Perfect Digraphs: Answers and Questions

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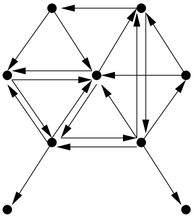
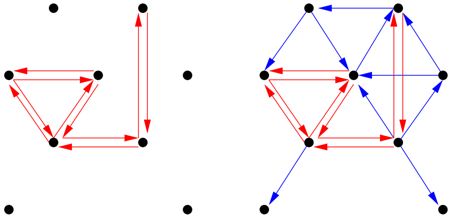
Perfect digraphs

Digraphs



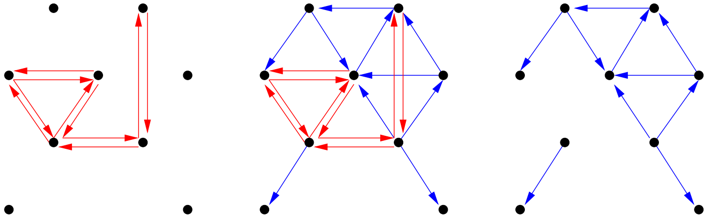
digraph D

Digraphs



symmetric part $S(D)$ digraph D

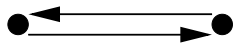
Digraphs



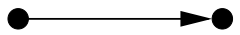
symmetric part $S(D)$ digraph D oriented part $O(D)$

Graphs as digraphs

A graph is a digraph D with $D = S(D)$, i.e. its arc set contains only edges, but no single arcs.



an edge



a single arc

graph = symmetric digraph

The dichromatic number of a digraph

The dichromatic number $\chi(D)$ of a digraph D is the smallest number of induced acyclic subdigraphs of D that cover the vertices of D. [Neumann-Lara 1982]

The dichromatic number of graphs is the chromatic number.

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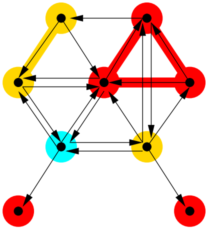
COLORING: color classes are acyclic subdigraphs



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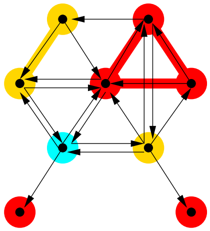
2-COLORING of digraphs is $\mathcal{NP}$-complete.



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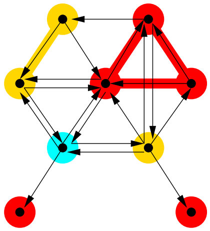
Conjecture (Neumann-Lara (1985))

Orientations of planar graphs are 2-colorable.



The clique number of a digraph

A symmetric clique is a digraph $D = (V, V \times V \setminus \{(v, v) \mid v \in V\})$.

The clique number $\omega(D)$ of a digraph D is the largest size of a symmetric clique in D.

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Theorem (Karp (1971))

Deciding whether a digraph has a symmetric clique of size k is $\mathcal{NP}$-complete.

Perfect digraphs

Observation

For any digraph D,

$$\omega(D) \leq \chi(D).$$

A digraph D is perfect if, for any induced subdigraph H of D,

$$\omega(H) = \chi(H).$$

Main result and a "Strong Perfect Digraph Theorem"

Technical requirements

Req. A)

For a digraph $D = (V, A)$ and $V' \subseteq V$, we denote by $D[V']$ the subdigraph of D induced by the vertices in V'.

Req. B)

Observation

For any digraph D, we have $\omega(D) = \omega(S(D))$.

The main result

Theorem

A digraph D is perfect if and only if $S(D)$ is a perfect graph and D does not contain any induced directed cycle $\vec{C}_n$ with $n \geq 3$.

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Proof (by contraposition).

"$\Longrightarrow$" 1) Assume $S(D)$ is not perfect.
$\Longrightarrow \exists$ induced subgraph $H = (V', E')$ of $S(D)$: $\omega(H) < \chi(H)$.
By the observation and from $S(D[V']) = H$ we get

$$\omega(D[V']) = \omega(S(D[V'])) = \omega(H) < \chi(H) = \chi(S(D[V'])) \leq \chi(D[V'])$$

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$\Longrightarrow$ D is not perfect.
2) Assume D contains induced directed cycle $\vec{C}_n$, $n \geq 3$.
$\Longrightarrow$ D is not perfect, since $\omega(\vec{C}_n) = 1 < 2 = \chi(\vec{C}_n)$.

The main result

Theorem

A digraph D is perfect if and only if $S(D)$ is a perfect graph and D does not contain any induced directed cycle $\vec{C}_n$ with $n \geq 3$.

