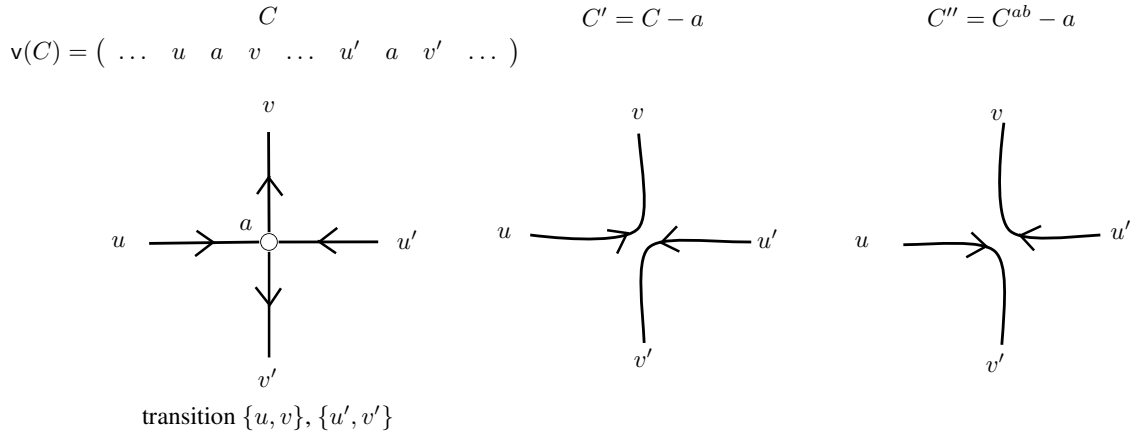


Figure 19: Transition at a vertex a , and contracting out a from an Euler tour.

Lemma 7.8. *If D has an Euler tour C with no interlaced pairs then D has only one Euler tour.*

PROOF. Induction on the number of vertices of D . The base case is where D is a single vertex with two loops, for which the assertion is true. Suppose it is true for all 2-in 2-out digraphs on $n - 1$ vertices and consider D on n vertices. By hypothesis D has an Euler tour C with no interlacements. This implies there is some vertex a with a loop. (Consider the word $\mathbf{v}(C)$ in which the two occurrences of a vertex u either enclose or are enclosed by a pair of occurrences of another vertex v , or the two occurrences are disjoint; for given vertex u , consider all the vertices that it encloses: either there are none, in which case take $a = u$, or there is a such a vertex v whose two occurrences are between those of u , and now repeat the argument with v in place of u and eventually an adjacent pair of vertices must be found.) The Euler tour $C - a$ of $D - \{a\}$ has no interlacements, hence $D - \{a\}$ has a unique Euler tour by induction hypothesis, and therefore so does D , as the only choice for an Euler tour at a is to traverse the loop between entering and leaving a . \square

Lemma 7.9. *If C and C' are Euler tours of D then there is a sequence of transpositions that transforms C to C' . In other words, the orbit of an Euler tour of D under the action of transposition along interlaced vertices is the set of all Euler tours of D .*

PROOF. The proof is by induction on the number of vertices of D . The case of a single vertex is vacuously true. Suppose the statement is true for digraphs on $n - 1$ vertices and let D be a 2-in 2-out digraph on n vertices. Suppose that D has distinct Euler tours C and C' . If there is a vertex a at which C and C' have the same transition, then by contracting at a the tours $C - a$ and $C' - a$ can be obtained one from the other by a sequence of transpositions, which upon reinserting a means the same is true of C and C' .

Suppose then that C and C' have different transitions at all vertices. Then, since C and C' are distinct, by Lemma 7.8 there is an interlaced pair a and b in C . The Euler tour

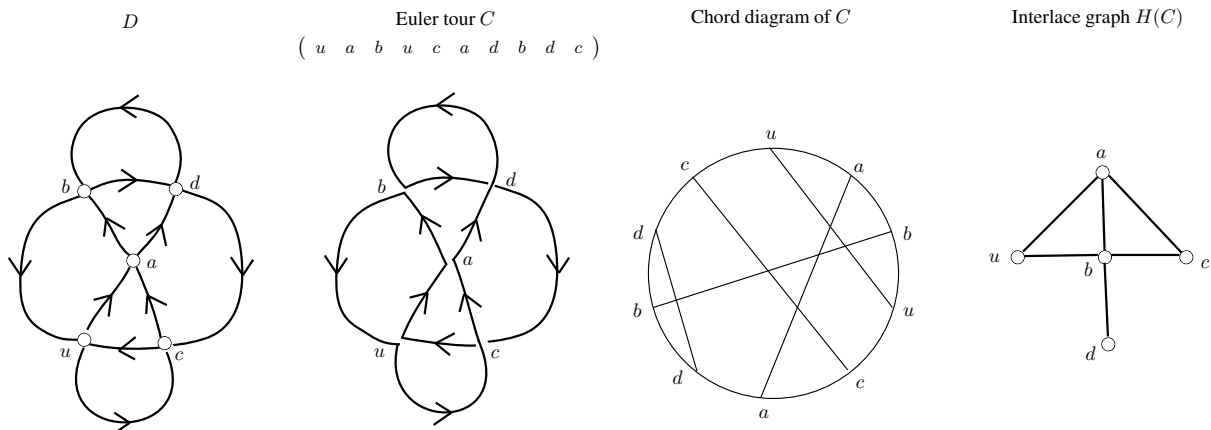


Figure 20: A 2-in 2-out digraph D with Euler tour C , its chord diagram, and the interlace graph of C , equal to the intersection graph of the chord diagram.

C^{ab} then has the same transition as C' at vertex a , and the previous argument shows that C^{ab} can be obtained from C' by a sequence of transpositions, and hence the same is true of $C = (C^{ab})^{ab}$. \square

7.4 Interlace polynomial

Definition 7.10. The interlace graph $H(C)$ of an Euler tour of 2-in 2-out digraph D is defined on the vertex set of D traversed by C in which vertices a and b are adjacent if a and b are interlaced in C , i.e., if the cyclic vertex word of C takes the form

$$v(C) = (\dots \ a \ \dots \ b \dots \ a \dots \ b \ \dots).$$

The interlace graph of C is the intersection graph of the *chord diagram* of C , in which the vertices of the cyclic vertex word of C are placed around a circle and each pair of like vertices is joined by a chord. This type of intersection graph is known as a *circle graph*. See Figure 20.

Question 38 Explain why the 5-wheel (the graph on six vertices formed by joining each vertex of a 5-cycle to a central vertex) is not a circle graph. (This is the smallest example of a graph that is not a circle graph.)

For a vertex a in graph H denote by $N(a)$ its open neighbourhood $\{c \in V(H) : ac \in E(H)\}$ and by $N[a]$ its closed neighbourhood $N(a) \cup \{a\}$.

