The probability of planarity of a random graph near the critical point

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CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS

Freie Universität Berlin

The material of this talk

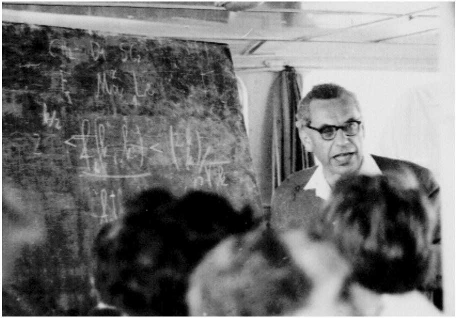
1.– **Planarity on the critical window for random graphs**

2.– **Our result. The strategy**

3.– **Cubic planar multigraphs**

4.– **Other applications**

Planarity on the critical window for random graphs



The model $G(n, M)$

There are $2^{\binom{n}{2}}$ labelled graphs with n vertices.

A random graph $G(n, M)$ is the probability space with properties:

- Sample space: set of labelled graphs with n vertices and $M = M(n)$ edges.
- Probability: Uniform probability $\left(\binom{\binom{n}{2}}{M}^{-1}\right)$

Properties:

♡ Fixed number of edges ✓

♣ The probability that a fixed edge belongs to the random graph is $p = \binom{n}{2}^{-1} M$. ✓

♠ There is not independence.

The Erdős-Rényi phase transition

Random graphs in $G(n, M)$ present a dichotomy for $M = \frac{n}{2}$:

1.– **(Subcritical)** $M = cn$, $c < \frac{1}{2}$: a.a.s. all connected components have size $O(\log n)$, and are either trees or unicyclic graphs.

2.– **(Critical)** $M = \frac{n}{2} + \lambda n^{2/3}$: a.a.s. the largest connected component has size of order $n^{2/3}$

3.– **(Supercritical)** $M = cn$, $c > \frac{1}{2}$: a.a.s. there is a **unique** component of size of order n.

Double jump in the creation of the ***giant component***.

The problem; what was known

ON THE EVOLUTION OF RANDOM GRAPHS

by

P. ERDŐS and A. RÉNYI

Dedicated to Professor P. Turán at his 50th birthday.

We can show that for $N(n) = \frac{n}{2} + \lambda\sqrt{n}$ with any real λ the probability of $\Gamma_{n,N(n)}$ not being planar has a positive lower limit, but we cannot calculate ts value. It may even be I, though this seems unlikely.

PROBLEM: Compute

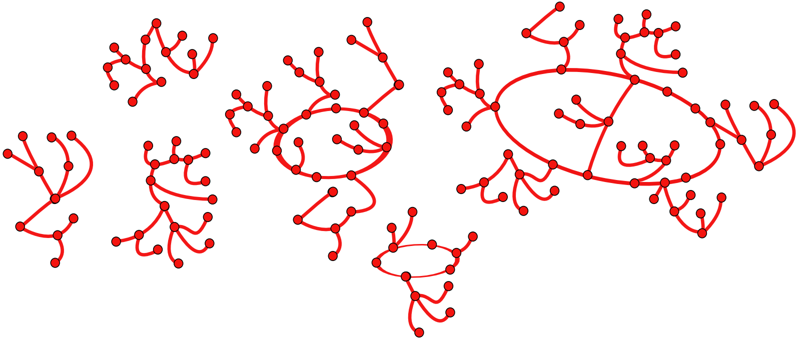
$$p(\lambda) = \lim_{n\to\infty} \Pr\left\{G\left(n, \tfrac{n}{2}(1+\lambda n^{-1/3})\right) \text{ is planar}\right\}$$

What was known:

- Janson, Łuczak, Knuth, Pittel (94): $0.9870 < p(0) < 0.9997$
- Łuczak, Pittel, Wierman (93): $0 < p(\lambda) < 1$

Our contribution: **the whole description of** $p(\lambda)$

Our result. The strategy



The main theorem

Theorem (Noy, Ravelomanana, R.) Let g_r be the number of cubic planar weighted multigraphs with $2r$ vertices. Write

$$A(y,\lambda) = \frac{e^{-\lambda^3/6}}{3^{(y+1)/3}} \sum_{k\geq 0} \frac{\left(\frac{1}{2}3^{2/3}\lambda\right)^k}{k!\,\Gamma\big((y+1-2k)/3\big)}.$$

