Graph polynomials and matroid invariants by counting graph homomorphisms

Delia Garijo[1] **Andrew Goodall**[2]
Patrice Ossona de Mendez[3] Jarik Nešetřil[2] Guus Regts[4]
and Lluís Vena[2]

[1]University of Seville

[2]Charles University, Prague

[3]CAMS, CNRS/EHESS, Paris

[4]University of Amsterdam

27 January 2016 Technion, Haifa

Chromatic polynomial

Definition by evaluations at positive integers

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$$uv \in E(G), \quad P(G;k) = P(G\backslash uv;k) - P(G/uv;k)$$

Independence polynomial

Definition by coefficients

$$I(G;x) = \sum_{1 \le j \le |V(G)|} b_j(G) x^j,$$

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(Chudnovsky & Seymour, 2006) $K_{1,3} \not\subseteq_i G \Rightarrow I(G;x)$ real roots

$$b_j(G)^2 \geq b_{j-1}(G)b_{j+1}(G), \qquad \text{(implies } b_1,\ldots,b_{|V(G)|} \text{ unimodal)}$$

Flow polynomial

Definition (Evaluation at positive integers)

$k \in \mathbb{N}, \qquad F(G;k) = \#\{\text{nowhere-zero } \mathbb{Z}_k\text{-flows of } G\}.$

$$F(G;k) = \begin{cases} F(G/e) - F(G\backslash e) & e \text{ ordinary} \\ 0 & e \text{ a bridge} \\ (k-1)F(G\backslash e) & e \text{ a loop} \end{cases}$$

Tutte polynomial

Definition

For graph $G = (V, E)$,

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)},$$

where $r(A)$ is the rank of the spanning subgraph (V, A) of G.

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$$T(G; x, y) = \begin{cases} T(G/e; x, y) + T(G \backslash e; x, y) & e \text{ ordinary} \\ xT(G/e; x, y) & e \text{ a bridge} \\ yT(G \backslash e; x, y) & e \text{ a loop}, \end{cases}$$

and $T(G; x, y) = 1$ if G has no edges.

Definition

Graphs G, H.
$f : V(G) \to V(H)$ is a homomorphism from G to H if
$uv \in E(G) \Rightarrow f(u)f(v) \in E(H)$.

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$$\hom(G, H) = \sum_{f:V(G)\to V(H)} \prod_{uv\in E(G)} a_{f(u),f(v)}.$$

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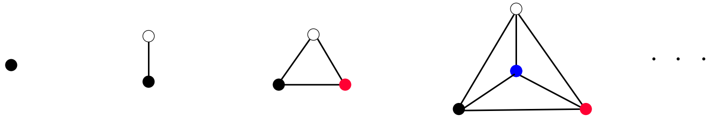
$$\mathrm{hom}(G,H) = \#\{\text{homomorphisms from } G \text{ to } H\}$$
$$= \#\{H\text{-colourings of } G\}$$

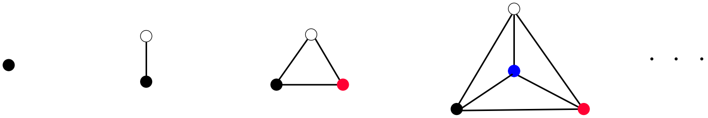
when H simple ($a_{s,t} \in \{0,1\}$) or multigraph ($a_{s,t} \in \mathbb{N}$)

Example 1



$$(K_k)$$

Example 1



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$$\mathrm{hom}(G, K_k) = P(G; k)$$

chromatic polynomial

Problem 1

Which sequences (H_k) of graphs are such that, for all graphs G, for each $k \in \mathbb{N}$ we have

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for polynomial $p(G)$?

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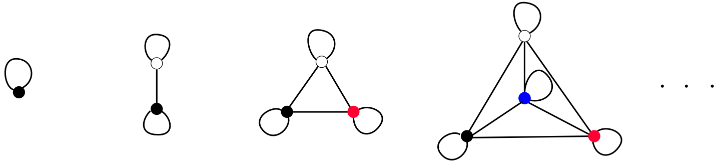
Example

For all graphs G, $\hom(G, K_k) = P(G; k)$ is the evaluation of the chromatic polynomial of G at k.

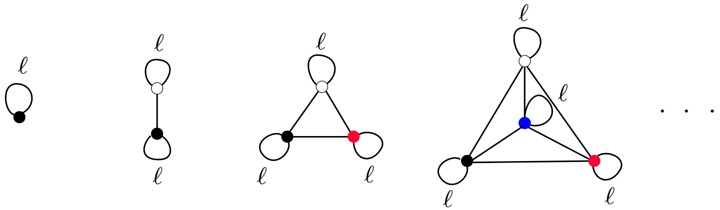
Example 2: add loops



$$(K_k^1)$$

$$\hom(G, K_k^1) = k^{|V(G)|}$$

Example 3: add ℓ loops



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$$\hom(G, K_k^\ell) = \sum_{f:V(G)\to[k]} \ell^{\#\{uv\in E(G)\,|\,f(u)=f(v)\}}$$

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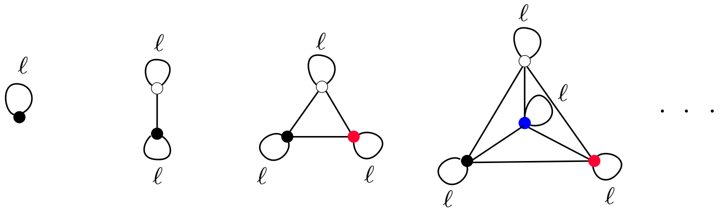