Lemma 7.11. Let a, b be interlaced in an Euler tour C of 2-in 2-out digraph D and C^{ab} the Euler tour of D obtained by transposition along a and b . Then the interlace graph $H(C^{ab})$ is obtained by applying the following two operations to $H(C)$:

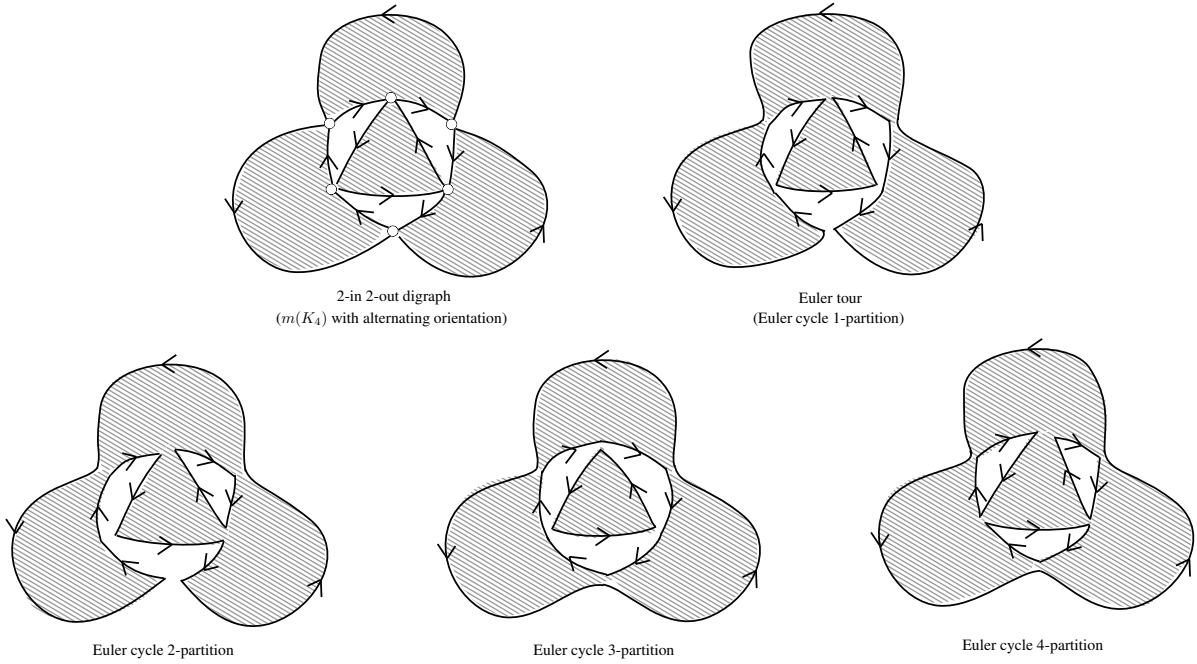


Figure 21: Canonical alternating orientation of $m(K_4)$ and examples of Euler cycle k -partitions for $k = 1, 2, 3, 4$.

- (i) Switch along ab : toggle adjacencies between $N(a) \cap N(b)$, $N(a) \setminus N[b]$ and $N(b) \setminus N[a]$ (but not within these sets).
- (ii) Swap a and b , i.e., ac is an edge in $H(C^{ab})$ if and only if bc is an edge in $H(C)$, and bc is an edge in $H(C^{ab})$ if and only if ac is an edge in $H(C)$.

PROOF. Case analysis. See [4]. \square

See Figure 22.

Denote the graph obtained from H by switching along (adjacent) vertices a and b by H^{ab} , and the graph obtained from H by swapping a and b by H_{ab} . Then Lemma 7.11 says that for interlaced vertices a and b we have $H(C^{ab}) = (H(C)^{ab})_{ab}$. Since we shall only be concerned with the interlace graph $H(C)$ up to isomorphism, it suffices to work with the fact that $H(C^{ab}) \cong H(C)^{ab}$. Note also that $(H^{ab})^{ab} = H$. We say two simple graphs H and H' are *switching equivalent* if there is a sequence of switches transforming one into the other. By Lemma 7.11 the interlace graphs of Euler tours C and C' of a 2-in 2-out digraph D are switching equivalent.

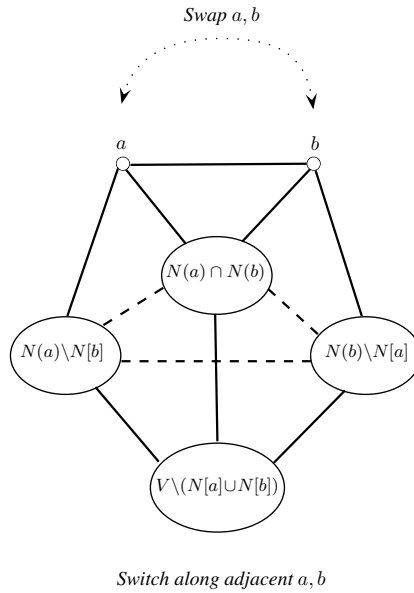


Figure 22: Switching along ab and then swapping a and b transforms $H(C)$ to $H(C^{ab})$. The dashed lines indicate where adjacencies need to be toggled. Remaining adjacencies in $H(C)$ are preserved.

Question 39

- (i) Which 2-in 2-out digraphs have an Euler tour with empty interlace graph \overline{K}_n ?
- (ii) Show that K_n is unaffected by switching along an edge. Which digraphs D have an Euler tour with interlace graph K_n ?

Define the function q_k on interlace graphs by $q_k(H(C)) = e_k(D(C))$, where C is an Euler tour of D . We have $q_1(K_1) = 1 = q_2(K_1)$ and $q_k(K_1) = 0$ for $k \geq 2$ since $D(C)$ in this case is the digraph on a single vertex with two loops.

Lemma 7.12. *The function q_1 satisfies the recurrence*

$$q_1(H(C)) = q_1(H(C) \setminus a) + q_1(H(C)^{ab} \setminus b),$$

when ab is an edge of $H(C)$, and $q_1(H(C)) = 1$ when $H(C)$ has no edges (C is the unique Euler tour of D).

PROOF. If $H(C)$ has no edges then by Lemma 7.8 there is a unique Euler tour and $q_1(H(C)) = 1 = e_1(D(C))$.

Otherwise, suppose ab is an edge of $H(C)$. Referring to Figure 19, let $D = D(C)$ be the 2-in 2-out digraph determined by C , $D' = D(C - a)$ that determined by the contraction of C at a and $D'' = D(C^{ab} - a)$ that determined by the contraction of C^{ab} at a . Since transposition at ab switches the transition at a , and $C \mapsto C^{ab}$ is a bijection on Euler tours of D , partitioning tours according to their transition at a we have

$$e_1(D(C)) = e_1(D(C - a)) + e_1(D(C^{ab} - a)).$$

The interlace graph of $D(C - a)$ is $H(C) \setminus a$ and the interlace graph of $D(C^{ab} - a)$ is $H(C)^{ab} \setminus b$ (where b is the vertex deleted since $H(C^{ab}) = (H(C)^{ab})_{ab}$ involves a swap of a and b which is not carried out in just the switch $H(C)^{ab}$). We have then $e_1(D(C - a)) = q_1(H(C) \setminus a)$ and $e_1(D(C^{ab} - a)) = q_1(H(C)^{ab} \setminus b)$ and the statement of the lemma is now proved. \square

Question 40 Prove that the function q_k , $k \geq 1$, satisfies the recurrence

$$q_k(H(C)) = q_k(H(C) \setminus a) + q_k(H(C)^{ab} \setminus b),$$

where C is an Euler tour of 2-in 2-out digraph D and a, b are interlaced in C .

Lemma 7.12, and the fact that the switching operation is defined on any simple graph, not just interlace graphs, prompted Arratia, Bollobás and Sorkin [4] to postulate the existence of a polynomial $Q(H; x)$ defined on simple graphs H as follows:

Definition 7.13. The interlace polynomial $Q(H; x)$ of a simple graph $H = (V, E)$ is defined by the recurrence

$$Q(H; x) = \begin{cases} Q(H \setminus a; x) + Q(H^{ab} \setminus b; x) & ab \in E \\ x^{|V|} & E = \emptyset. \end{cases}$$

Example 7.14. Take $H = K_n$ and edge ab , for which $K_n \setminus a \cong K_n^{ab} \setminus b \cong K_{n-1}$. By the defining recurrence for $Q(K_n; x)$ we have, by induction on n , $Q(K_n; x) = 2^{n-1}x$.

Note that $Q(H; 1) = q_1(H)$ when $H = H(C)$ is the interlace graph of an Euler tour C . The authors of [4] prove that the order of edges ab chosen in the switching and vertex-deletion recurrence defining $Q(H; x)$ does not affect the resulting polynomial, i.e., that the polynomial $Q(H; x)$ is well-defined. This is analogous to the situation for the recurrence defining the Tutte polynomial, where we have independence of the order of edge deletions and contractions. Also analogous to the case of the Tutte polynomial, it is possible to circumvent this somewhat tedious verification by producing a bona fide polynomial that does indeed satisfy the given recurrence.