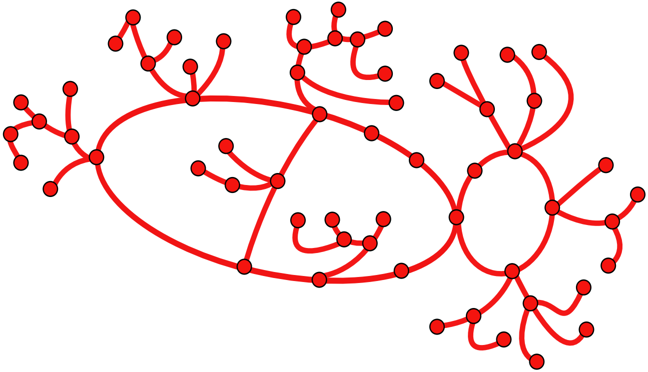
Then the limiting probability that the random graph $G\left(n, \frac{n}{2}(1+\lambda n^{-1/3})\right)$ is planar is

$$p(\lambda) = \sum_{r\geq 0} \frac{\sqrt{2\pi}}{(2r)!}\, g_r A\left(3r+\frac{1}{2}, \lambda\right).$$

In particular, the limiting probability that $G\left(n, \frac{n}{2}\right)$ is planar is

$$p(0) = \sum_{r\geq 0} \sqrt{\frac{2}{3}} \left(\frac{4}{3}\right)^r g_r \frac{r!}{(2r)!^2} \approx 0.99780.$$

The strategy (I): pruning a graph



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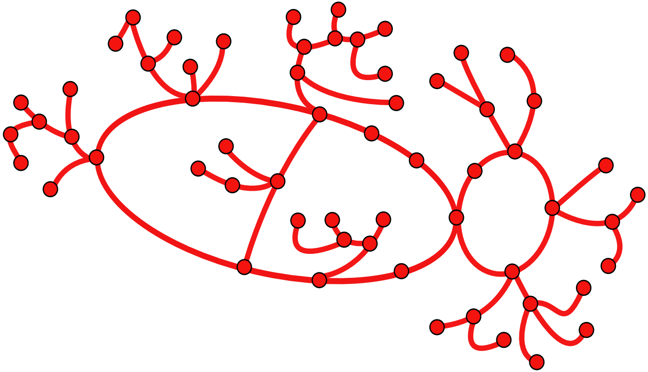

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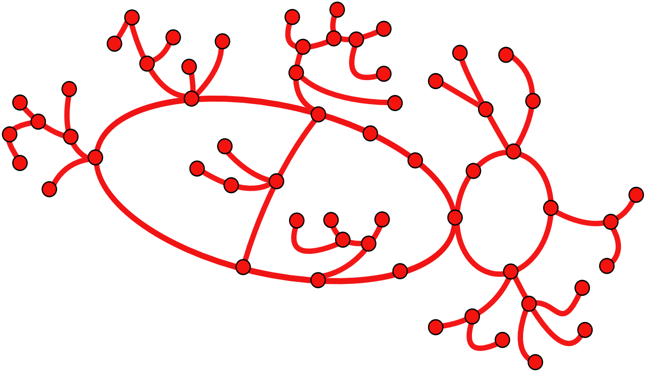

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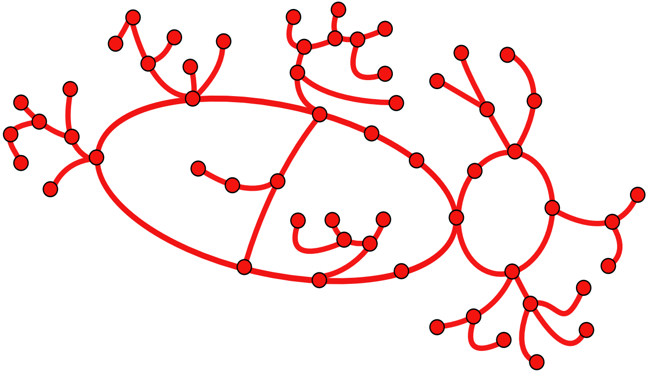