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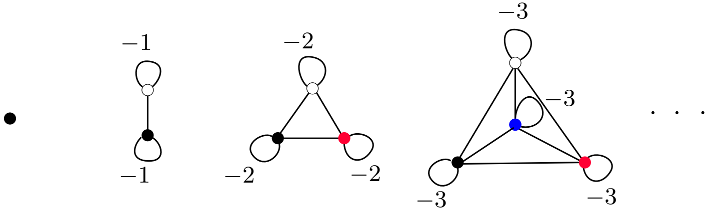
$$\hom(G, K_k^\ell) = \sum_{f:V(G)\to[k]} \ell^{\#\{uv\in E(G)\,|\,f(u)=f(v)\}}$$

$$= k^{c(G)}(\ell-1)^{r(G)}\,T(G; \tfrac{\ell-1+k}{\ell-1}, \ell)$$ (Tutte polynomial)

Example 4: add loops weight $1-k$



$$(K_k^{1-k})$$

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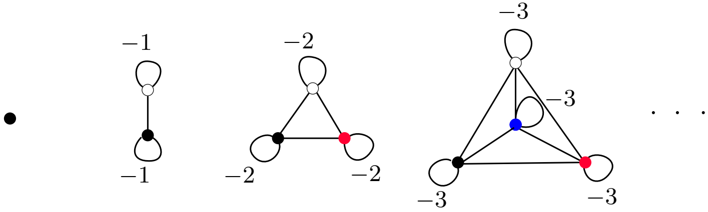

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$$\hom(G, K_k^{1-k}) = \sum_{f:V(G)\to[k]} (1-k)^{\#\{uv\in E(G)\,|\,f(u)=f(v)\}}$$

$$= (-1)^{|E(G)|} k^{|V(G)|} F(G; k) \quad \text{(flow polynomial)}$$

Example 5

$$(K_1^1 + K_{1,k})$$

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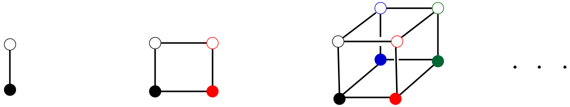
$$\hom(G, K_1^1 + K_{1,k}) = I(G;k)$$

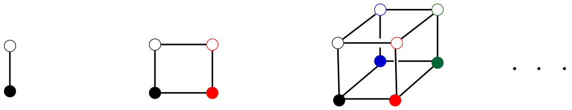
independence polynomial

Example 6



$$(K_2^{\square k}) = (Q_k) \text{ (hypercubes)}$$

Example 6

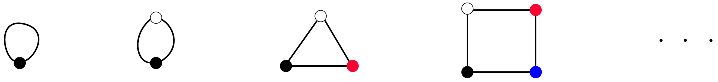


$(K_2^{\square k}) = (Q_k)$ *(hypercubes)*

Proposition (Garijo, G., Nešetřil, 2015)

$\mathrm{hom}(G, Q_k) = p(G; k, 2^k)$ *for bivariate polynomial* $p(G)$

Example 7



(C_k)

$\hom(C_3, C_3) = 6$, $\hom(C_3, C_k) = 0$ when $k = 2$ or $k \geq 4$

Definition

(H_k) is *strongly polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for all $k \in \mathbb{N}$.

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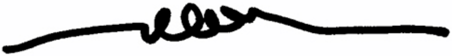
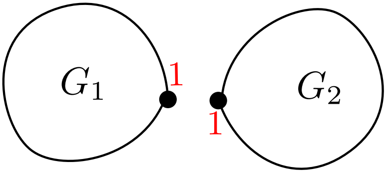
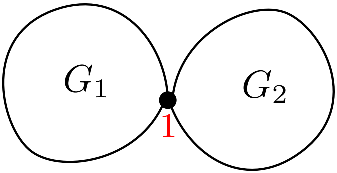
De la Harpe & Jaeger (1995) construct families of strongly polynomial sequences, extended by Garijo, G. & Nešetřil (2015), and further by G., Nešetřil & Ossona de Mendez (2016) using quantifier-free interpretation schemes for finite relational structures (digraphs with added unary relations).



Gluing 1-labelled graphs

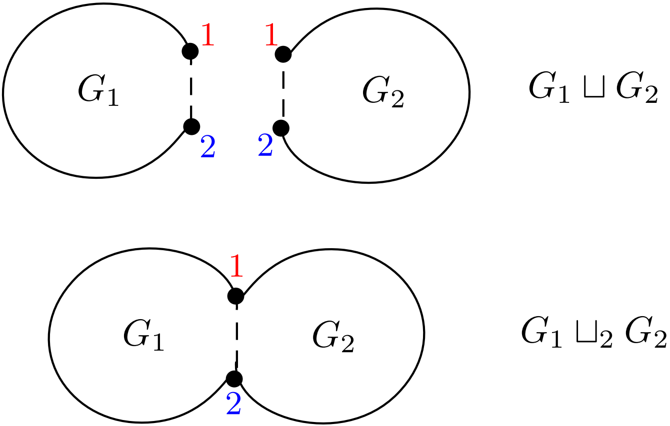


$G_1 \sqcup G_2$

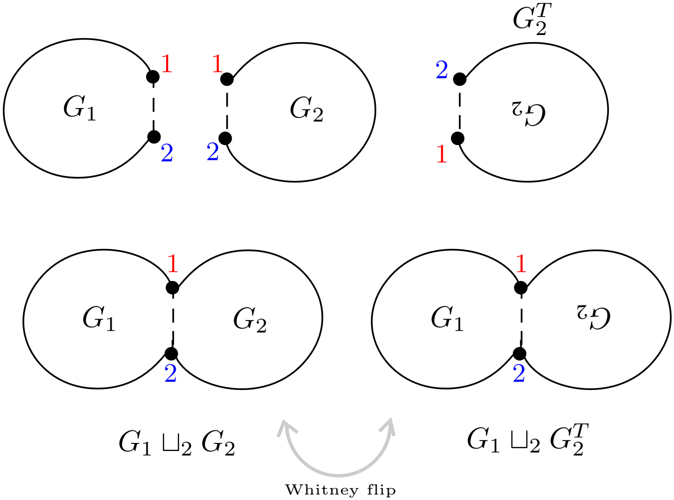


$G_1 \sqcup_1 G_2$

Gluing 2-labelled graphs



Whitney flip



Whitney 2-isomorphism theorem

Theorem (Whitney, 1933)