Theorem 7.15. *The interlace polynomial of a simple graph $H = (V, E)$ is given by the induced subgraph expansion*

$$Q(H; x) = \sum_{U \subseteq V} (x - 1)^{|U| - \text{rk}(A_U)},$$

where A is the adjacency matrix of H , A_U its restriction to rows and columns indexed by U , and $\text{rk}(A_U)$ the rank of the matrix A_U over \mathbb{F}_2 (where $\text{rk}(A_\emptyset) = 0$ by fiat).

PROOF. See [2, Ch. 9]. \square

Proposition 7.16. *Switching equivalent graphs have the same interlace polynomial.*

PROOF. For edge ab of H ,

$$\begin{aligned} Q(H^{ab}; x) &= Q(H^{ba}; x) \\ &= Q(H^{ba} \setminus b; x) + Q(H \setminus a; x) \quad \text{since } (H^{ba})^{ab} = H, \\ &= Q(H; x) \end{aligned}$$

\square

Proposition 7.17. *The interlace polynomial is multiplicative over disjoint unions.*

PROOF. We wish to prove that if $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are vertex-disjoint graphs then

$$Q(H_1 \cup H_2; x) = Q(H_1; x)Q(H_2; x).$$

If neither H_1 nor H_2 has any edges then the assertion is trivial, with $Q(H_1 \cup H_2; x) = x^{|V_1|+|V_2|} = x^{|V_1|}x^{|V_2|} = Q(H_1; x)Q(H_2; x)$. Without loss of generality then, suppose $ab \in E_1$. Then, by induction on the number of vertices of $H_1 \cup H_2$, and using the fact that deleting a or b or switching on $ab \in E_1$ does not affect H_2 ,

$$\begin{aligned} Q(H_1 \cup H_2) &= Q((H_1 \setminus a) \cup H_2) + Q((H_1^{ab} \setminus b) \cup H_2) \\ &= Q(H_1 \setminus a)Q(H_2) + Q(H_1^{ab} \setminus b)Q(H_2) \\ &= Q(H_1)Q(H_2), \end{aligned}$$

\square

Question 41

- (i) Prove that $Q(H; 2) = 2^n$ for a graph H on n vertices.
- (ii) Show that if H is a forest and a a leaf of H (degree 1) attached to vertex b then $Q(H; x) = Q(H \setminus a; x) + xQ(H \setminus \{a, b\}; x)$.

Proposition 7.18. For graph $H = (V, E)$ and any $U \subseteq V$ we have $\deg Q(H) \geq \deg Q(H[U])$. In particular, $\deg Q(H) \geq \alpha(H)$, where $\alpha(H)$ is the size of the largest independent (stable) set of vertices in H .

PROOF. It suffices to prove that $\deg Q(H) \geq \deg Q(H \setminus a)$ for $a \in V$.

If a is isolated then $q(H; x) = xQ(H \setminus a)$ by multiplicativity over disjoint unions. Otherwise, for $ab \in E$, $Q(H; x) = Q(H \setminus a; x) + Q(H^{ab} \setminus b; x)$, and since the interlace polynomial of a graph has nonnegative coefficients the result follows. \square

Question 42

- (i) Prove that, more generally than the first statement of Proposition 7.18, for a connected graph H the coefficient of x^i in $Q(H[U]; x)$ is less than or equal to that of x^i in $Q(H; x)$.
- (ii) Prove that $c(H)$ (number of connected components of H) is the smallest index i for which the coefficient of x^i in $Q(H; x)$ is non-zero.

Recall that for 2-in 2-out digraph D we denote by $e_k(D)$ the number of Euler cycle k -partitions of D . Let

$$e(D; x) = \sum_{k \geq 1} e_k(D)x^{k-1}.$$

Theorem 7.19. For Euler tour C of 2-in 2-out digraph D we have

$$e(D(C); x) = Q(H(C); x + 1).$$

PROOF. The proof is by induction on the number of vertices of D (number of vertices of H).

For an interlaced pair of vertices a and b in Euler tour C of D ,

$$e_k(D(C)) = e_k(D(C - a)) + e_k(D(C^{ab} - a)),$$

since transposing C along a and b switches transition at a , and to each Euler cycle k -partition there corresponds one of two possible transitions at a (either given by an interlacement of one of the constituent Euler cycles, or by the transition obtained by taking the union of the two Euler cycles containing a). Euler cycle k -partitions of D associated with the one type of transition at a correspond bijectively to Euler cycle k -partitions of $D(C - a)$, while those with the other transition at a corresponds bijectively to Euler cycle k -partitions of $D(C^{ab} - a)$. Hence,

$$\begin{aligned} e(D; x) &= e(D(C - a); x) + e(D(C^{ab} - a); x) \\ &= Q(H(C) \setminus a; x + 1) + Q(H^{ab} \setminus b; x + 1) && \text{by induction hypothesis,} \\ &= Q(H(C); x + 1). \end{aligned}$$

If there is no interlaced pair of vertices in C , then $H(C)$ has no edges and C has loop on some vertex a (see proof of Lemma 7.8). In this case, by either keeping the loop as a separate Euler cycle or gluing it to the Euler cycle passing through a , we have

$$\begin{aligned} e(D(C); x) &= xe(D(C - a); x) + e(D(C - a); x) \\ &= (x + 1)e(D(C - a); x) \\ &= (x + 1)Q(H(C) \setminus a; x + 1) \quad \text{by induction hypothesis,} \\ &= Q(H(C); x + 1) \quad \text{by Prop. 7.17 (multiplicativity over disjoint unions).} \end{aligned}$$

□

We finish with the relationship between the interlace polynomial and the Tutte polynomial of a plane graph. First it will be useful to describe how $e(D; x)$ behaves over connected components and blocks:

Lemma 7.20. (i) *If D_1 and D_2 are 2-in 2-out digraphs on disjoint vertex sets then*

$$e(D_1 \cup D_2; x) = xe(D_1; x)e(D_2; x).$$

(ii) *If D is a 2-in 2-out digraph with cut-vertex a , C an arbitrary Euler tour of D , and D_1 and D_2 are the two connected components of the 2-in 2-out digraph $D(C - a)$, then*

$$e(D_1 \cup D_2; x) = (x + 1)e(D_1; x)e(D_2; x).$$

PROOF. For (i) we have

$$\begin{aligned} e(D_1 \cup D_2; x) &= \sum_{k \geq 2} \left(\sum_{i+j=k} e_i(D_1)e_j(D_2) \right) x^{k-1} \\ &= xe(D_1; x)e(D_2; x). \end{aligned}$$

For (ii), if a is the cut-vertex of $D_1 \cup D_2$, there are two possibilities for a given Euler cycle k -partition of $D_1 \cup D_2$: either it has a transition at a making it the union of an Euler cycle i -partition of D_1 and Euler cycle $(k - i)$ -partition of D_2 , or its transition at a joins a cycle in a Euler cycle i -partition of D_1 and a cycle in an Euler cycle $(k + 1 - i)$ -partition of D_2 . By partitioning Euler cycle k -partitions of $D_1 \cup D_2$ into these two cases, we have

$$\begin{aligned} e(D_1 \cup D_2; x) &= \sum_{k \geq 2} \left(\sum_{i+j=k} e_i(D_1)e_j(D_2) \right) x^{k-1} + \sum_{k \geq 1} \left(\sum_{i+j=k+1} e_i(D_1)e_j(D_2) \right) x^{k-1} \\ &= (x + 1)e(D_1; x)e(D_2; x). \end{aligned}$$

□

Theorem 7.21. *When G is a plane graph with medial graph $\vec{m}(G)$ given the alternating orientation which orients black faces anticlockwise,*

$$e(\vec{m}(G); x) = T(G; x + 1, x + 1).$$

Hence if C is an Euler tour of $\vec{m}(G)$ and $H(C)$ its interlace graph, then $Q(H(C); x) = T(G; x, x)$.