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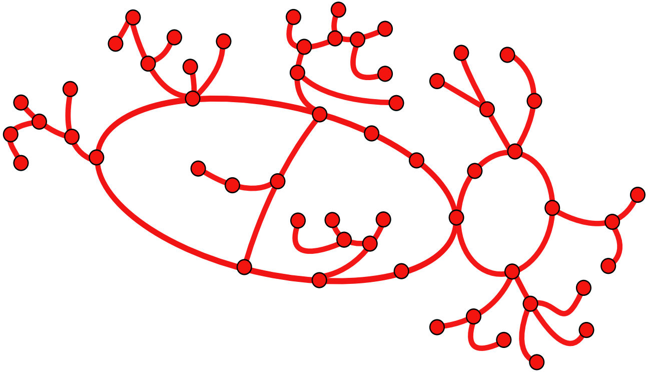

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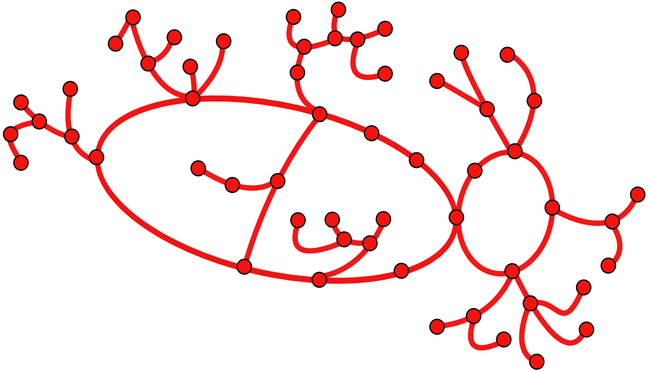

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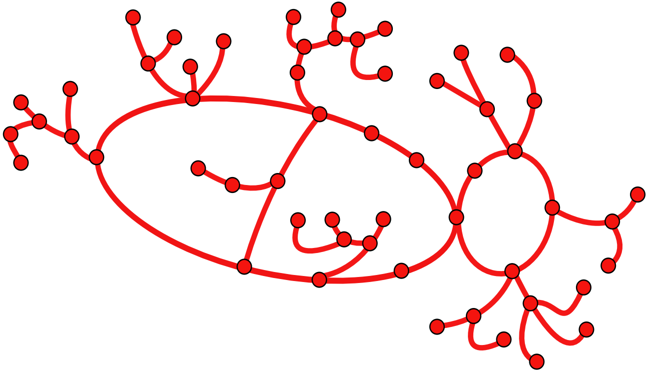

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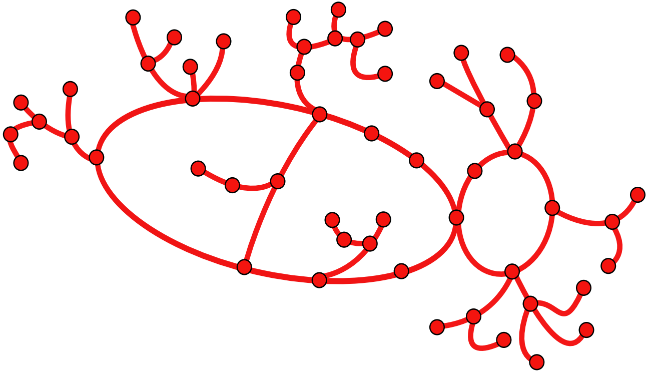

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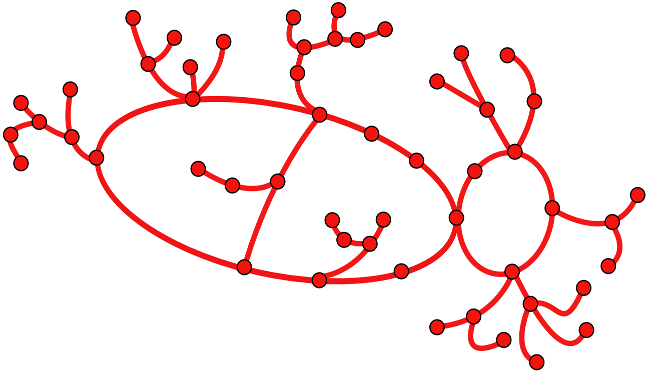