Two graphs G and G' have the same cycle matroid if and only if G' can be obtained from G by a sequence of operations of the following three types:

(cut) $G_1 \sqcup_1 G_2 \longmapsto G_1 \sqcup G_2$

(glue) $G_1 \sqcup G_2 \longmapsto G_1 \sqcup_1 G_2$

(flip) $G_1 \sqcup_2 G_2 \longmapsto G_1 \sqcup_2 G_2^T$

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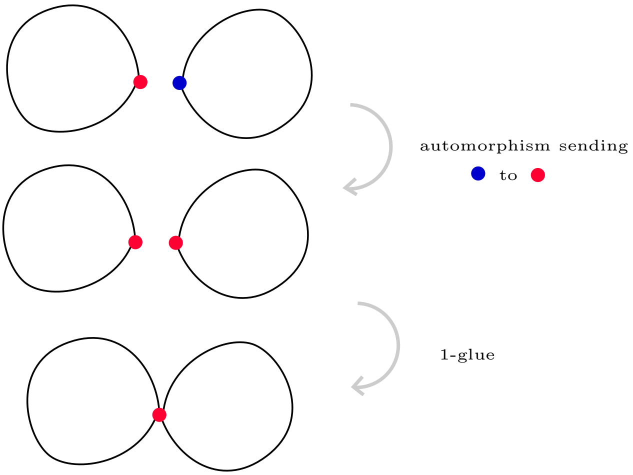
Example

Any two forests with the same number of edges have the same cycle matroid.

Proper colourings and 1-gluing



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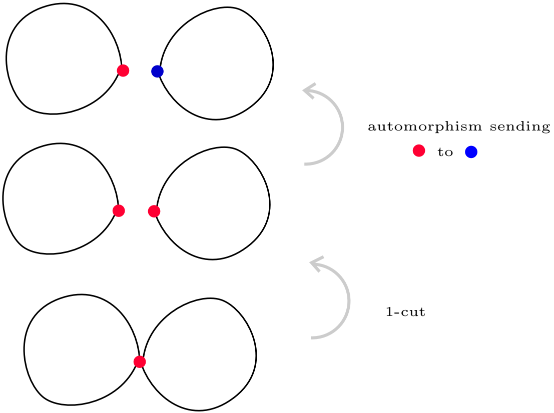
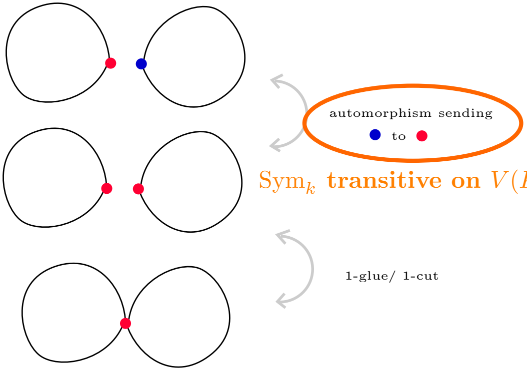


$$P(G_1 \sqcup_1 G_2; k) = P(G_1 \sqcup G_2; k)/k = \frac{P(G_1; k)P(G_2; k)}{P(K_1; k)}$$

Proper colourings and 1-gluing



Sym_k **transitive on** $V(K_k)$

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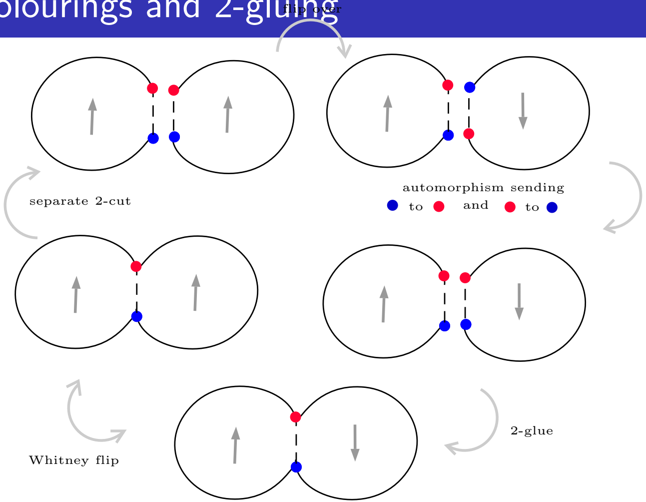
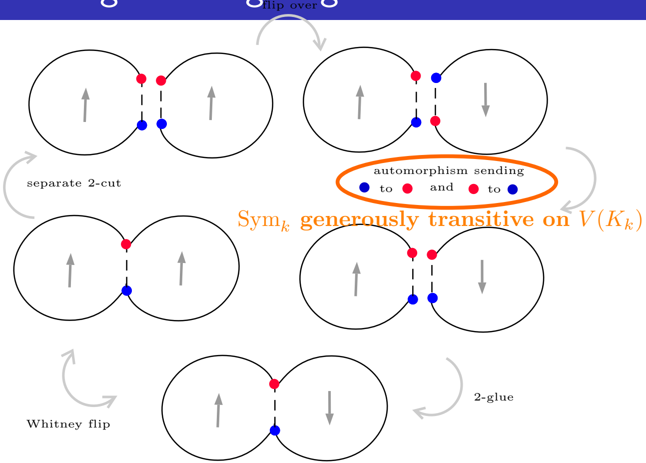


$$P(G_1 \sqcup_2 G_2) = P(G_1 \sqcup_2 G_2^T)$$

Proper colourings and 2-gluing



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Chromatic polynomial as a cycle matroid invariant

Proposition

The graph invariant

$$\frac{P(G;k)}{k^{c(G)}} = \frac{\hom(G, K_k)}{k^{c(G)}}$$

depends just on the cycle matroid of G.