PROOF. We prove that $e(\vec{m}(G); x) = T(G; x + 1, x + 1)$ for plane graph $G = (V, E)$ by induction on the number of edges of G . The statement is true for $G = K_2$, where $m(G)$ is a single vertex with two loops and we have $e(\vec{m}(G); x) = 1 + x = T(K_2; x + 1, x + 1)$. (The case where $G = K_1$ is also vacuously true.)

When e is an ordinary edge of G (neither bridge nor loop),

$$\begin{aligned} T(G; x + 1, x + 1) &= T(G/e; x + 1, x + 1) + T(G \setminus e; x + 1, x + 1) \\ &= e(\vec{m}(G)/e; x) + e(\vec{m}(G \setminus e); x) \quad \text{by induction hypothesis,} \\ &= e(\vec{m}(G); x), \end{aligned}$$

the last line by the fact that (cf. proof of Theorem 7.19 and Figure 23) $\vec{m}(G/e)$ and $\vec{m}(G \setminus e)$ correspond to $\vec{m}(G)$ with vertex e contracted out according to the two possible transitions at e that an Euler tour may take.

If e is a bridge, then e is a cut-vertex in $\vec{m}(G)$ and

$$\begin{aligned} T(G; x + 1, x + 1) &= (x + 1)T(G/e; x + 1, x + 1) \\ &= (x + 1)T(G_1; x + 1, x + 1)T(G_2; x + 1, x + 1) \quad \text{blocks } G_1 \text{ and } G_2 \text{ of } G/e, \\ &= (x + 1)e(\vec{m}(G_1); x)e(\vec{m}(G_2); x) \quad \text{by induction hypothesis,} \\ &= e(\vec{m}(G); x) \quad \text{cut-vertex } e \text{ of } m(G), \text{ Lemma 7.20(ii).} \end{aligned}$$

If e is a loop, then in $\vec{m}(G)$ the vertex e is a cut-vertex with a loop. Let $\vec{m}(G)'$ denote the digraph obtained by contracting out the vertex e (see Figure 24). Then $\vec{m}(G)' = \vec{m}(G \setminus e)$ and we have

$$\begin{aligned} T(G; x + 1, x + 1) &= (x + 1)T(G \setminus e; x + 1, x + 1) \\ &= (x + 1)e(\vec{m}(G)'; x) \quad \text{by induction hypothesis,} \\ &= e(\vec{m}(G); x) \quad \text{cut-vertex } e \text{ of } m(G), \text{ Lemma 7.20(ii).} \end{aligned}$$

(The last line can also be seen by considering the two cases where the loop is separate from the other Euler cycles comprising the Euler cycle partition, or joined to an existing Euler cycle.) \square

Corollary 7.22. *If G is a connected plane graph then the number of Euler tours of $\vec{m}(G)$ is equal to the number of spanning trees of G .*

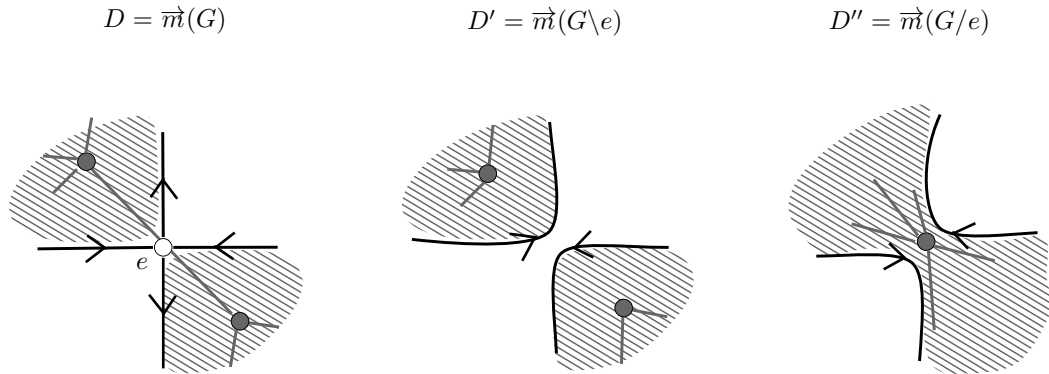


Figure 23: The two types of transition at a vertex of a plane 2-in 2-out digraph D , equal to $\vec{m}(G)$ for some plane graph G , with alternating orientation anticlockwise around black faces. The black transition corresponds to edge deletion in G , and the white transition to edge contraction.

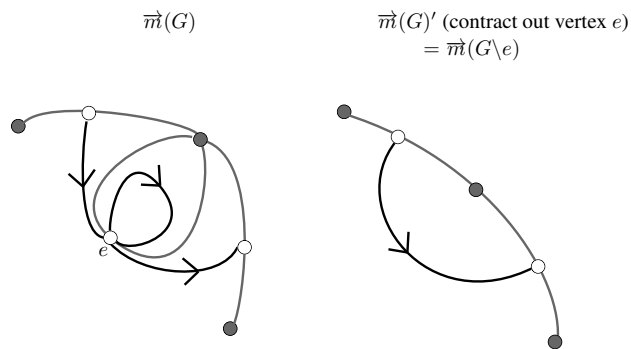


Figure 24: Contracting out a vertex with a loop in the oriented medial graph (proof of Theorem 7.21).

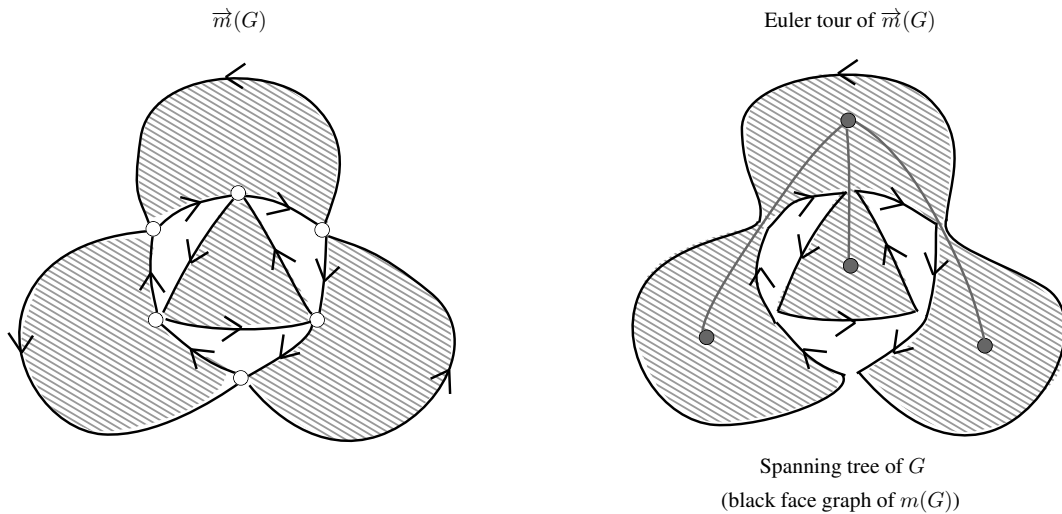


Figure 25: Correspondence between Euler tours of an alternating orientation of the medial graph of a plane graph G and spanning trees of G . A tour of the medial graph forms a simple closed curve in the plane unifying all the black faces into one region.

Question 43 Prove Corollary 7.22 directly. (See Figure 25. Also [52].)

Given a 2-in 2-out digraph D there is a unique *anticycle partition* formed by following two out-edges then two in-edges, and repeating this until all edges have been traversed (once an edge is encountered again this closes a component of the anticycle, which is a cycle which when traversed alternates in the orientation of its edges forward and backward). The number of components in this anticycle partition is denoted by $a(D)$. For the example $D = \vec{m}(K_4)$ see Figure 26, which interprets this anticycle partition as the diagram of a link (in this case three unknots linked as Borromean rings). We shall develop this connection in the next section about the Kauffman bracket and Jones polynomial.