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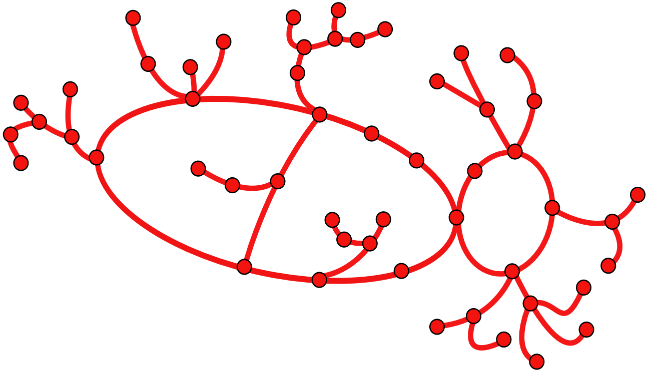

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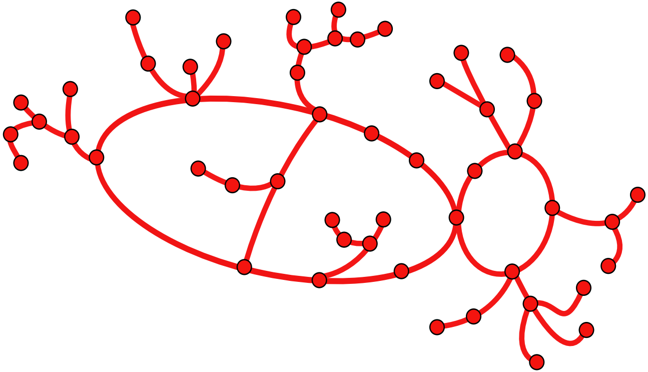

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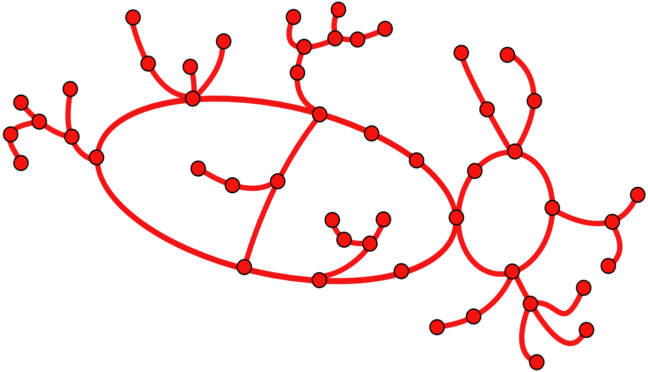

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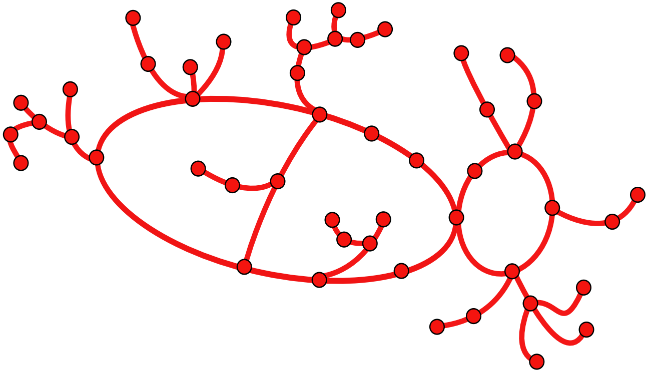

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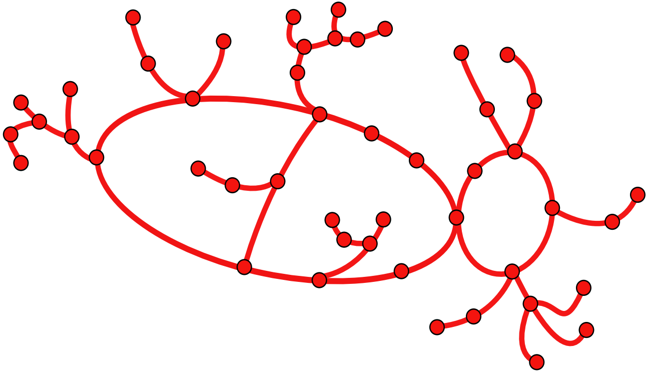

