Extension complexity of combinatorial polytopes

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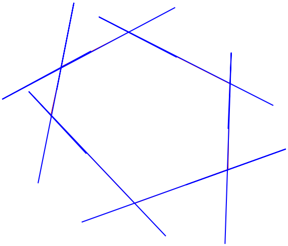
Joint Work with David Avis (McGill University, Kyoto University) and Samuel Fiorini (ULB)

Introduction

Polytope: Bounded intersection of finitely many halfspaces



$$P := \{x \in \mathbb{R}^d \mid Ax \leqslant b\}$$

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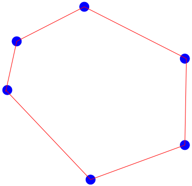
Polytope: Bounded intersection of finitely many halfspaces
Alternatively: Convex hull of finitely many points



$$P := \{x \in \mathbb{R}^d \mid Ax \leqslant b\} = \mathrm{conv}(V)$$

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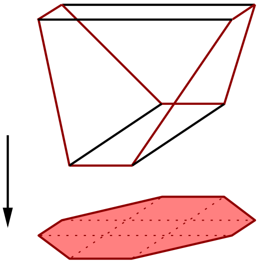

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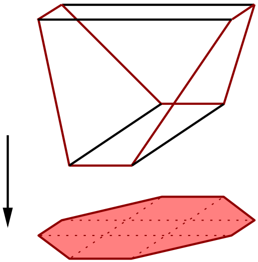

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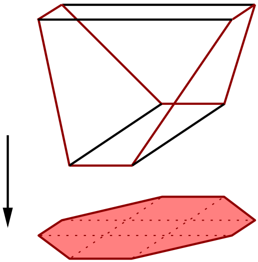

• Optimizing over a polytope can be achieved by optimizing over an EF.
• Number of inequalities in an EF may be substantially fewer!

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Extension complexity denoted **ex**(P) is the **minimum** number of inequalities representing any EF of P.

Example: $xc(P_n) = \Theta(\log n)$ where P_n is a regular n-gon.

Extended formulations for NP-hard problems: Backdrop

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• $xc(TSP(n)) \geqslant 2^{\sqrt{n}}$ Fiorini, Massar, Pokutta, T., de Wolf (STOC 2012)

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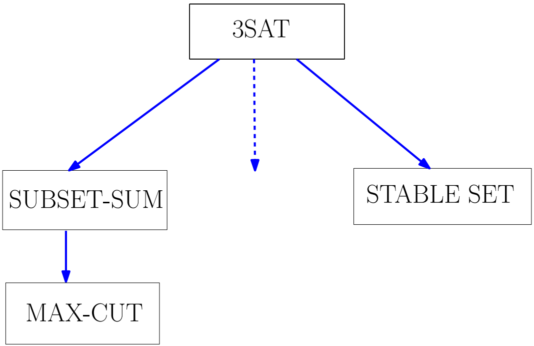
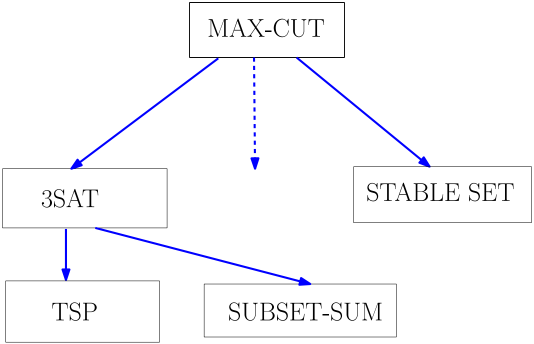
NP-hardness: the usual game



Extension complexity: the game so far



Two observations

Two observations:

- If P is a face of Q, then $\text{xc}(P) \leqslant \text{xc}(Q)$.
- If P is a projection of Q, then $\text{xc}(P) \leqslant \text{xc}(Q)$.