Chromatic polynomial as a cycle matroid invariant

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Problem 2

Which graphs H are such that, for a graph G, we have

$$\frac{\hom(G,H)}{|V(H)|^{c(G)}} = p(G)$$

where $p(G)$ depends only on the cycle matroid of G?

Definition

The action of a group Γ on a set S is transitive if for each $s, t \in S$ there is $\gamma \in \Gamma$ such that $s\gamma = t$.
The action of a group Γ on a set S is generously transitive if for each $s, t \in S$ there is $\gamma \in \Gamma$ such that $s\gamma = t$ and $s = t\gamma$.

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Theorem (de la Harpe & Jaeger, 1995)

The graph invariant

$$G \mapsto \frac{\hom(G, H)}{|V(H)|^{c(G)}}$$

depends just on the cycle matroid of G if H has generously transitive automorphism group.

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Theorem (G., Regts & Vena, 2016)

The graph invariant

$$G \mapsto \frac{\hom(G, H)}{|V(H)|^{c(G)}}$$

depends just on the cycle matroid of G *only if* H *has generously transitive automorphism group.*

Connection matrices

Definition

Graph invariant $G \mapsto f(G)$ has ℓth connection matrix

$$(\ f(G_1 \sqcup_\ell G_2)\ )_{G_1, G_2}$$

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H is twin-free if its adjacency matrix has no two rows equal.

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Theorem (Lovász, 2005)

Let H be a twin-free graph. Then the ℓth connection matrix of $G \mapsto \hom(G,H)$ has rank $\mathrm{orb}_\ell(H)$.

Definition

For labelling $\phi : [\ell] \to V(H)$ and ℓ–labelled G,

$$\hom_\phi(G,H) = \sum_{\substack{f:V(G)\to V(H) \\ f \text{ extends } \phi}} \prod_{uv\in E(G)} a_{f(u),f(v)}.$$

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$$\hom_\phi(G_1 \sqcup_\ell G_2, H) = \hom_\phi(G_1,H)\hom_\phi(G_2,H).$$
- the ℓth connection matrix of $G \mapsto \hom(G,H)$ is
$$(\ \hom_\phi(G,H)\ )^T_{\phi,G}\ (\ \hom_\phi(G,H)\ )_{\phi,G}.$$

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Let H be a twin-free graph. Then the column space of $(\,\hom_\phi(G,H)\,)_{\phi,G}$ is precisely the set of vectors invariant under automorphisms of H.

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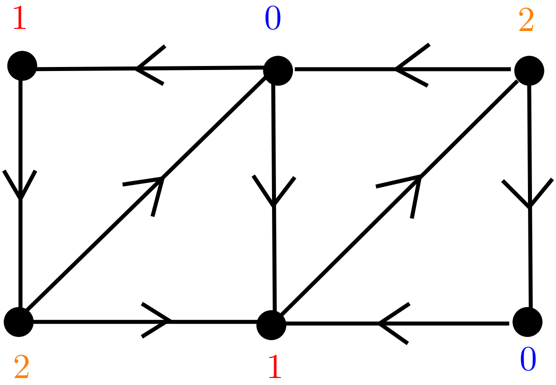
Proof sketch of our result

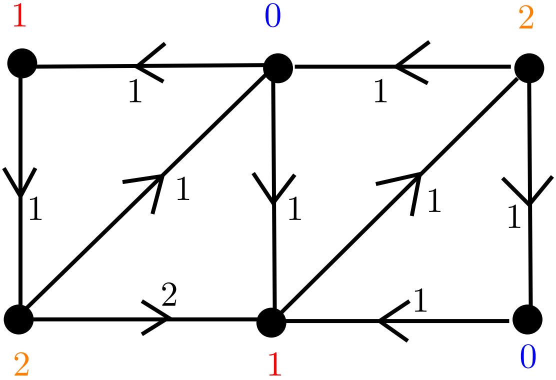
Use Lovász' theorem[a] and the fact that when $\frac{\hom(G,H)}{|V(H)|^{c(G)}}$ depends just on the cycle matroid of G the column space of $(\,\hom_\phi(G,H)\,)_{\phi,G}$ is invariant under (generously) transitive action of a subgroup of $\mathrm{Aut}(H)$. (Taking connection matrices with $\ell=1$ and $\ell=2$.)

[a] Actually, an extension of it by Guus Regts

Proper vertex 3-colouring



Nowhere-zero $\mathbb{Z}_3$-tension



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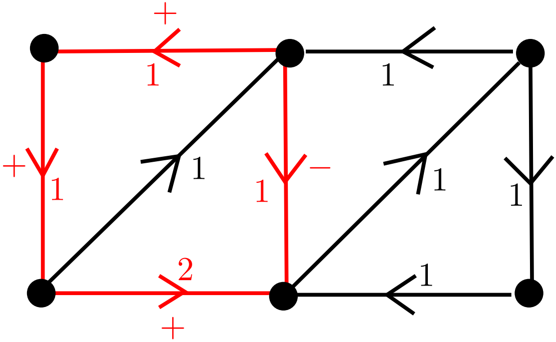

$1 + 1 + 2 - 1 = 0$ in $\mathbb{Z}_3$

Tensions

Graph G with arbitrary orientation of its edges.
Traverse edges around a circuit C and let C^+ be its forward edges and C^- its backward edges.

