Parameterized Complexity of Directed Steiner Tree and Domination Problems on Sparse Graphs

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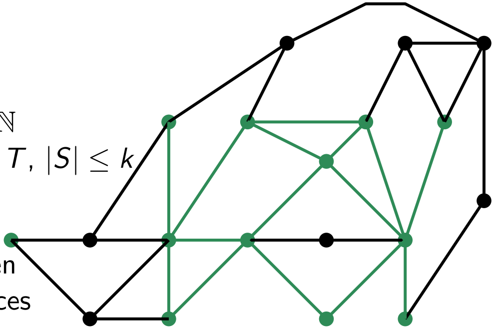
Prague, 31st July 2013

Steiner Tree

STEINER TREE
Input: $G = (V, E)$, $T \subseteq V$, $k \in \mathbb{N}$
Question: Is there a set $S \subseteq V \setminus T$, $|S| \leq k$ such that $G[S \cup T]$ is connected?

Vertices in T ... terminals ... green
Vettices in $V \setminus T$ ... Steiner vertices



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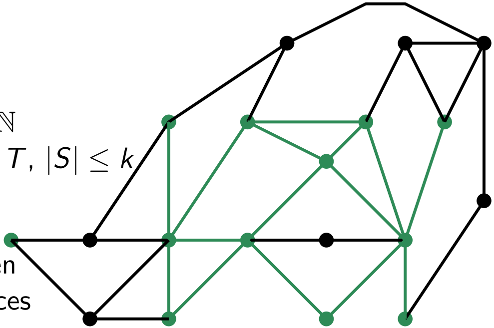

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- FPT wrt $|T|$ ($O^*(3^{|T|})$) [Dreyfuss & Wagner 1972]
- $O^*(2^{|T|})$ time, poly-space algorithm [Nederlof 2009]
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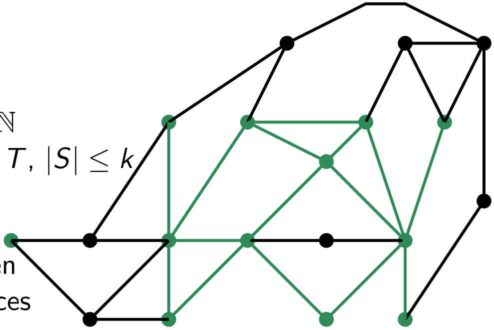

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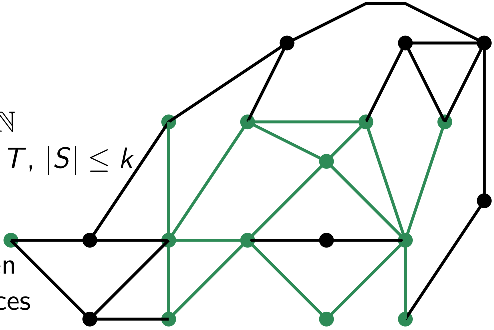

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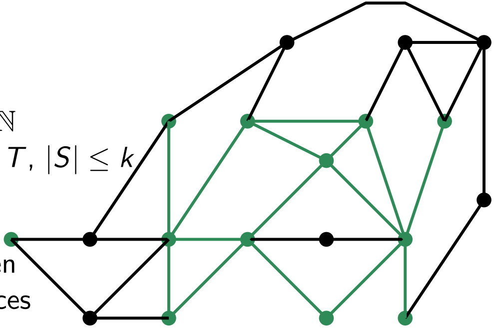

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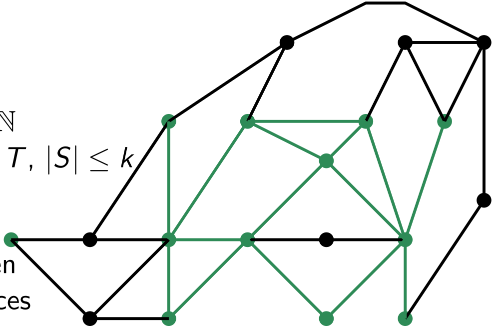