The following evaluation of the interlace polynomial of the interlace graph of a 2-in 2-out digraph D gives another interpretation of the Tutte polynomial evaluation $T(G; -1, -1)$ when $D = \vec{m}(G)$ for plane G :

Theorem 7.23. *Let D be a 2-in 2-out digraph on n vertices with Euler tour C and $a(D)$ the number of components in its anticycle partition. Then*

$$e(D; -2) = (-1)^n (-2)^{a(D)-1}.$$

PROOF. The proof is by induction on n . When $n = 1$ the digraph D is a single vertex with two loops and we have $a(D) = 1$ and $e(D; x) = 1 + x$, so $e(D; -2) = -1 = (-1)(-2)^0$ and the base case is true.

Consider D on $n > 1$ vertices and a vertex a of D . Let D'_a denote the digraph D with a contracted out according to one possible transition for Euler cycles at a and D''_a the digraph D with a contracted out according to the other possible transition for Euler cycles at a . (Given an Euler tour C interlaced at a and b we can take $D'_a = D(C - a)$ and $D''_a = D(C^{ab} - a)$.)

If the vertex a belongs to two anticycles then $a(D'_a) = a(D''_a) = a(D) - 1$ and

$$\begin{aligned} e(D; -2) &= e(D'_a; -2) + e(D''_a; -2) \\ &= (-1)^{n-1}(-2)^{a(D)-2} + (-1)^{n-1}(-2)^{a(D)-2} && \text{by induction hypothesis,} \\ &= (-1)^n(-2)^{a(D)-1}. \end{aligned}$$

If on the other hand the vertex a belongs to a single anticycle component then $\{a(D'_a), a(D''_a)\} = \{a(D), a(D) + 1\}$ and

$$\begin{aligned} e(D; -2) &= (-1)^{n-1}(-2)^{a(D)-1} + (-1)^{n-1}(-2)^{a(D)} \\ &= (-1)^n(-2)^{a(D)-1}. \end{aligned}$$

□

Corollary 7.24. *If G is a plane graph and $\vec{m}(G)$ its medial graph with alternating orientation then*

$$T(G; -1, -1) = (-1)^{|E(G)|}(-2)^{a(\vec{m}(G))-1},$$

where $a(\vec{m}(G))$ is the number of components in the anticyle partition of $\vec{m}(G)$.

Actually, for a general graph G we have the evaluation

$$T(G; -1, -1) = (-1)^{|E(G)|}(-2)^{\dim(\mathcal{Z} \cap \mathcal{Z}^\perp)},$$

where \mathcal{Z} is the (binary) cycle space of G and \mathcal{Z}^\perp the cutset space of G . The space $\mathcal{Z} \cap \mathcal{Z}^\perp$ is called the *bicycle space* of G . See [32] and Section 3.10 above for an elucidation of the correspondence.

7.5 The Kauffman bracket of a link

Lecture notes for the relation between the Tutte polynomial and the Kauffman bracket of a link follow as a scan of handwritten notes.

1. Knots and links

KNOT : subset of \mathbb{R}^3 homeomorphic to circle S^1

LINK : several disjoint knots

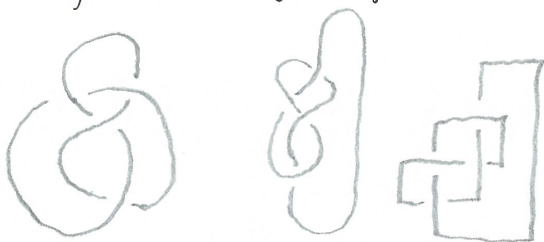
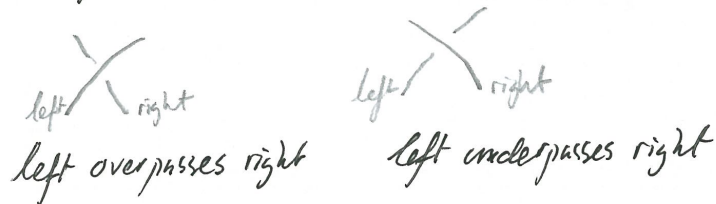
AMBIENT ISOTOPY taking knot K to knot K' :
 continuous map $h: \mathbb{R}^3 \times [0,1] \rightarrow \mathbb{R}^3$ such that h_0 is identity map and $h_1(K) = K'$

TAME KNOTS : ambient isotopic to simple closed polygons (piecewise linear simple closed curves)

REGULAR PROJECTION of knot onto plane:

- finitely many multiple points
- these points are double points (marking crossings)

DIAGRAM of a knot or link : regular projection together with underpass/overpass indicated at each double point -

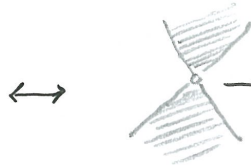


← three diagrams of Figure of Eight knot

2.

diagram D

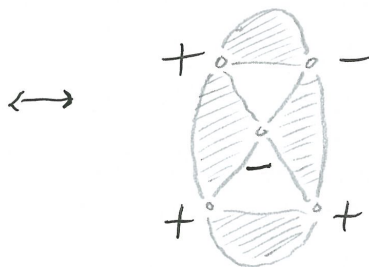
\leftrightarrow plane 4-regular \tilde{D}
face 2-coloured, vertex signs



example:



diagram D
with face 2-colouring
(outer face white)



face 2-coloured vertex-signed
plane 4-regular \tilde{D}

face 2-coloured vertex-signed
plane 4-regular \tilde{D}

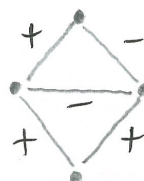
\leftrightarrow

edge-signed plane
graph G



medial

\leftrightarrow



3.

Lemma

Bijection between edge-signed plane graphs G
and link diagrams D

(G is the TAIT GRAPH of link L with diagram D)

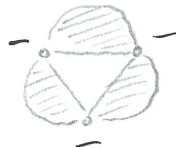
ALTERNATING LINK : crossings alternate under-over

— L is alternating iff in its Tait graph G all signs are same

examples



left trefoil
 D



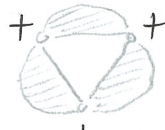
\tilde{D}



Tait graph
 G



right trefoil



+



+



Hopf link



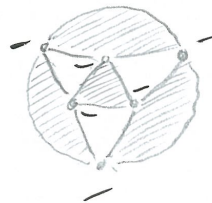
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-



Borromean rings



-



-

In fact every link with crossing number ≤ 7 is equivalent to an alternating link (with minimum number of crossings)

4.

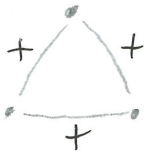
Propⁿ

G Tait graph of link L ,

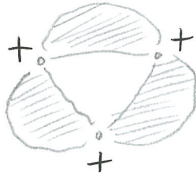
G^* plane dual of G with signs reversed

G^* is Tait graph of a link equivalent to L

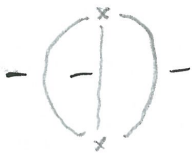
example:



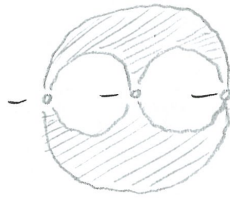
Tait graph G



right trefoil



dual G^* with opposite signs



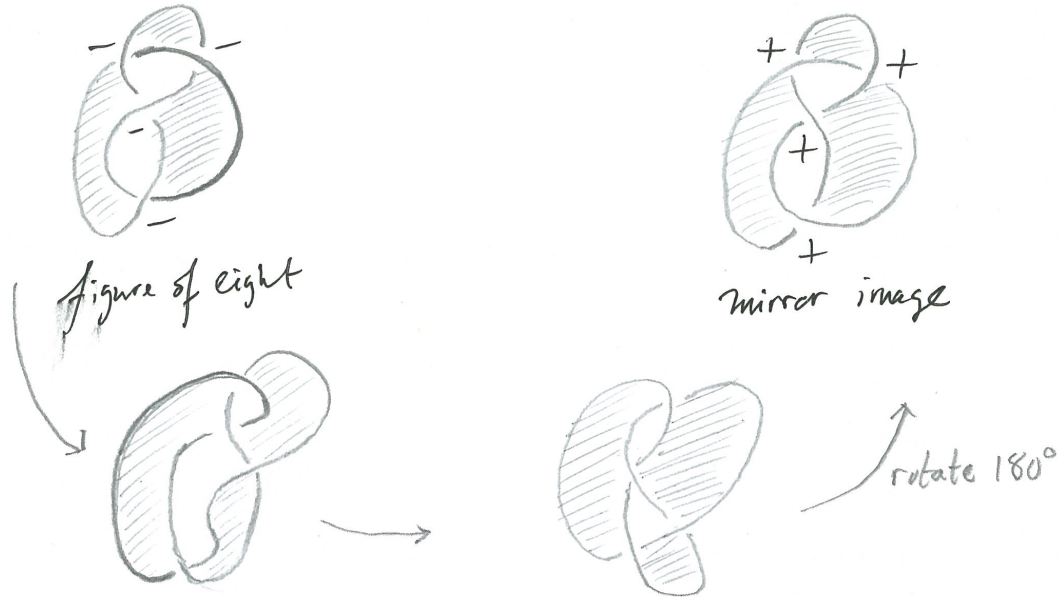
MIRROR IMAGE of a link: interchange overcrossings with undercrossings, i.e., switch signs in Tait graph

example: left trefoil is mirror image of right trefoil.

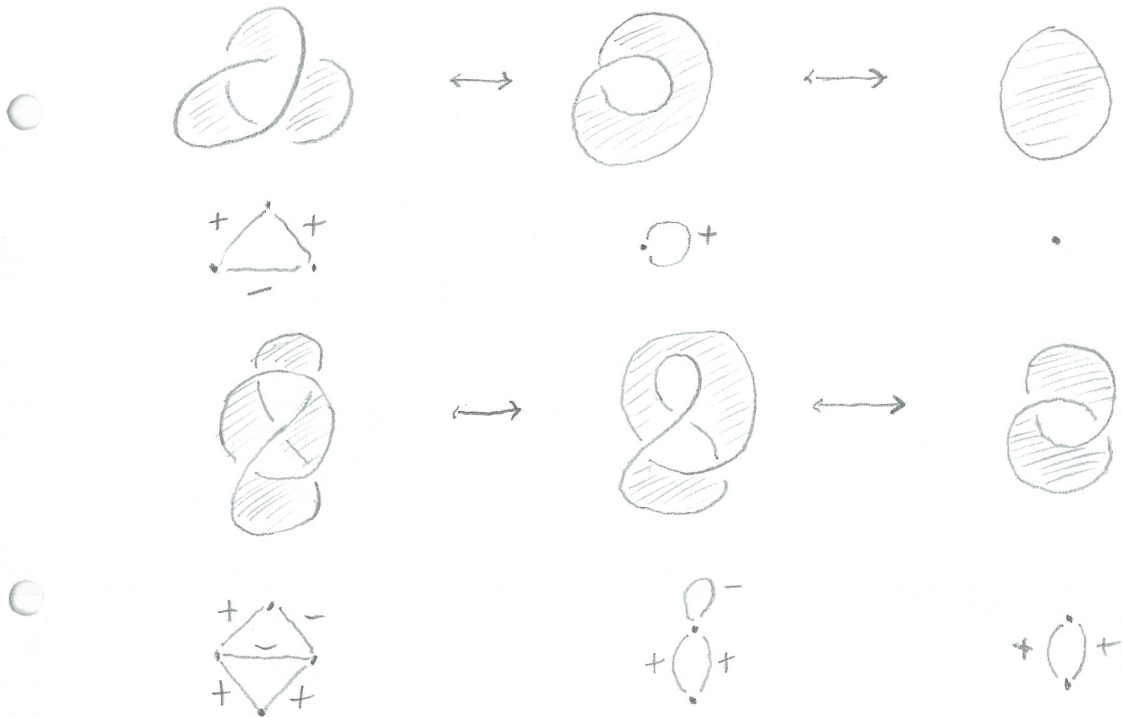
Defⁿ link is AMPHICHEIRAL (or ACHIRAL) if ambient isotopic to its mirror image.

5.

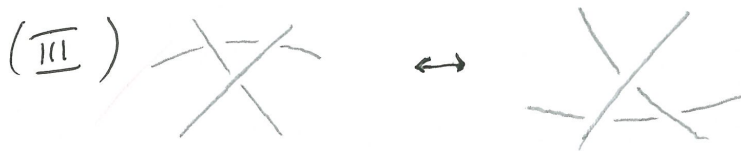
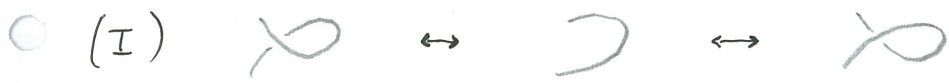
example: Figure of Eight is amphicheiral:



Equivalence of Links, Reidemeister Moves



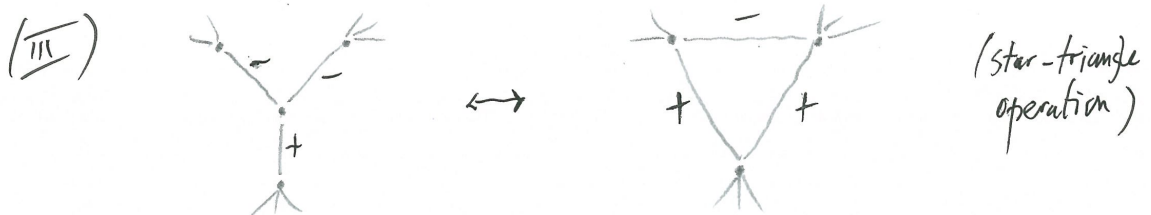
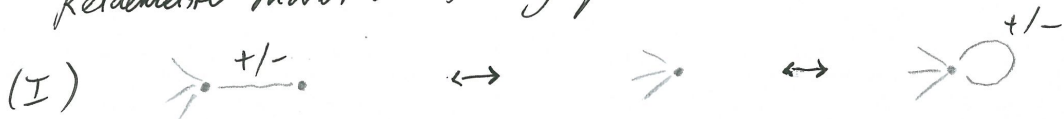
6. REIDEMEISTER MOVES on a link diagram:



Theorem (Reidemeister, 1935)

Two links are ambient isotopic if and only if the diagram of one can be transformed to the other by a finite sequence of Reidemeister moves (I), (II), (III).

Reidemeister Moves on Tait graph:



7.

Problem Complexity status of deciding whether a given knot is equivalent (ambient isotopic) to the unknot?
(size of input: number of crossings, ie number of edges in Tait graph)

Definition LINK INVARIANT f has property that $f(L) = f(L')$ whenever L and L' are ambient isotopic.

examples: - number of component knots of a link
- minimum number of crossings in a diagram representing a link

Knowing $f(L) \neq f(L')$ for link invariant f means that L and L' are not ambient isotopic. In particular, $f(L) \neq f(\text{unknot})$ tells us that L is not equivalent to the unknot, and $f(L) \neq f(\bar{L})$ where \bar{L} is the mirror image of L tells us that L is chiral (not amphicheiral).

By Reidemeister's Theorem, f is a link invariant if and only if it is invariant under Reidemeister moves (I), (II) and (III).