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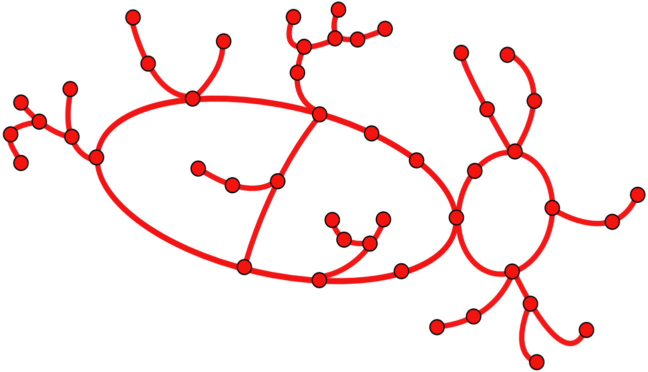

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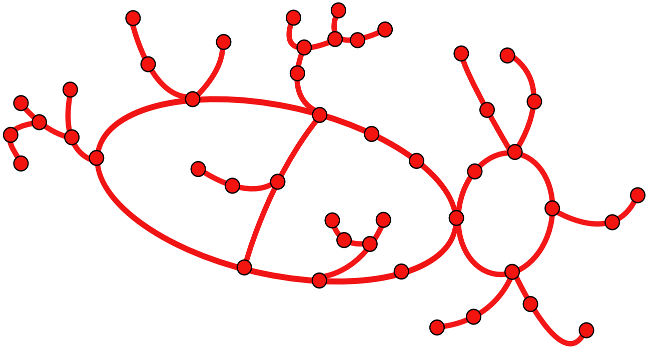

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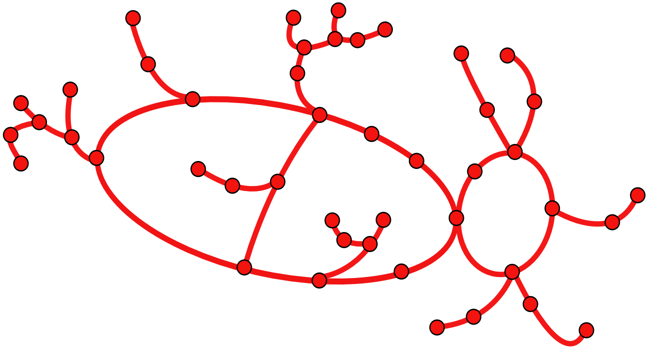

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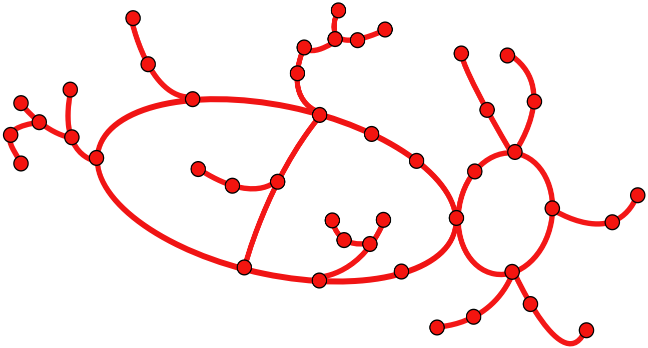

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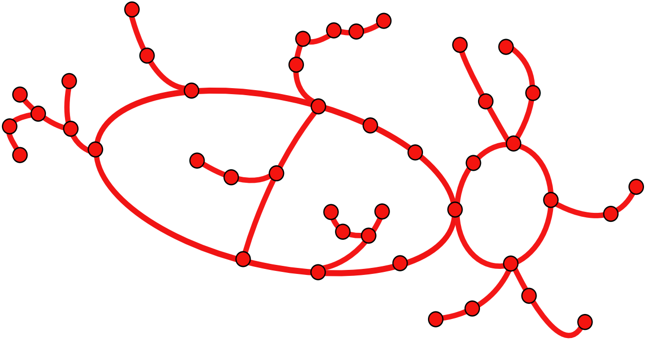

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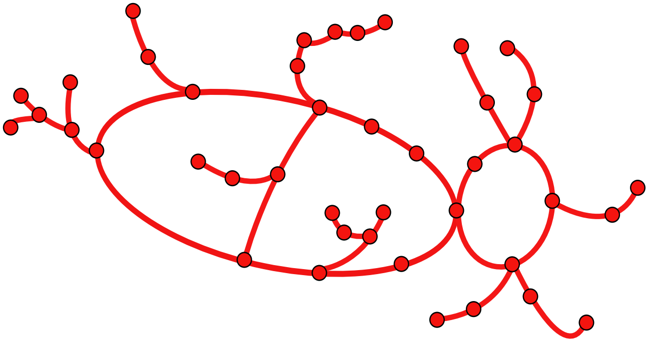