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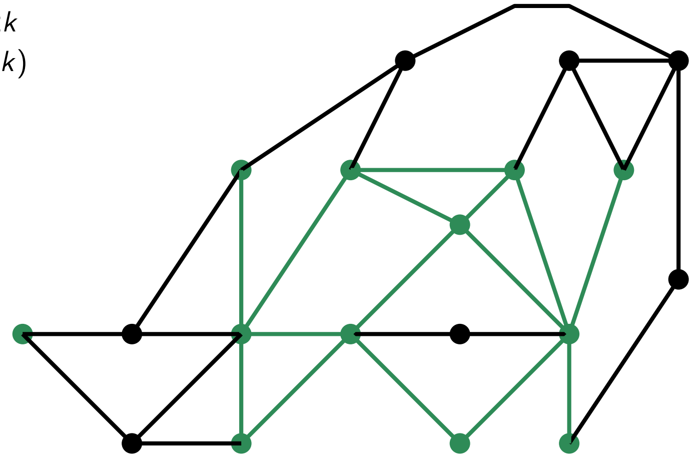
[4] Treewidth of a graph is the minimum width of a tree decomposition of the

Steiner Tree on Planar graphs

On planar graphs

- contract edges between terminals
- on a path at least every second vertex is Steiner
- diameter $\leq 2k$
- treewidth $O(k)$
- FPT wrt k



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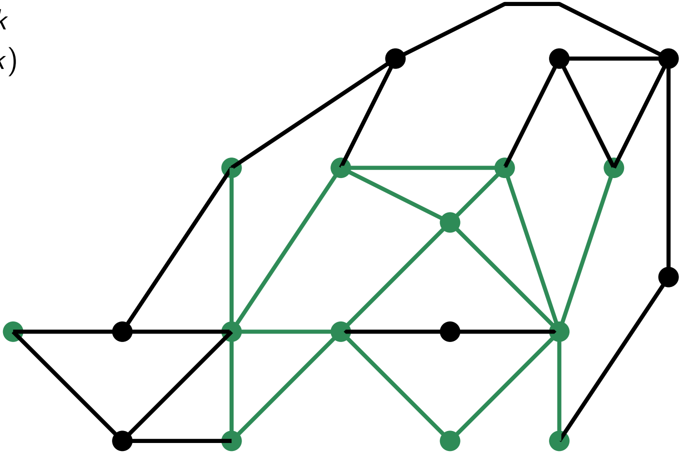

- kernel of size $O((k + |T|)^{142})$ [Pilipczuk et al. 2013]

Intermezzo on sparse graphs

Sparse graph classes often studied:

- planar graphs
- K_h-minor free
- K_h-topological minor free
- d-degenerate

Sparse directed = sparse underlying undirected

Directed Steiner Tree

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Input: $D = (V, A)$, root $r \in V$, $T \subseteq V$, k
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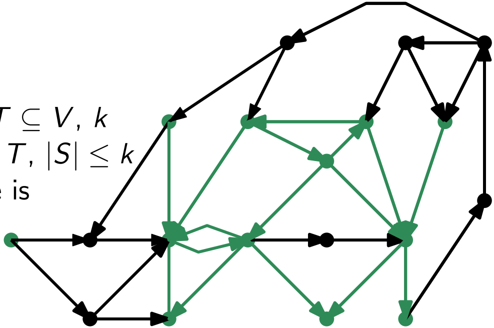

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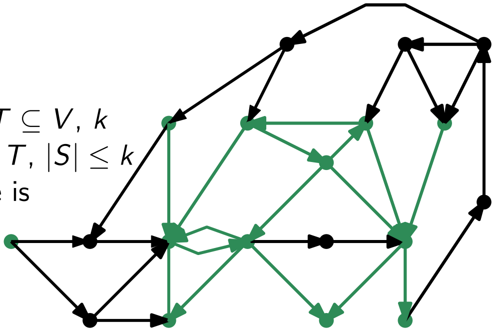

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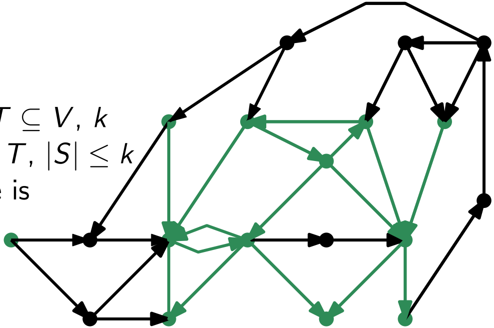