8. The Kauffman bracket

defined on links in terms of their diagrams:

$$(K1) \quad \langle \bigcirc \rangle = 1 \quad \bigcirc = \text{unknot}$$

$$(K2) \quad \langle \bigcirc \cup D \rangle = d \langle D \rangle \quad \text{disjoint union}$$

$$(K3) \quad \langle \text{crossing} \rangle = A \langle \text{smooth 1} \rangle + B \langle \text{smooth 2} \rangle$$

A, B, d commuting indeterminates

(K3) reduces the number of crossings by one.

Example: $\langle \text{trefoil} \rangle = A \langle \text{smooth 1} \rangle + B \langle \text{smooth 2} \rangle$

Invariance under Reidemeister Move (II): $\langle \text{crossing} \rangle = \langle \text{crossing} \rangle$

$$\langle \text{crossing} \rangle = A \langle \text{smooth 1} \rangle + B \langle \text{smooth 2} \rangle$$

$$= A^2 \langle \text{smooth 1} \rangle + AB \langle \text{smooth 1} \rangle +$$

$$AB \langle \text{smooth 2} \rangle + B^2 \langle \text{smooth 2} \rangle$$

$$= (A^2 + B^2 + dAB) \langle \text{smooth 1} \rangle + AB \langle \text{smooth 2} \rangle$$

In order that $\langle \text{crossing} \rangle = \langle \text{crossing} \rangle$ need $AB=1$ $A^2 + B^2 + dAB = 0$

ie. $B = A^{-1}$ $d = -(A^2 + A^{-2})$

9. Invariant under Reidemeister move (III): 


$$\langle \text{III} \rangle = A \langle \text{III}' \rangle + A^{-1} \langle \text{III}'' \rangle$$

$$\stackrel{\text{by (II)}}{=} A \langle \text{III} \rangle + A^{-1} \langle \text{III}'' \rangle$$

and $\langle \text{III}'' \rangle = A \langle \text{III} \rangle + A^{-1} \langle \text{III}'' \rangle$

$$\stackrel{\text{by (II)}}{=} A \langle \text{III} \rangle + A^{-1} \langle \text{III}'' \rangle$$

Hence $\langle \text{III} \rangle = \langle \text{III}'' \rangle$

Reidemeister move (I): 

$$\langle \text{I} \rangle = A \langle \text{I}' \rangle + A^{-1} \langle \text{I}'' \rangle$$

$$= -A^{-3} \langle \text{I}' \rangle$$

$$\langle \text{I}'' \rangle = A \langle \text{I}' \rangle + A^{-1} \langle \text{I}'' \rangle$$

$$= -A^3 \langle \text{I}' \rangle$$

So not invariant under (I). To fix this move to ORIENTED LINKS.

Side-note:




$$\langle \text{I} \rangle = -A^{-3} \langle \text{I}' \rangle = -A^{-3}$$

$$\langle \text{I}'' \rangle = -A^3$$

10.

 $\langle \text{circle with counter-clockwise arrow} \rangle = -A^3 \langle \text{circle} \rangle = -A^3$

Q+

 $\langle \text{circle with clockwise arrow} \rangle = -A^{-3} \langle \text{circle} \rangle = -A^{-3}$

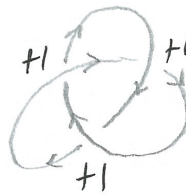
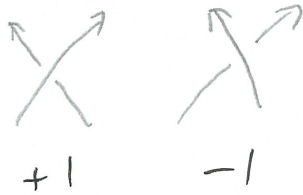
Q-

ORIENTED LINK : for each component knot give a direction of traversal

assign weight to crossings :



right trefoil



right trefoil
with reverse
orientation

for a knot, weight is independent of orientation — a reversal of orientation reverses local orientation on both segments.

for a link, different components can be independently reoriented so there is a change of sign weight.

For diagram D of oriented link,
 $\boxed{\text{Def}^n}$ $\text{writhe}(D) = w(D) = \text{sum of sign weights on crossings}$

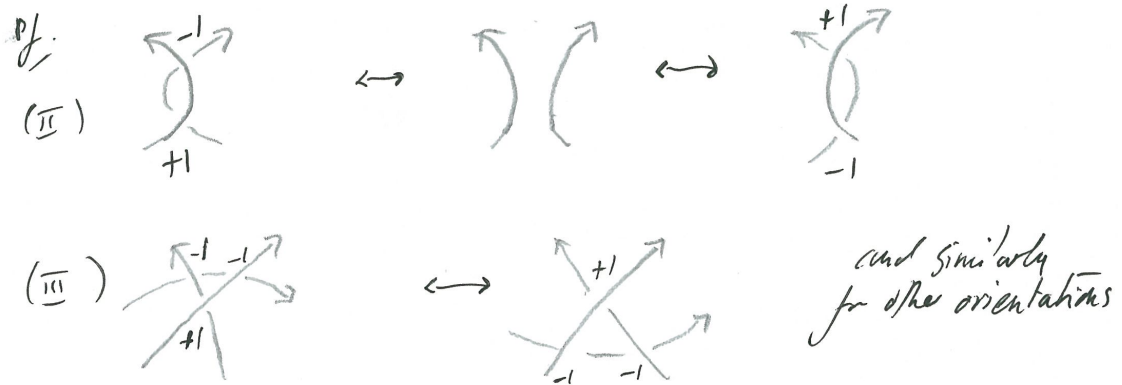
$$\text{Self-writhe}(D) = w(D_1) + \dots + w(D_k)$$

when D is link with components D_1, \dots, D_k

$$= s(D)$$

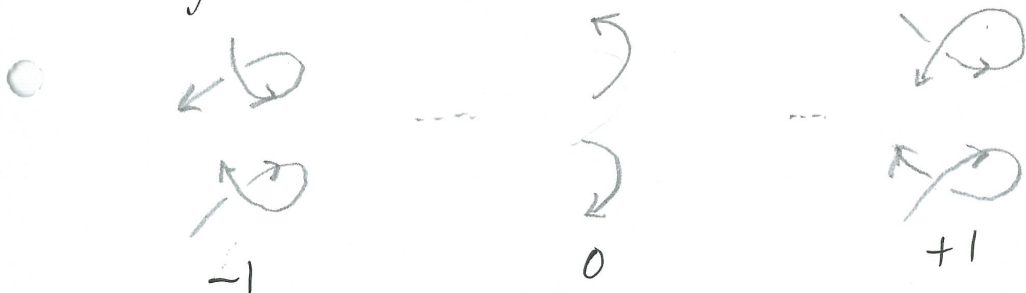
note : $s(D)$ is independent of orientation of D : so can be defined for unoriented D

11. Lemma $w(D)$ and $s(D)$ are regular isotopy invariants
 (invariant under Reidemeister moves (II) and (III))



Theorem The Laurent polynomial $f[D] = (-A)^{-3s(D)} \langle D \rangle$
 in $\mathbb{Z}[A, A^{-1}]$ is an invariant of ambient isotopy
 for unoriented links. (KAUFMAN POLYNOMIAL)

Pf. Since $\langle D \rangle$ and $s(D)$ are invariant under (II) and (III)
 only need check invariance under (I).



so: $s(\text{loop}) = s(\text{straight}) - 1$

$s(\text{loop}) = s(\text{straight}) + 1$

and we have (from previous)

$$\langle \text{loop} \rangle = -A^{-3} \langle \text{straight} \rangle$$

$$\langle \text{loop} \rangle = -A^3 \langle \text{straight} \rangle$$

12.