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$f : E \to \mathbb{Z}_k$ is a $\mathbb{Z}_k$-tension of G if, for each signed circuit $C = C^+ \sqcup C^-$,

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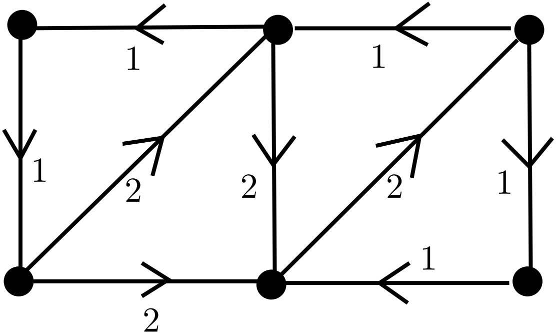

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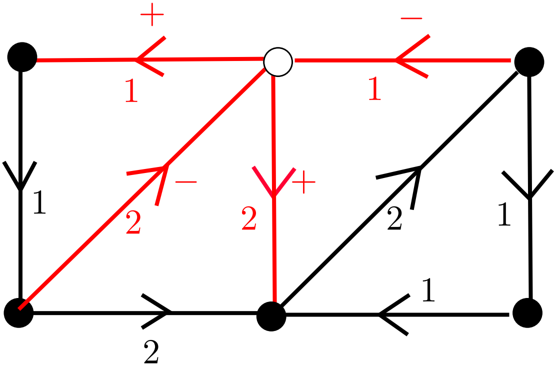
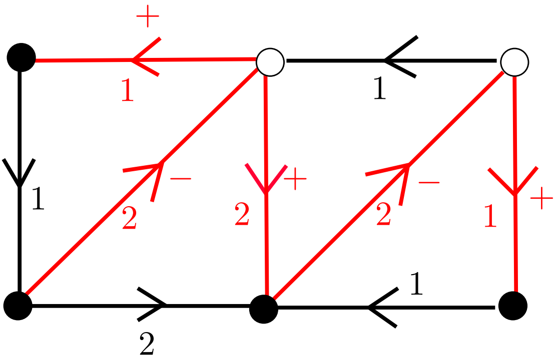


$1 + 2 - 1 - 2 = 0$ in $\mathbb{Z}_3$

Nowhere-zero $\mathbb{Z}_3$-flow



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For a cutset $B = E(U, V\backslash U)$, let B^+ be edges of B directed out from U to $V\backslash U$ and B^- edges of B directed in to U from $V\backslash U$.

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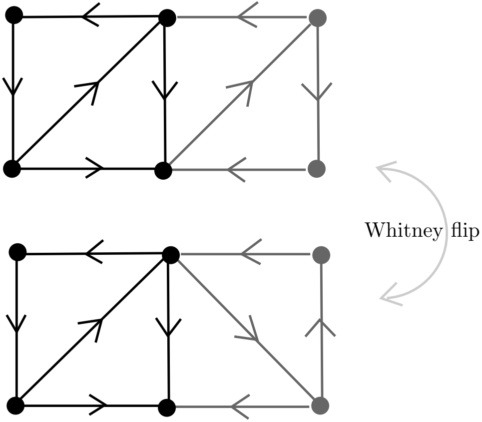
$$F(G; k) = \#\{\text{nowhere-zero } \mathbb{Z}_k\text{-flows of } G\}$$

Whitney flip preserves cycle matroid



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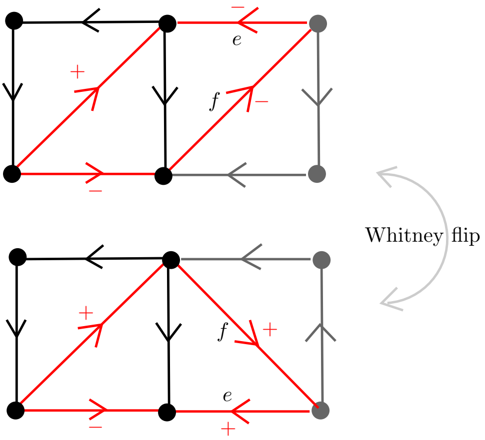
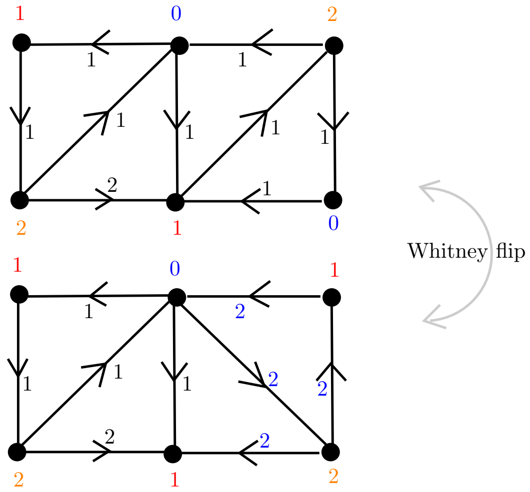


edges in flipped half are traversed in reverse order and opposite sign

Tensions



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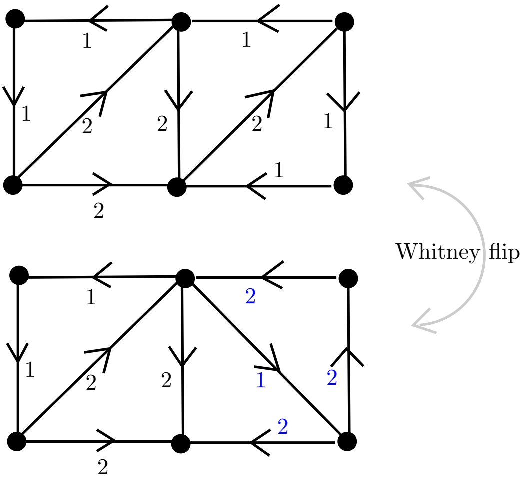