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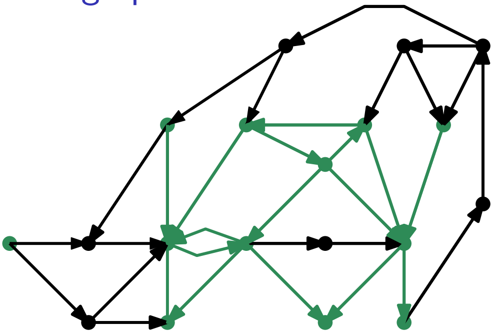
On planar graphs

- cannot contract the arcs between terminals
- need a different approach

Directed Steiner Tree on planar graphs



Directed Steiner Tree on planar graphs

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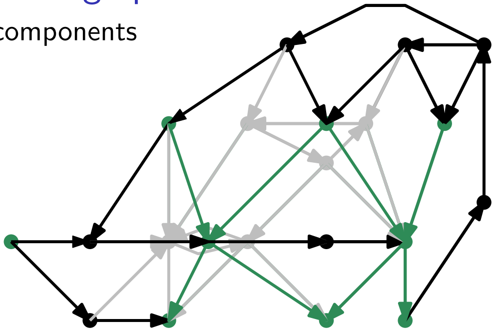
  -$D[T]$ becomes a DAG



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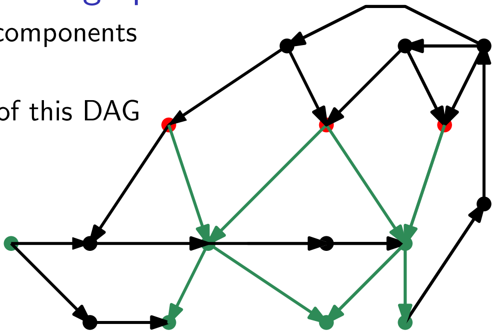
  - *source-terminals, sources*



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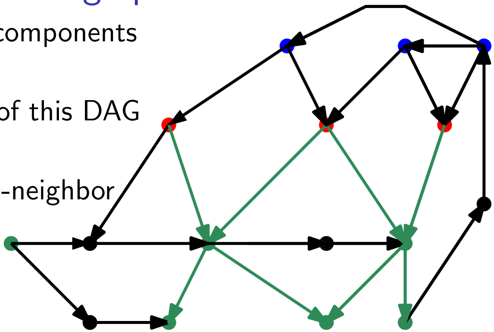

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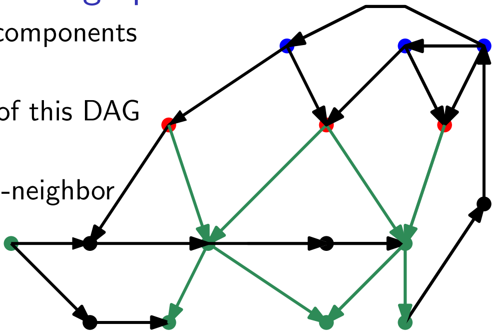

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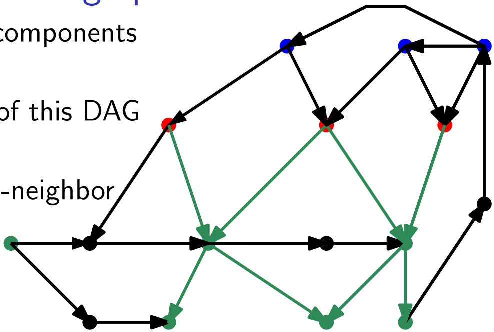

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- *big dominator* - dominates at least $d + 1$ sources
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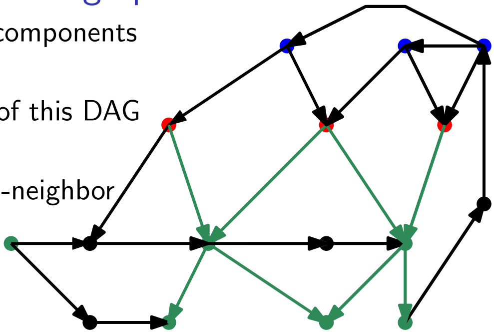