Hence $f[\cup](A) = (-A)^{-3s(\cup)} \langle \cup \rangle$
 $= (-A)^{-3(s(\cup)+1)} (-A^{-3}) \langle \cup \rangle$
 $= (-A)^{-3s(\cup)} \langle \cup \rangle$

Similarly, $f[\cap](A) = f[\cup](A) \quad \square$

Propⁿ if \bar{L} is mirror image of L then

$$f[\bar{L}](A) = f[L](A^{-1})$$

Pf: From recursive definition of bracket

$$\langle \overline{\text{crossing}} \rangle = A \langle \text{crossing} \rangle + A^{-1} \langle \text{crossing} \rangle$$

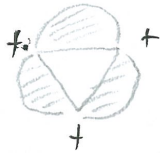
Mirror: $\langle \text{crossing} \rangle = A \langle \overline{\text{crossing}} \rangle + A^{-1} \langle \text{crossing} \rangle$

Also weights are switched in mirror image \square

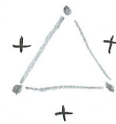
Kauffman polynomial for alternating links



L alternating link



\tilde{D} diagram
all crossings same sign



Tait graph G
all edges same sign

Take all signs $+$ in plane graph G
to give Tait graph of alternating link

13. / Recursion (k3)

on diagram D: $\langle \text{diagram with edge } e \text{ and shading} \rangle = A \langle \text{diagram with crossing} \rangle + A^{-1} \langle \text{diagram with two crossings} \rangle$

on (Tait) graph G : $\langle G \rangle = A \langle G/e \rangle + A^{-1} \langle G|e \rangle$
 ↪ plane



Example: $\langle \text{triangle with edge } e \text{ and shading} \rangle = A \langle \text{circle with shading} \rangle + A^{-1} \langle \text{two vertices with shading} \rangle$



Also, $\langle \text{infinity symbol} \rangle = -A^{-3}$ $\langle \text{circle with a dot} \rangle = -A^3$
 . + . . ⊕

So by Recipe Theorem for Tutte polynomial:

Theorem For diagram D with Tait graph the plane graph $G_{-} = (V, E)$ with all edges + alternating link

$$\langle D \rangle = A^{2|V| - |E| - 2} T(G; -A^{-4}, -A^4)$$

$G = (V, E, F)$ faces $= A^{|V| - |F|} T(G; -A^{-4}, -A^4)$
 where $T(G)$ is the Tutte polynomial of plane graph G ,

and mirror image \bar{D} has

$$\langle \bar{D} \rangle = A^{|V^*| - |F^*|} T(G^*; -A^{-4}, -A^4)$$

14.

where $G^* = (V^*, E^*, F^*)$ is plane dual of G

Corollary Kauffman polynomial of alternating link with diagram D

is given by $f[D](A) = (-A)^{-3s(D)} \langle D \rangle$

$$= (-1)^{|E|} A^{|V|-|E|-3s(D)} T(G; -A^{-4}, A^4)$$

Pf. $E =$ vertices of $D =$ edges of G

Let E_+ be those weight $+1$ in arbitrary orientation of link with diagram D
 E_- " " -1

$$|E| = |E_+| + |E_-| \quad s(D) = |E_+| - |E_-|$$

$$\text{So } (-1)^{s(D)} = (-1)^{|E|} \quad \square$$

Example right trefoil has diagram with Tait graph K_3 (all edges $+$), with $e = 3$

$$T(K_3; x, y) = x^2 + x + y$$

$$\langle \text{right trefoil} \rangle = A^{2 \cdot 3 - 3 - 2} (A^{-8} - A^{-4} - A^4) = A^{-7} - A^{-3} - A^5$$

and $f[\text{right trefoil}](A) = -A^{-16} + A^{-12} + A^{-4}$

Minor image is left trefoil (with $G = \text{circle with dot} = K_3^*$)

and $f[\text{left trefoil}](A) = -A^{16} + A^{12} + A^4 \neq f[\text{right trefoil}](A)$

implying left and right trefoils inequivalent, i.e. chiral

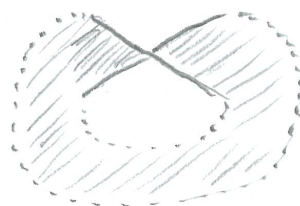
15. Tait conjecture

Defⁿ A crossing is NEGATIVE if some two of the local regions appearing at the crossing are parts of the same region in the whole diagram.

A negative crossing ^{in an alternating link} appears as a bridge or loop in the Tait graph.



bridge



loop

Theorem (conj. Tait, proved Murasugi & Thistlethwaite ca. century later)

The number of crossings of a connected alternating link diagram without negative crossings is an ambient isotopy invariant.

If diagram D with Tait graph all signs + (take mirror image if all signs -), plane graph G this without signs.

Laurent polynomial $f[D](A) = (-A)^{-3s(D)} \langle D \rangle$

has span maximum degree - minimum degree := span $\langle D \rangle$

$\langle D \rangle = A^{2|V|-|E|-2} \prod (5_i^{-4}, A^4)$ has

$\text{span } \langle D \rangle = 4(r(G) + |E| - r(G)) = 4|E|$

$= 4 \# \text{ crossings of } D$ (edges $G \leftrightarrow$ vertices medial \leftrightarrow crossings of D)

when G has no loops or bridges (i.e. ~~no~~ no negative crossings in D)

16.

Lemma

The Tutte polynomial $T(G; x, y)$ of a bridgeless, loopless graph $G = (V, E)$ has degree $r(G)$ in x and degree $|E| - r(G)$ in y . *

(By induction on $|E|$.)

iff - When $E = \emptyset$, $T(G; x, y) = 1$ and $r(G) = 0 = |E|$.

The recurrence $T(G) = T(G/e) + T(G \setminus e)$ for $e \in E$ (not a bridge or loop by hypothesis) and

$$r(G/e) = \begin{cases} r(G) - 1 \\ r(G) \end{cases} \quad e \text{ loop} : \quad \begin{cases} |E(G/e)| - r(G/e) \\ = |E| - r(G) \end{cases} \text{ for non-loop } e$$

$$r(G \setminus e) = \begin{cases} r(G) \\ r(G) - 1 \end{cases} \quad e \text{ bridge}$$

provides the inductive step. □

* Moreover $T(G; x, y) = x^{r(G)} + y^{|E| - r(G)} + \text{lower degree terms}$

So that span of $T(G; -A^{-4}, -A^4)$ is equal to $4(|E| - r(G) + r(G)) = 4|E|$.

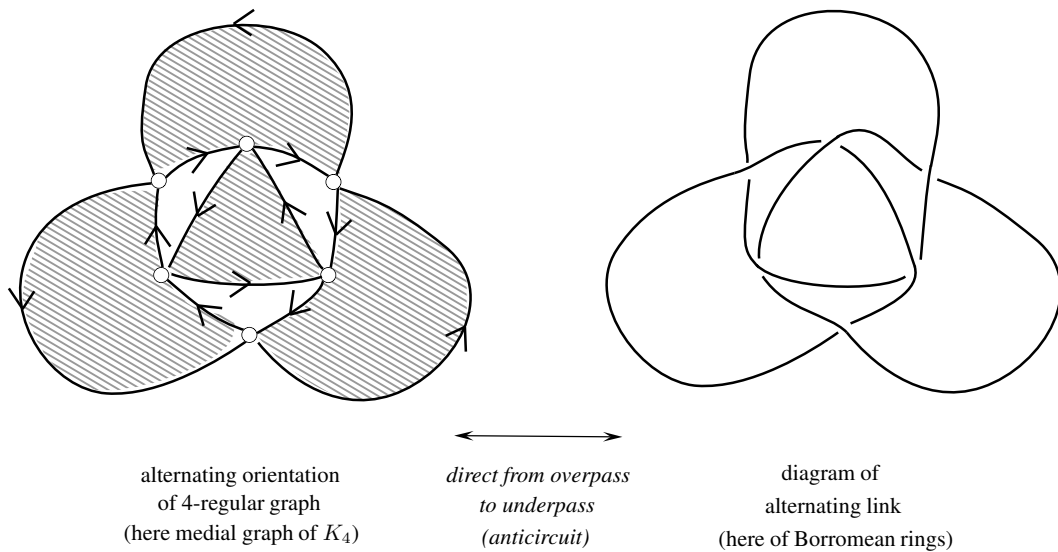


Figure 26: Canonical alternating orientation of $m(K_4)$ and interpretation of the anticycle partition (into three components) as the diagram of an alternating link, which in this case represents the Borromean rings.

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