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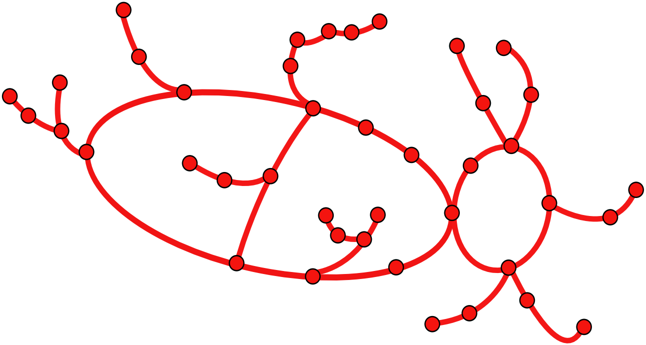

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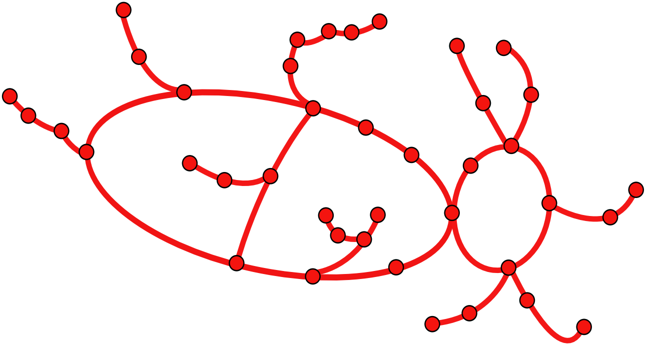

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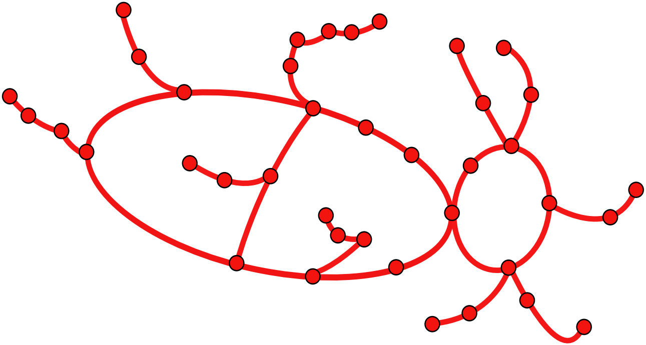

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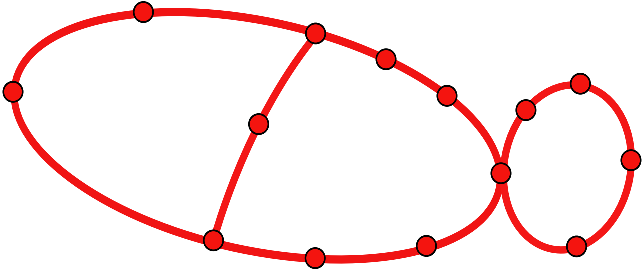

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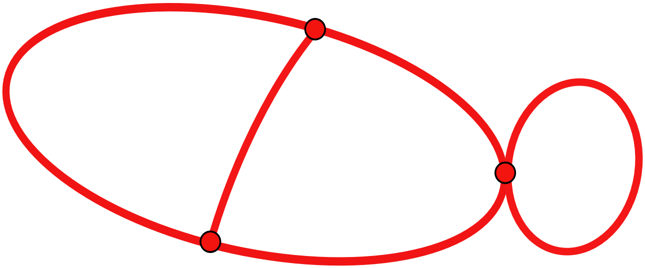
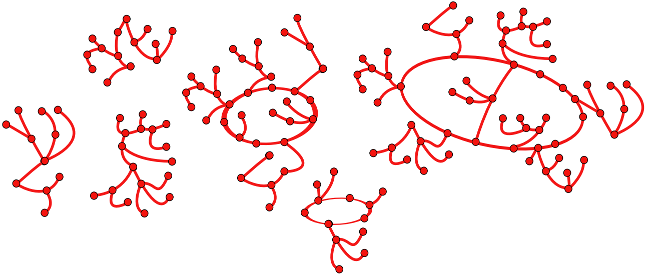