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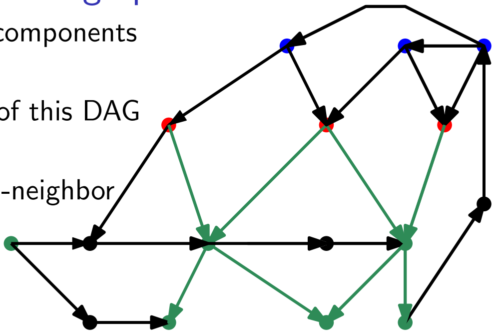

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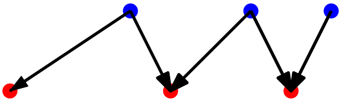
- solution: ignore the small dominators !

Algorithm for d-degenerate



- look on the bipartite graph between big dominators and sources dominated by them
- D is d-degenerate $\Rightarrow$ there is a source v dominated by $\leq d$ dominators

Solution options:

- v is dominated by some big dominator in $N^{-}(v)$
  -branch on which of them
- v is dominated by some small dominator
  -save it for later
  -delete the big dominators in $N^{-}(v)$

Progress measure:

- at most dk sources can be dominated by k small dominators

Algorithm formally

the algorithm keeps 5 disjoint vertex sets:

- Y - partial solution - size $\leq k$.
- B_h - dominators which dominate $> d$ sources not dominated by Y.
- B_l - dominators which dominate $\leq d$ of them.
- W_h - sources not dominated by Y, but dominated by B_h.
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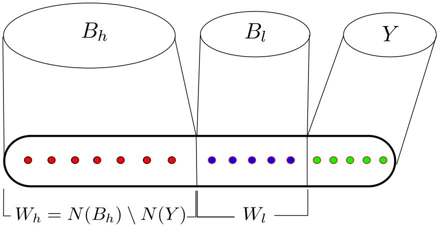

the algorithm:

- Find vertex $v \in W_h$ with the least neighbors in B_h.
- For each $u \in B_h \cap N^-(v)$ add u to Y, update the other sets, and recurse
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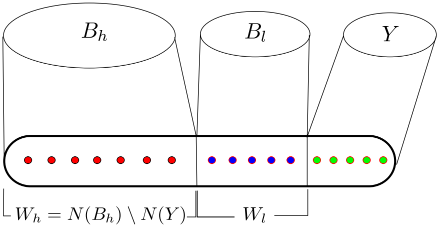

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- the measure $d(k - |Y|) - |W_l|$ drops in each case
- if W_h empty, apply Nederlof's algo with $W_l \cup Y$ as terminals

Consequences of the algorithm

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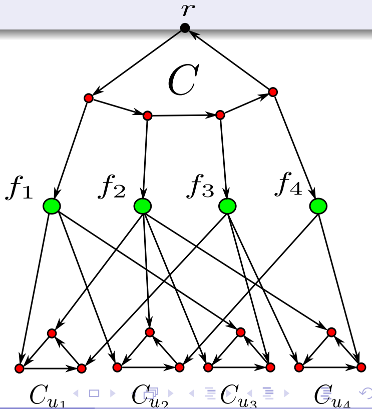
Reduction from Set Cover.

Set Cover

Input: A universe U, a family $\mathcal{F} \subseteq 2^U$, $k \in \mathbb{N}$

Question: Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$, such that $|\mathcal{F}'| \leq k$ and $\bigcup \mathcal{F}' = U$?



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- above bounds actually for K_h-topological minor free

Our results for DST

$D[T]$ arbitrary

- $O^*(3^{hk+o(hk)})$-time on K_h-minor free digraphs
- $O^*(f(h)^k)$-time on K_h-topological minor free digraphs.
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- $O^*(3^{dk+o(dk)})$-time on d-degenerate graphs
  - DST is FPT wrt k on $o(\log n)$-degenerate graph classes
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  - first FPT algorithm for undirected Steiner Tree on d-degenerate
- For any constant $c > 0$, no $f(k)n^{o(\frac{k}{\log k})}$-time algorithm on graphs of degeneracy $c \log n$ unless ETH[5] fails.
  - no $O^*(2^{o(d)f(k)})$-time algorithm unless ETH fails
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[5] Exponential time hypothesis — 3-SAT cannot be solved in time $2^{o(n)}$