Proof (by contraposition).

"$\Longleftarrow$" Assume that $S(D)$ is perfect but D is not perfect.
Suffices to show: D contains induced $\vec{C}_n$, $n \geq 3$.
Let $H = (V', A')$ be induced subdigraph of D: $\omega(H) < \chi(H)$.
$\Longrightarrow \exists$ proper coloring of $S(H) = S(D)[V']$ with $\omega(S(H))$ colors, i.e., by Observation 1, with $\omega(H)$ colors.
$\Longrightarrow$ is not a feasible coloring for H.
$\Longrightarrow \exists$ (not necessarily induced) monochromatic $\vec{C}_n$ ($n \geq 3$) in $O(H)$
Let C be such a cycle of minimal length. C is induced!

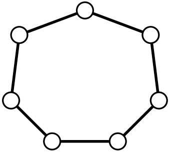
The Strong Perfect Graph Theorem

Theorem (Chudnovsky, Robertson, Seymour, Thomas (2006))

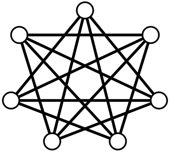
A graph is perfect if and only if it does not contain induced subgraphs of the following types:

(1) *odd holes: i.e. cycles of odd length* ≥ 5 *resp.*

(2) *odd antiholes: i.e. complements of type (1).*



type (1)



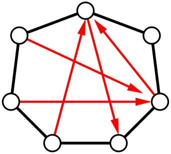
type (2)

The Strong Perfect Digraph Theorem

Theorem

A digraph is perfect if and only if it does not contain induced subdigraphs of the following types:

1. *filled odd holes: i.e. D with $S(D)$ is odd hole resp.*
2. *filled odd antiholes: i.e. D with $S(D)$ is odd antihole resp.*
3. *directed cycles of length ≥ 3.*



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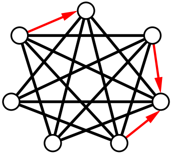
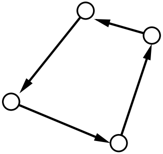


type (2)



type (3)

Complexity results

COLORING and MAX induced ACYCLIC SUBDIGRAPH

Corollary (from main result)

If D is a perfect digraph, then a coloring is feasible for D if and only if it is feasible for $S(D)$.

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Proofs use:

Theorem (Grötschel, Lovász, Schrijver (1981))

The COLORING resp. MAX INDEPENDENT SET problem for perfect graphs is polynomially solvable.

Question 1

Open Question

Are there other interesting $\mathcal{NP}$-hard problems on digraphs that are polynomially solvable for perfect digraphs?

RECOGNITION of perfect digraphs

In order to test, whether a digraph D is perfect, by our Theorem we have to test

1.) whether $S(D)$ is perfect, and

2.) whether D does not contain an induced $\vec{C}_n$, $n \geq 3$.

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Theorem (Chudnovsky, Cornuéjols, Liu, Seymour, Vušković (2005))

Recognizing graphs of types (1) resp. (2) ("Berge graphs") is in $\mathcal{P}$.

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Theorem

The PERFECT DIGRAPH RECOGNITION problem is co-$\mathcal{NP}$-complete.

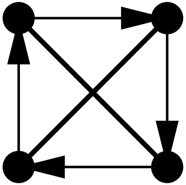
Question 2

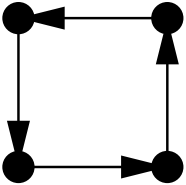
Open Question

Are there other interesting efficiently solvable problems on perfect graphs that have generalizations to perfect digraphs which are $\mathcal{NP}$-hard?

A "Weak Perfect Digraph Theorem"

Complements of perfect digraphs may be not perfect





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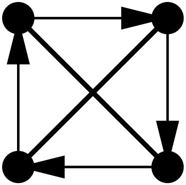

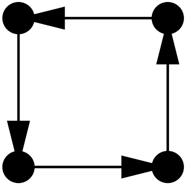

So there is no direct analog to Lovasz' Perfect Graph Theorem.

Theorem (Lovasz (1972))

A graph is perfect if and only if its complement is perfect.

A weak perfect digraph theorem

Def 1: A *superorientation* of an undirected graph $G = (V, E)$ is a digraph $D = (V, A)$, so that for any $e = vw \in E$ there is an arc (v, w) or (w, v) or both in A, and for any $vw \notin E$ there is none of the arcs (v, w) and (w, v) in A. We write $G(D) := G$.

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A digraph D is perfect if and only if its (loopless) complement $\overline{D}$ is a clique-acyclic superorientation of a perfect graph.