The resulting *multigraph* is the **core** of the initial graph

The strategy (and II): shape in the critical window

Łuczak, Pittel, Wierman (94):
the structure of a random graph in the critical window



$$p(\lambda) = \frac{\text{number of planar graphs with } \frac{n}{2}(1+\lambda n^{-1/3}) \text{ edges}}{\binom{\binom{n}{2}}{\frac{n}{2}(1+\lambda n^{-1/3})}}$$

Hence...**We need to count!**

The symbolic method à la Flajolet

COMBINATORIAL RELATIONS between CLASSES

$$\Updownarrow$$

EQUATIONS between GENERATING FUNCTIONS

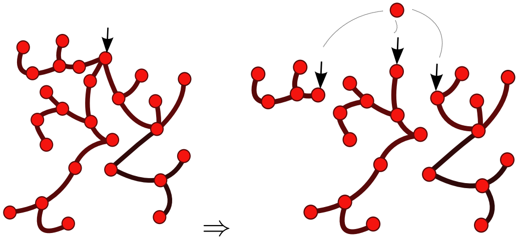
Class	Relations
$\mathcal{C} = \mathcal{A} \cup \mathcal{B}$	$C(x) = A(x) + B(x)$
$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$C(x) = A(x) \cdot B(x)$
$\mathcal{C} = \mathrm{Seq}(\mathcal{B})$	$C(x) = (1 - B(x))^{-1}$
$\mathcal{C} = \mathrm{Set}(\mathcal{B})$	$C(x) = \exp(B(x))$
$\mathcal{C} = \mathcal{A} \circ \mathcal{B}$	$C(x) = A(B(x))$

All GF are *exponential* $\equiv$ *labelled* objects

$$A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$$

Trees

We apply the previous grammar to count *rooted* trees



$$\mathcal{T} = \bullet \times \text{Set}(\mathcal{T}) \to T(x) = xe^{T(x)}$$

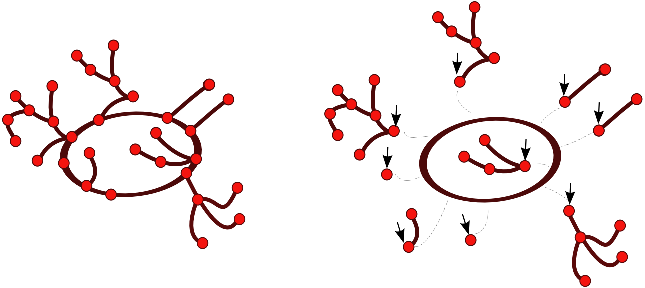
To forget the root, we just integrate: $(xU'(x) = T(x))$

$$\int_0^x \frac{T(s)}{s} ds = \left\{ \begin{array}{c} T(s) = u \\ T'(s)\, ds = du \end{array} \right\} = \int_{T(0)}^{T(x)} 1 - u \, du = T(x) - \frac{1}{2} T(x)^2$$

and the general version

$$e^{U(x)} = e^{T(x)} e^{-\frac{1}{2} T(x)^2}$$

Unicyclic graphs

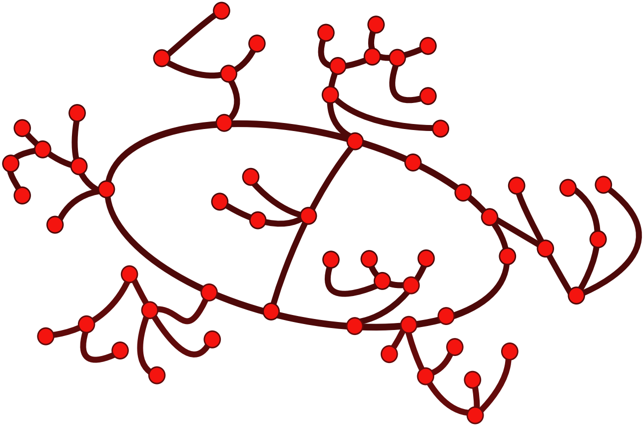


$$\mathcal{V} = \bigcirc_{\geq 3}(\mathcal{T}) \to V(x) = \sum_{n=3}^{\infty} \frac{1}{2} \frac{(n-1)!}{n!} (T(x))^n$$