Application to Dominating Set

- there is a parameterized reduction from DOMINATING SET to (DIRECTED) STEINER TREE, preserving the degeneracy
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The sets

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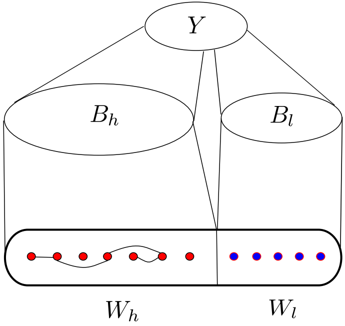
- B - vertices dominated by Y.
- W - not dominated by Y.
- B_h - vertices of B dominating $\geq d + 1$ vertices of W.
- B_l - vertices of B dominating $\leq d$ vertices of W.
- W_h - vertices in W with neighbor in $B_h \cup W$.
- W_l - remaining vertices of W.



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graph class	running time
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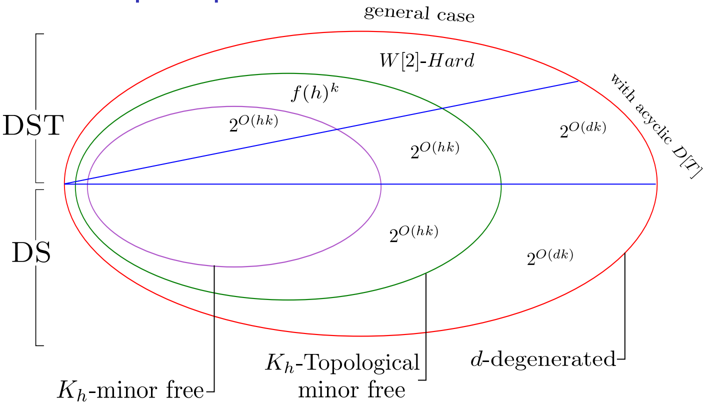
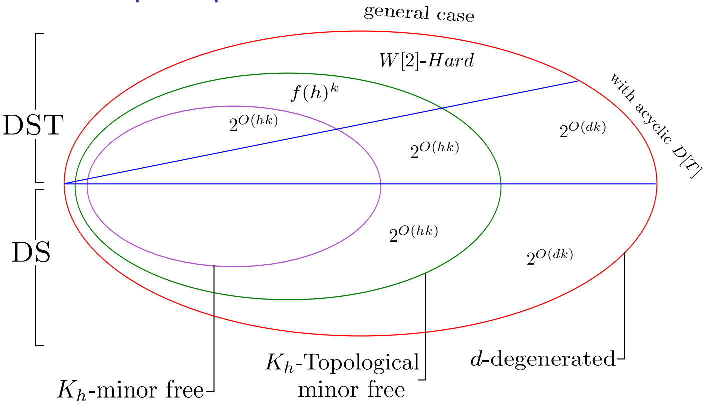
- no algorithm $O^*(2^{f(d)o(k)})$, unless ETH fails.
- $O(dn\log n)$-time d^2-approximation algorithm on d-degenerate graphs.

Future research, Open questions



- Asymptot. optimal running times for d-degenerate graphs - $2^{O(kd)}$
  - SETH lower bound on the basis?
  - Improving the upper bound - currently $3^{kd+o(kd)}$
- STRONGLY CONNECTED STEINER SUBGRAPH, DIRECTED STEINER NETWORK in planar graphs
- kernel for planar STEINER TREE wrt $|T|$? wrt k?

Future research, Open questions



- Asymptot. optimal running times for d-degenerate graphs - $2^{O(kd)}$
  - SETH[6] lower bound on the basis?
  - Improving the upper bound - currently $3^{kd+o(kd)}$
- Strongly Connected Steiner Subgraph, Directed Steiner Network in planar graphs
- kernel for planar Steiner Tree wrt $|T|$? wrt k?

[6] Strong ETH — SAT cannot be solved in time $(2-\varepsilon)^n$

Thank you for your attention!