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Proof. By main result, D being perfect is equivalent to $S(D)$ perfect ($\overset{Lovasz}{\iff}$ $\overline{S(D)}$ perfect $\iff G(\overline{D})$ perfect) and D contains no induced directed cycle ($\iff \overline{D}$ is clique-acyclic)

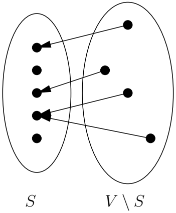
Consequences (I): Recognition of $\overline{D}$

Corollary

The recognition of clique-acyclic superorientations of perfect graphs is co-$\mathcal{NP}$-complete.

Consequences (II): Kernels

A kernel S of a digraph $D = (V, A)$:



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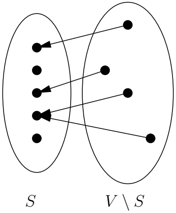

Theorem (Boros, Gurvich (2006))

"Perfect graphs are kernel-solvable," i.e. every clique-acyclic superorientation of a perfect graph has a kernel.

Corollary

For any perfect digraph D, the complement $\overline{D}$ has a kernel.

Kernels: The contrast

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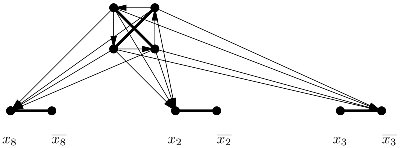
Proof. Reduction from 3-SAT like in Chvátal's classical proof that existence of kernels in digraphs is $\mathcal{NP}$-complete:



Open questions

Open questions on perfect digraphs

Open Question (1)

Are there other interesting efficiently solvable problems on perfect graphs that have generalizations to perfect digraphs which are $\mathcal{NP}$-hard?

Open Question (2)

Are there other interesting $\mathcal{NP}$-hard problems on digraphs that are polynomially solvable for perfect digraphs?

Open Question (3)

Are there other problems that are $\mathcal{NP}$-complete or co-$\mathcal{NP}$-complete for graphs in general as well as for perfect digraphs?

Open Question (4)

What is the complexity of recognizing superorientations of perfect graphs that have a kernel?

Other questions on dichromatic numbers

Conjecture (Neumann-Lara (1985))

Orientations of planar graphs are 2-colorable.

Open Question

Determine the maximum dichromatic number $M(n)$ of a tournament of order n.

Known values

n	0	1	2	3	4	5	6	7	8	9	10	11
$M(n)$	0	1	1	2	2	2	2	3	3	3	3	4

Thank you!

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Recognition of non-perfect digraphs

NON-PERFECT DIGRAPH RECOGNITION: Given a digraph, decide whether it is not perfect.

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The NON-PERFECT DIGRAPH RECOGNITION problem is $\mathcal{NP}$-complete.

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Proof. 1.) NON-PERFECT DIGRAPH RECOGNITION is in $\mathcal{NP}$.
Certificate is an induced directed cycle, a filled odd hole/antihole.

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Proof. 1.) NON-PERFECT DIGRAPH RECOGNITION is in $\mathcal{NP}$. Certificate is an induced directed cycle, a filled odd hole/antihole.

2.) Now we prove $\mathcal{NP}$-completeness. This proof is very similar to the proof of Bang-Jensen, Havet, and Trotignon (2010) of the $\mathcal{NP}$-completeness of testing whether a digraph has an induced directed cycle.

$\mathcal{NP}$-completeness proof I

We describe a reduction from 3-SAT to NON-PERFECT DIGRAPH RECOGNITION.

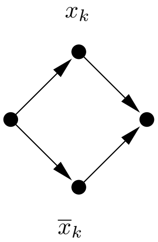
Instance of 3-SAT: Boolean formula of type

$$F = \bigwedge_{i=1}^{m} C_i = \bigwedge_{i=1}^{m} (l_{i1} \vee l_{i2} \vee l_{i3}) \qquad \text{with } l_{ij} \in \{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}.$$

Given such an instance, we construct a digraph in the following way.