Duality

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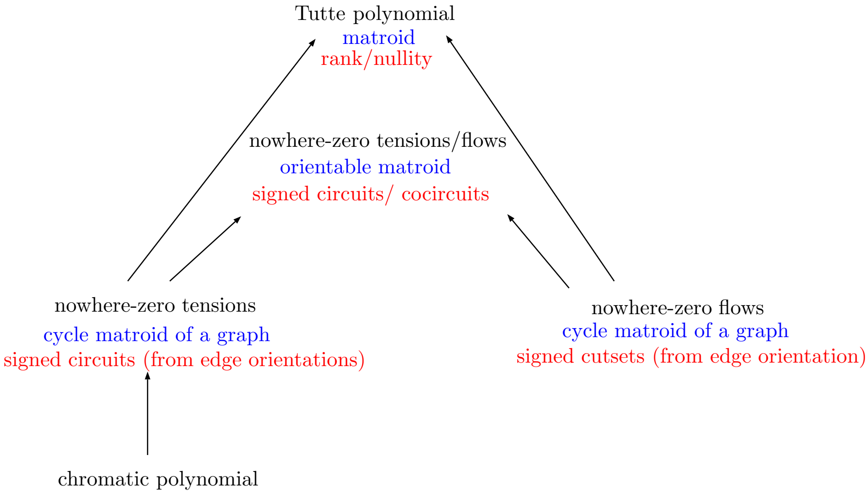
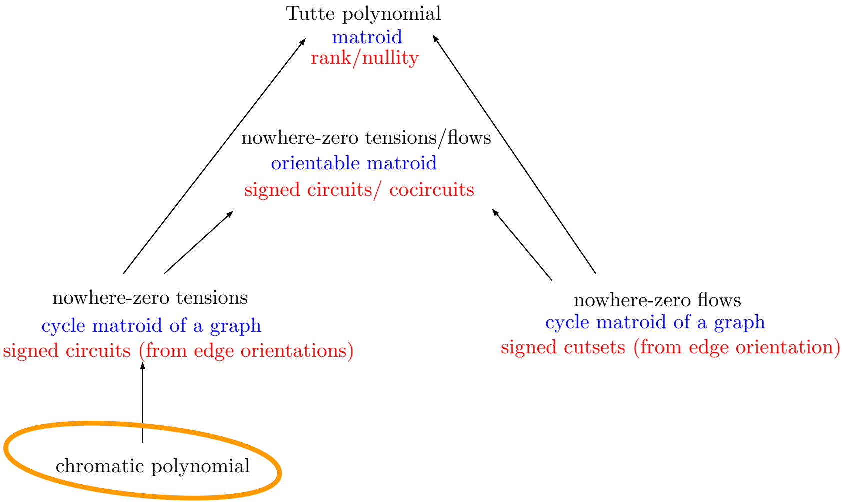
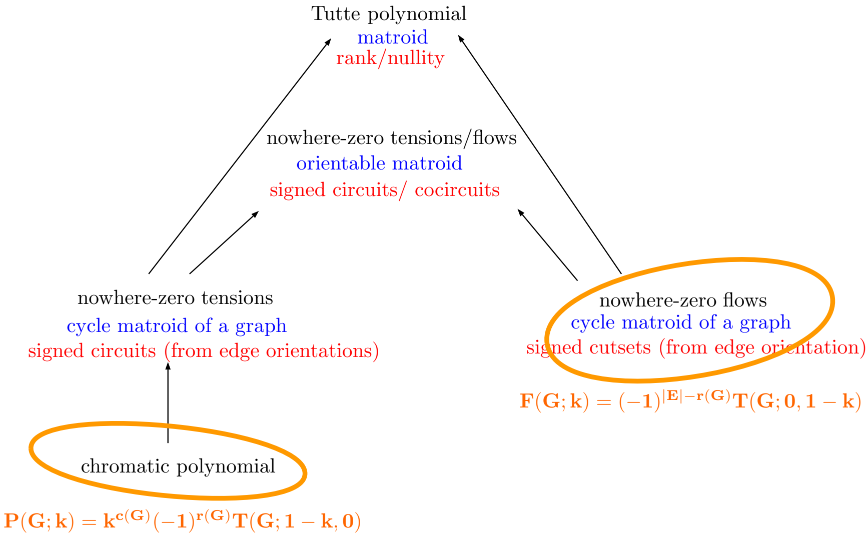
- Tensions/flows defined for orientable matroids.