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Heuristic:

- Define the "natural" polytope with your favorite problem.
- Inspect the available textbook NP-hardness reduction for your problem.
- Do any of the previous two observations apply?
- Use the lower-bound for SAT polytope.

Proving lower bounds (Ad-hoc inspection of reductions)

Example 1: Subset sum

$$SUBSETSUM(A, b) := \text{conv}\left(\left\{x \in [0,1]^n \mid \sum_{i=1}^{n} a_i x_i = b\right\}\right)$$

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Reduction from SAT

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	1	0
v_1'	=	1	0	0	0	1	0	1
v_2	=	0	1	0	0	1	1	0
v_2'	=	0	1	0	1	0	0	1
v_3	=	0	0	1	1	1	0	0
v_3'	=	0	0	1	0	0	1	1
s_1	=	0	0	0	1	0	0	0
s_1'	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s_2'	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s_3'	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s_4'	=	0	0	0	0	0	0	2
b	=	1	1	1	4	4	4	4

Table : The base 10 numbers created as an instance of subset-sum for the 3SAT formula $(x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)$.

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Observation: x is a vertex of $SUBSETSUM(A(\Phi), b)$ if and only if x restricted to variables $(v_1, \ldots, v_n)$ is a vertex of $SAT(\Phi)$

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SAT polytope is a projection of the SUBSETSUM polytope

- $\text{xc}(SUBSETSUM) \geqslant \text{xc}(SAT)$

Some other polytopes

1. **3-dimensional Matching**

$$3DM(G) := conv(\{\chi(E') \mid E' \subseteq E \quad is\ a\ 3d-matching\})$$

2. **Stable set for cubic planar graphs**

$$STAB(G) := conv(\{\chi(V') \mid V' \subseteq V \quad is\ a\ stable\ set\})$$

3. **Cut polytope for K_6 minor-free graphs**

$$CUT(G) := conv(\{\chi(E') \mid E' \subseteq E \quad is\ a\ cut\})$$

4. **Cut polytope for $K_{1,n,n}$ aka Bell Polytope**

Some other polytopes

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$$3DM(G) := conv(\{\chi(E') \mid E' \subseteq E \quad is\ a\ 3d - matching\})$$

2. **Stable set for cubic planar graphs** $xc(STAB(G)) \geqslant 2^{\Omega(\sqrt{n})}$

$$STAB(G) := conv(\{\chi(V') \mid V' \subseteq V \quad is\ a\ stable\ set\})$$

3. **Cut polytope for K_6 minor-free graphs** $xc(CUT(G)) \geqslant 2^{\Omega(n^{1/4})}$

$$CUT(G) := conv(\{\chi(E') \mid E' \subseteq E \quad is\ a\ cut\})$$

4. $CUT(K_{1,n,n})$ **i.e. Bell Polytope** $xc(CUT(K_{1,n,n})) \geqslant 2^{\Omega(n)}$

Some other polytopes

We also show a general result about cut polytopes of minors of a graph.

Theorem

Let G be a graph and H be a minor of G. Then,

$$xc(CUT(G)) \geqslant xc(CUT(H)).$$

Concluding remarks / questions

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- How to canonically associate polytopes with "problems"?
- What kind of reductions allow for translation of lower bounds?
- What is the class of problems that are captured by small extended formulations?

Thank You!

Associating polytopes to problems

- Associate polytopes to a “verifier”

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- Captures our intuition of "natural" for many polytopes.

- Instead of talking about different problems, we can talk about different verifiers for the same problem.

Linear reductions

A reduction from an optimization problem A to B is called **linear** iff there exists a matrix R such

$$\forall (c, K) \exists x \in P(A), c^T x \geqslant B \Leftrightarrow \exists y \in P(B), w^T y \geqslant K'$$

where, (w,K')=(c,K)R.

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where, (w,K')=(c,K)R.

Theorem: $P(A)$ is a projection of some face of $P(B)$ if and only if there is a linear reduction from A to B.