Clumsy Packings with Polyominoes

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packed packings

(many applications, papers, results . . .)
clumsy packings
(very little known,
This talk . . . )

packed packings
(many applications,
papers, results . . . )
I. Introduction
- Definitions & Examples

II. Results
- Extremal Questions
- Aperiodic Clumsy Packings
- Undecidability
Example & Definitions

palette $\mathcal{D}$
Example & Definitions

packing $P$ = maximal set of disjoint copies from $\mathcal{D}$
Example & Definitions

- **packing** $P = \text{maximal set of disjoint copies from } \mathcal{D}$

- **density** $(P) = \lim_{n \to \infty} \frac{\text{area covered by } P \text{ in } n\text{-ball}}{\text{area of } n\text{-ball}}$

- **clumsiness** $(\mathcal{D}) = \text{minimum density of a packing}$
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- **clumsiness**($\mathcal{D}$) = *minimum* density of a packing
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- **density** $(P) = \lim_{n \to \infty} \frac{\text{area covered by } P \text{ in } n\text{-ball}}{\text{area of } n\text{-ball}}$

- **clumsiness** $(\mathcal{D}) = \text{minimum density of a packing}$

**Thm.** (Gyárfás, Lehel, Tuza 1988)

$\text{clumsiness}(\begin{array}{c}
\square \\
\square
\end{array}) = 2/3$. 
More Examples

\[
\text{clumsiness}(\underbrace{\phantom{\text{\text{\text{\text{\text{}}}\\}}}^{k}}) = \frac{k}{2k-1} \approx \frac{1}{2}
\]
$$\text{clumsiness}\left(\begin{array}{c} k \\
\end{array}\right) = \frac{k}{2^k - 1} \approx \frac{1}{2}$$
More Examples

\[
\text{clumsiness}(k) = \frac{k}{2k-1} \approx \frac{1}{2}
\]

\[
\text{clumsiness}(\overbrace{\begin{array}{c}
\vdots
\end{array}}^{k},\quad) = 1
\]

\[
\leq \frac{2k}{k(k-1)+1} \approx \frac{2}{k}
\]
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The Clumsiest Polyomino

What is the smallest clumsiness if \( D \) is a single polyomino of size \( k \)?

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\text{clumsiness}(\underbrace{\quad}_{k}) = \frac{k}{2k-1} \approx \frac{1}{2}
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The Clumsiest Polyomino

What is the smallest clumsiness if $\mathcal{D}$ is a single polyomino of size $k$?

clumsiness($\overset{k}{\rule{2cm}{0.5pt}}$) = $\frac{k}{2k-1} \approx 1/2$

clumsiness($\overset{k}{\rule{2cm}{0.5pt}}$) = $k^2 - \left\lfloor \frac{(k-1)}{2} \right\rfloor^2 - \left\lceil \frac{(k-1)}{2} \right\rceil^2 \approx \frac{2}{k}$
What is the smallest clumsiness if $D$ is a single polyomino of size $k$?

$$\text{clumsiness}(\begin{array}{|c|c|}
\hline
\vline & \vline \\
\vline & \vline \\
\hline
\end{array}) = \frac{k}{k^2 - k + 1} \approx \frac{1}{k}$$

$$\text{clumsiness}(\begin{array}{|c|c|c|}
\hline
\vline & \vline & \vline \\
\vline & \vline & \vline \\
\hline
\end{array}) = \frac{k}{k^2 - \left\lfloor \frac{k-1}{2} \right\rfloor^2 - \left\lceil \frac{k-1}{2} \right\rceil^2} \approx \frac{2}{k}$$
The Clumsiest Polyomino

**Theorem.** Let $D$ be a single polyomino of size $k$. Then

- $\text{clumsiness}(D) \geq k/(k^2 - k + 1)$.
- $\text{clumsiness}(D) \geq k/(k^2 - \lceil (k - 1)/2 \rceil^2 - \lfloor (k - 1)/2 \rfloor^2)$, if $D$ is connected.

Both bounds are best possible.
The Clumsiest Polyomino

**Theorem.** Let $D$ be a single polyomino of size $k$. Then

- $\text{clumsiness}(D) \geq k / (k^2 - k + 1)$. \(\approx 1/k\)
- $\text{clumsiness}(D) \geq k / (k^2 - \lceil (k - 1)/2 \rceil^2 - \lceil (k - 1)/2 \rceil^2)$, if $D$ is connected. \(\approx 2/k\)

Both bounds are best possible.
The Clumsiest Polyomino

**Theorem.** Let $D$ be a single polyomino of size $k$. Then

- $\text{clumsiness}(D) \geq \frac{k}{(k^2 - k + 1)} \approx \frac{1}{k}$
- $\text{clumsiness}(D) \geq \frac{k}{(k^2 - \lfloor (k - 1)/2 \rfloor^2 - \lceil (k - 1)/2 \rceil^2)},$ if $D$ is connected.

Both bounds are best possible.

**Lemma.** Let $D$ be a single polyomino of size $k$. Then

- $D$ intersects $\leq k^2 - k + 1$ copies of itself. $\approx \frac{k^2}{2}$
- $D$ intersects $\leq k^2 - \lfloor (k - 1)/2 \rfloor^2 - \lceil (k - 1)/2 \rceil^2$ copies, if $D$ is connected. $\approx k^2$

Both bounds are best possible.
**Proof of Lemma**

**Lemma.** A polyomino of size $k$ intersects at most $k^2 - k + 1$ copies of itself (including itself). 

fix $D$
Proof of Lemma

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Lemma. A polyomino of size $k$ intersects at most $k^2 - k + 1$ copies of itself (including itself).
**Proof of Lemma**

**Lemma.** A polyomino of size $k$ intersects at most $k^2 - k + 1$ copies of itself (including itself).

$D'$ intersects $D$ if and only if cell $c_i$ of $D$ coincides with cell $c_j$ of $D'$, where $c_i = (x_i, y_i)$ and $c_j = (x_j, y_j)$. The difference $c_i - c_j$ is a difference in $D$. 