Tutte polynomial
matroid
rank/nullity
nowhere-zero tensions/flows
orientable matroid
signed circuits/ cocircuits
nowhere-zero tensions
cycle matroid of a graph
signed circuits (from edge orientations)
nowhere-zero flows
cycle matroid of a graph
signed cutsets (from edge orientation)
chromatic polynomial



$$\mathbf{P(G;k) = k^{c(G)}(-1)^{r(G)}T(G;1-k,0)}$$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$\mathbf{F(G;k) = (-1)^{|E|-r(G)}T(G;0,1-k)}$
chromatic polynomial
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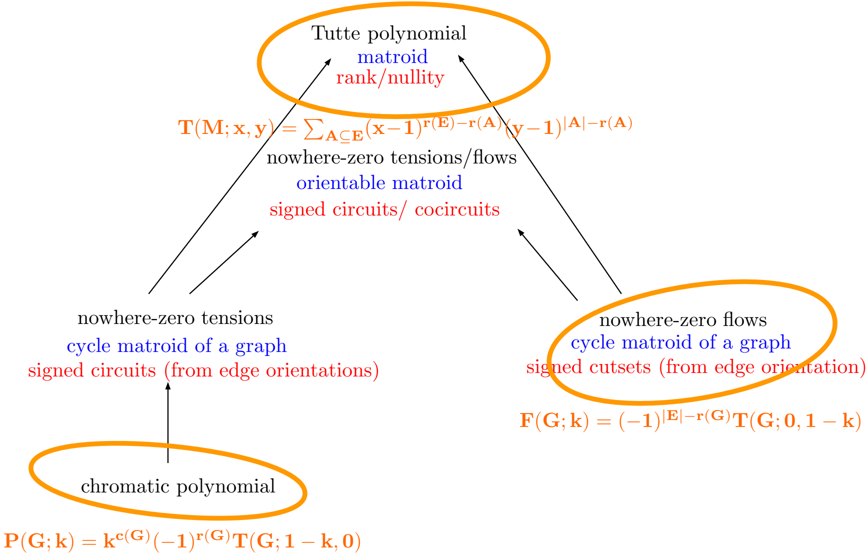
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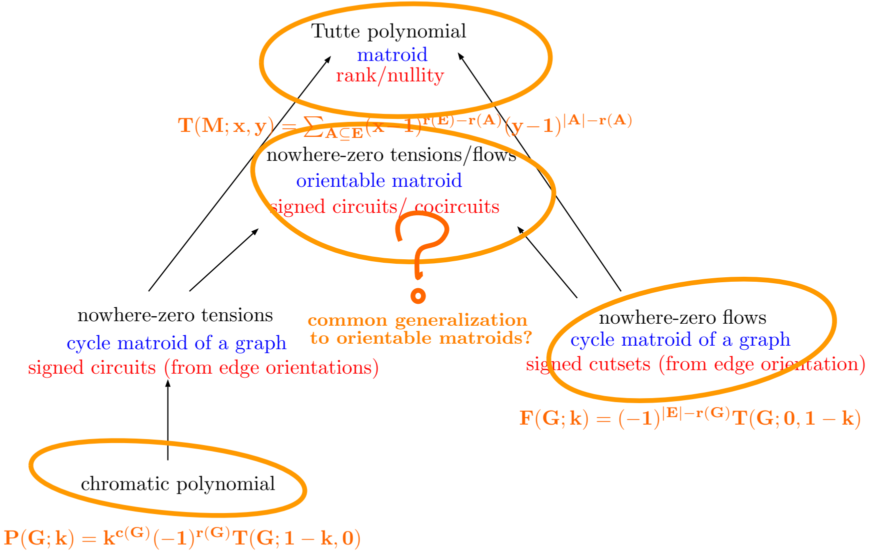
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signed circuits/ cocircuits
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common generalization
to orientable matroids?
nowhere-zero tensions
cycle matroid of a graph
signed circuits (from edge orientations)
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cycle matroid of a graph
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Definition

Generalized Johnson graph $J_{k,r,D}$, $D \subseteq \{0, 1, \ldots, r\}$
vertices $\binom{[k]}{r}$, edge uv when $|u \cap v| \in D$

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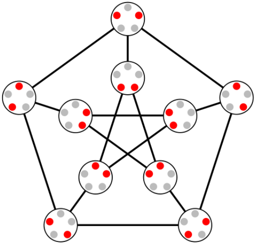
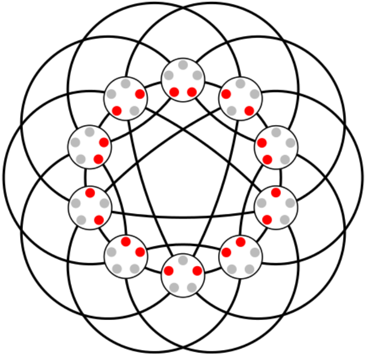


Petersen graph $= K_{5:2}$

Figure by Watchduck (a.k.a. Tilman Piesk). Wikimedia Commons



Johnson graph $J(5, 2)$

Figure by Watchduck (a.k.a. Tilman Piesk). Wikimedia Commons

Fractional chromatic number of graph G:

$$\chi_f(G) = \inf\{\frac{k}{r} : k, r \in \mathbb{N},\ \hom(G, K_{k:r}) > 0\},$$

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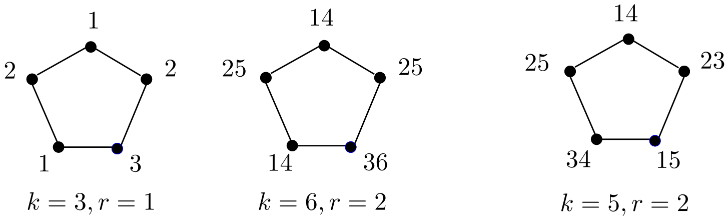
For $k \geq 2r$, $\chi(K_{k:r}) = k - 2r + 2$, while $\chi_f(K_{k:r}) = \frac{k}{r}$

Fractional colouring example: C_5 to $K_{k:r}$



$\chi(C_5) = 3$ but by the homomorphism from C_5 to Kneser graph $K_{5:2}$ (Petersen graph) $\chi_f(C_5) \leq \frac{5}{2}$ (in fact with equality)

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For a graph G and $k, r \geq 1$,
$\hom(G, K_{k:r}) = (r!)^{-|V(G)|} P(G[K_r]; k)$.