$\mathcal{NP}$-completeness proof II (gadgets)

For each variable x_k we construct a variable gadget:



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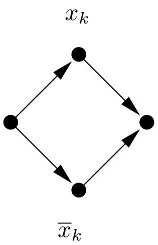
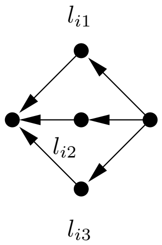
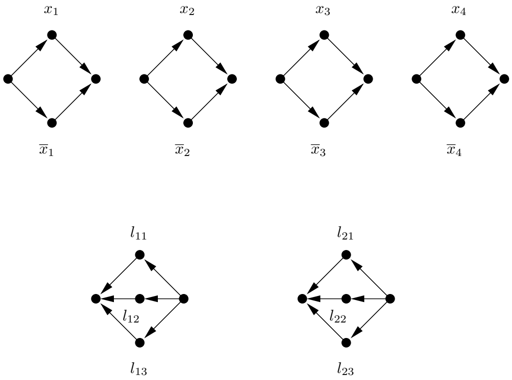


For each clause C_i we construct a clause gadget:



$\mathcal{NP}$-completeness proof III (construction)



$(\overline{x}_1 \vee x_3 \vee x_2) \wedge (x_3 \vee \overline{x}_2 \vee \overline{x}_4)$ The gadgets

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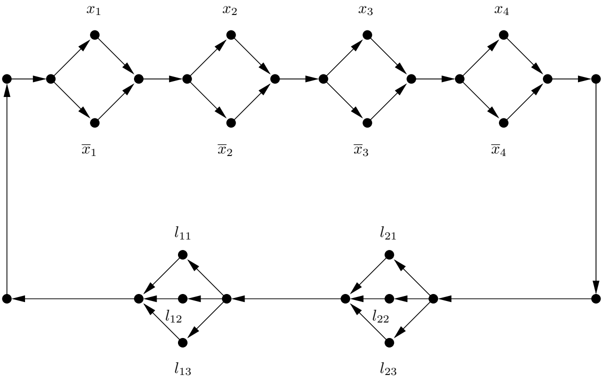

$(\overline{x}_1 \vee x_3 \vee x_2) \wedge (x_3 \vee \overline{x}_2 \vee \overline{x}_4)$ Form a ring

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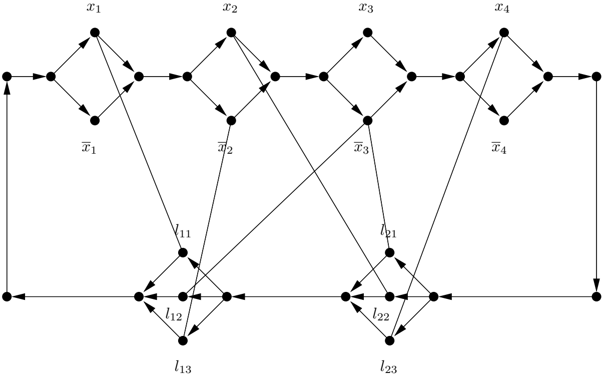
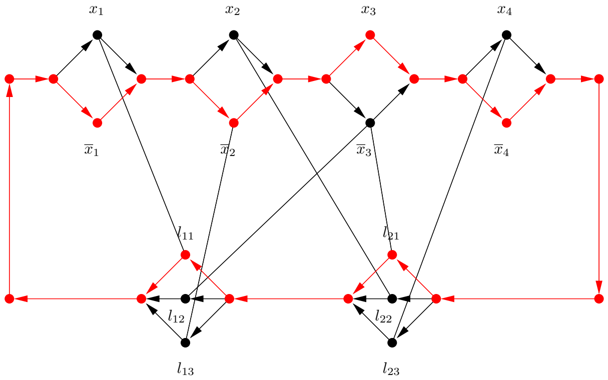


$(\overline{x}_1 \vee x_3 \vee x_2) \wedge (x_3 \vee \overline{x}_2 \vee \overline{x}_4)$ Insert edges

$\mathcal{NP}$-completeness proof III (construction)



$(\overline{x}_1 \vee x_3 \vee x_2) \wedge (x_3 \vee \overline{x}_2 \vee \overline{x}_4)$ Induced directed cycle iff satisfyable

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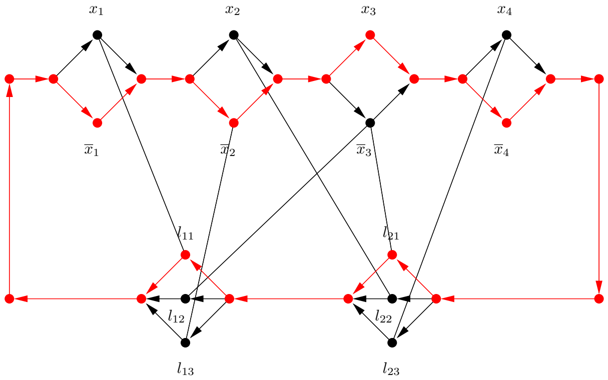

$(\overline{x}_1 \vee x_3 \vee x_2) \wedge (x_3 \vee \overline{x}_2 \vee \overline{x}_4)$ not perfect iff satisfyable