![Diagram of intersecting polyominos](image)
Examples of intersections of polyominoes with themselves.
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Examples of intersections of polyominoes with themselves.
Proof of Lemma (general case)

**Lemma.** In a polyomino of size $k$ there are at most $k^2 - k + 1$ distinct differences.

- There are $k^2$ differences $c_i - c_j$.
- The $k$ differences $c_i - c_i$ are the same.

**Thm.** (Singer 1938) For every prime power $n$ there is a set $S_n$ of $n + 1$ numbers in $\mathbb{Z}_{n^2 + n + 1}$ such that every non-zero difference modulo $n^2 + n + 1$ occurs exactly once in $S_n$.

E.g. $S_2 = \{1, 2, 4\}$ and $S_3 = \{1, 2, 4, 8, 13\}$
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Proof of Lemma (connected case)

**Lemma.** A connected polyomino of size $k$ has at most $k^2 - \lfloor (k - 1)/2 \rfloor^2 - \lceil (k - 1)/2 \rceil^2$ distinct differences.

Fix a tree $T$. 
Lemma. A connected polyomino of size $k$ has at most $k^2 - \lfloor (k - 1)/2 \rfloor^2 - \lceil (k - 1)/2 \rceil^2$ distinct differences.

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- Move \( d \) along \( T \) if possible on both ends.
- Record all starting points.
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$\# \text{ differences} \leq k^2 - \#(\bullet \bullet)^2 - \#(\bullet \bullet \bullet \bullet)$
The Clumsiest Set of Polyominoes

The clumsiest polyomino of size $k$ has clumsiness

$$\frac{k^2 - \lfloor (k-1)/2 \rfloor^2 - \lceil (k-1)/2 \rceil^2}{k} \leq \frac{2k}{k(k-1)+1} \approx \frac{2}{k}.$$

**Open Question:** What is the clumsiest set of polyominoes each of size at most $k$?
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Almost-Clumsy Periodic Packings

- Does there always exist a periodic clumsy packing?
Almost-Clumsy Periodic Packings

- Does there always exist a periodic clumsy packing?

**Theorem.** For every palette $\mathcal{D}$ and every $\varepsilon > 0$ there exists a **periodic** packing $P$ s.t. $\text{density}(P) \leq \text{clumsiness}(\mathcal{D}) + \varepsilon$. 
Almost-Clumsy Periodic Packings

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clumsy packing
Almost-Clumsy Periodic Packings

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(sparse square)

clumsy packing
Almost-Clumsy Periodic Packings

Does there always exist a periodic clumsy packing?

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Almost-Clumsy Periodic Packings

Does there always exist a periodic clumsy packing?

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Clausal packing

$\varepsilon$-clumsy periodic packing
Aperiodic Clumsy Packings

- Does there always exist a periodic clumsy packing?
Aperiodic Clumsy Packings

Does there always exist a periodic clumsy packing?

NO!
Aperiodic Clumsy Packings

- Does there always exist a periodic clumsy packing?

**Theorem.** There exists a palette $\mathcal{D}$, $|\mathcal{D}| = 14$ such that every clumsy packing is aperiodic.
Aperiodic Clumsy Packings

- Does there always exist a periodic clumsy packing? **NO!**

**Theorem.** There exists a palette $\mathcal{D}$, $|\mathcal{D}| = 14$ such that every clumsy packing is aperiodic.

**Wang Tiles**

![Wang Tiles Image]
Wang Tiles

Wang Tiling
Thm. (Culik 1996) There exists a set of 13 Wang tiles such that every Wang tiling is aperiodic.

Thm. (Berger 1966) It is undecidable whether a given set of Wang tiles tiles the plane.
From Wang Tiles to Polyominoes
From Wang Tiles to Polyominoes

\[ 2x + 5 \]
From Wang Tiles to Polyominoes

- Palette = Wang-polyominoes + bad $x$-by-$x$ square
From Wang Tiles to Polyominoes

- Palette = Wang-polyominoes + bad $x$-by-$x$ square

- Wang tiling exists
  \[ \text{density}(P) = \frac{20x-29}{(2x+5)^2}. \]

- Wang tiling does not exist
  \[ \Rightarrow \text{"many" bad squares} \]
  \[ \Rightarrow \text{density}(P) > \frac{20x-29}{(2x+5)^2}. \]
**Wang Tiles**

- **Thm.** (Culik 1996) There exists a set of 13 Wang tiles such that every Wang tiling is aperiodic.

- **Thm.** (Berger 1966) It is undecidable whether a given set of Wang tiles tiles the plane.

**Polyominoes**

- **Theorem.** There exists a set of 14 polyominoes such that every clumsy packing is aperiodic.

- **Theorem.** For some $q \in \mathbb{Q}$ it is undecidable whether a given set of polyominoes has clumsiness at most $q$. 
Thank you for your attention!

Clumsy Packings with Polyominoes

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Summary and Open Problems

**Thm.** The clumsiest connected polyomino of size $k$ has clumsiness $\approx \frac{2}{k}$.

**Open:** What is the clumsiest set of polyominoes each of size $k$?

**Open:** What if we allow rotations?

**Thm.** For every $\varepsilon > 0$ there exist a periodic packing $P$ such that density$(P) \leq$ clumsiness $+\varepsilon$.

**Thm.** Sometimes all clumsy packings are aperiodic.

**Thm.** Computing clumsiness is undecidable for some $q \in \mathbb{Q}$.

**Open:** What if we allow rotations?