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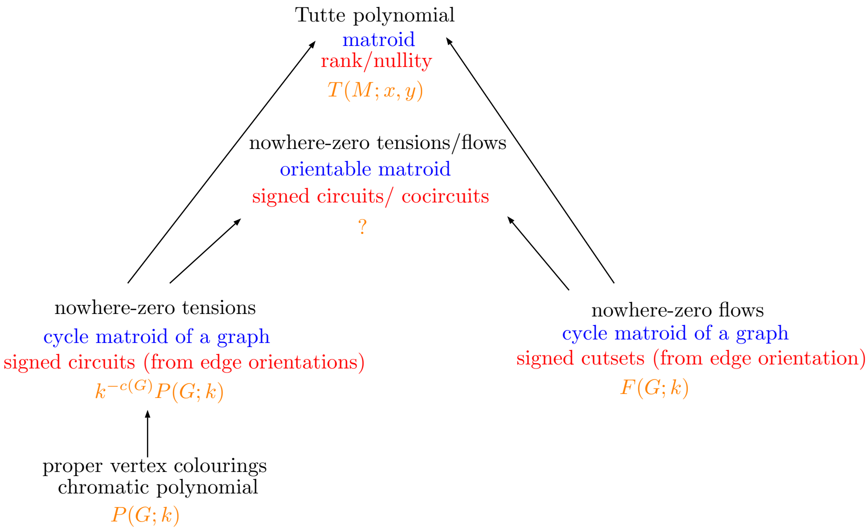
Interpret $\binom{k}{r}^{-c(G)} \hom(G, J_{k,r,D})$ in terms of the cycle matroid of G alone. E.g what is its evaluation at $k = -1$ (acyclic orientations for the chromatic polynomial $= 1, D = \{0\}$).


Tutte polynomial
matroid
rank/nullity
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nowhere-zero tensions/flows
orientable matroid
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proper vertex colourings
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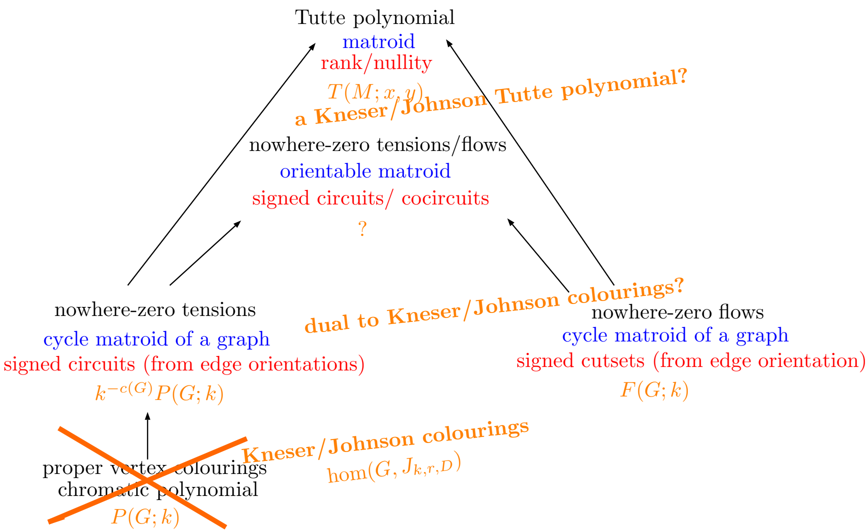
Tutte polynomial
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dual to Kneser/Johnson colourings?
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Current thoughts...

?

delta matroid?

?

nowhere-zero tensions/flows

“rotatable” orientable matroid?

signed/rotated circuits/ cocircuits?

?

non-Abelian tensions?

cycle matroid of a graph

signed circuits (from edge orientations)

circuit rotations (graph traversal)

$\binom{k}{r}^{-c(G)}\hom(G, J_{k,r,D})$

Kneser/Johnson colourings

$\hom(G, J_{k,r,D})$

non-Abelian flows?

cycle matroid of a graph

signed cutsets (from edge orientation)

vertex rotations (orientable embedding)

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- Which graph polynomials defined by strongly polynomial sequences of graphs satisfy a reduction formula (size-decreasing recurrence) like the chromatic polynomial and independence polynomial?

Beyond polynomials? Rational generating functions

- For strongly polynomial sequence (H_k),

$$\sum_k \hom(G, H_k)t^k = \frac{P_G(t)}{(1-t)^{|V(G)|+1}}$$

with polynomial $P_G(t)$ of degree at most $|V(G)|$.

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- For eventually polynomial sequence (H_k) such as (C_k),

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with polynomial $P_G(t)$.

Beyond polynomials? Rational generating functions

- For quasipolynomial sequence of Turán graphs $(T_{k,r})$

$$\sum_k \hom(G, T_{k,r}) t^k = \frac{P_G(t)}{Q(t)^{|V(G)|+1}}$$

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- For sequence of hypercubes (Q_k),

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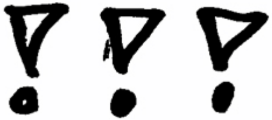
with polynomial $P_G(t)$ of degree at most $|V(G)|$ and polynomial $Q(t)$ with zeros powers of 2.

Beyond polynomials? Algebraic generating functions

- For sequence of odd graphs $O_k = J_{2k-1,k-1,\{0\}}$

$$\sum_k \mathrm{hom}(G, O_k) t^k$$

is algebraic (e.g. $\frac{1}{2}(1-4t)^{-\frac{1}{2}}$ when $G = K_1$).



Four papers

- P. de la Harpe and F. Jaeger, Chromatic invariants for finite graphs: theme and polynomial variations, *Lin. Algebra Appl.* **226–228** (1995), 687–722

  Defining graphs invariants from counting graph homomorphisms. Examples. Basic constructions.

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Four papers

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