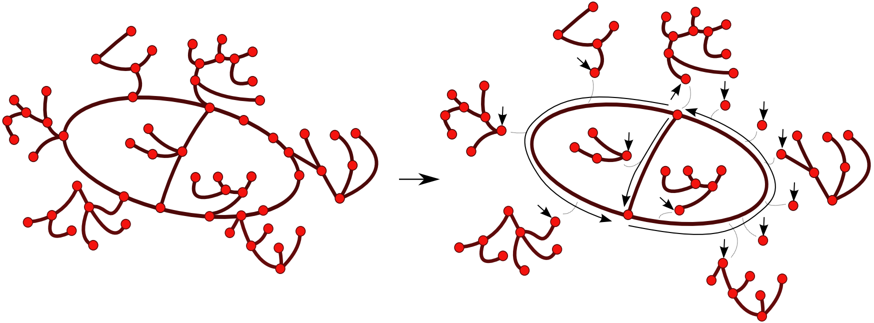
We can write $V(x)$ in a compact way:

$$\frac{1}{2}\left(-\log\left(1-T(x)\right) - T(x) - \frac{T(x)^2}{2}\right) \to e^{V(x)} = \frac{e^{-T(x)/2-T(x)^2/4}}{\sqrt{1-T(x)}}.$$

Cubic planar multigraphs



Planar graphs arising from cubic multigraphs



In an informal way:

$$\mathcal{G}(\bullet \leftarrow \mathcal{T}, \bullet - \bullet \leftarrow \mathrm{Seq}(\mathcal{T}))$$

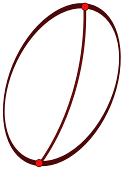
Weighted planar cubic multigraphs

Cubic multigraphs have $2r$ vertices and $3r$ edges (Euler's Relation)

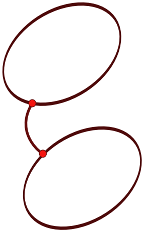
$$G(x,y) = \sum_{r \geq 1} \frac{g_r}{(2r)!} x^{2r} y^{3r} = g(x^2 y^3)$$

We need to remember the number of loops and the number of multiple edges to avoid symmetries:

weights $2^{-f_1 - f_2}(3!)^{-f_3}$



$\frac{1}{2!}\frac{1}{6}x^2y^3$



$\frac{1}{2!}\frac{1}{2^2}x^2y^3$

The equations

We have equations defining $G(z)$:

$$\begin{aligned}
G(z) &= \exp G_1(z) \\
3z\,\frac{dG_1(z)}{dz} &= D(z) + C(z) \\
B(z) &= \frac{z^2}{2}(D(z) + C(z)) + \frac{z^2}{2} \\
C(z) &= S(z) + P(z) + H(z) + B(z) \\
D(z) &= \frac{B(z)^2}{z^2} \\
S(z) &= C(z)^2 - C(z)S(z) \\
P(z) &= z^2C(z) + \frac{1}{2}z^2C(z)^2 + \frac{z^2}{2} \\
2(1 + C(z))H(z) &= u(z)(1 - 2u(z)) - u(z)(1 - u(z))^3 \\
z^2(C(z) + 1)^3 &= u(z)(1 - u(z))^3.
\end{aligned}$$

$u(z)$ is the **INPUT**: arising from map enumeration

The equations: an appetizer

All GF obtained (except $G(z)$) are *algebraic* GF; for instance:

$$
\begin{aligned}
&1048576\,z^6 + 1034496\,z^4 - 55296\,z^2 + \\
&\left(9437184\,z^6 + 6731264\,z^4 - 1677312\,z^2 + 55296\right) C + \\
&\left(37748736\,z^6 + 18925312\,z^4 - 7913472\,z^2 + 470016\right) C^2 + \\
&\left(88080384\,z^6 + 30127104\,z^4 - 16687104\,z^2 + 1622016\right) C^3 + \\
&\left(132120576\,z^6 + 29935360\,z^4 - 19138560\,z^2 + 2928640\right) C^4 + \\
&\left(132120576\,z^6 + 19314176\,z^4 - 12429312\,z^2 + 2981888\right) C^5 + \\
&\left(88080384\,z^6 + 8112384\,z^4 - 4300800\,z^2 + 1720320\right) C^6 + \\
&\left(37748736\,z^6 + 2097152\,z^4 - 614400\,z^2 + 524288\right) C^7 + \\
&\left(9437184\,z^6 + 262144\,z^4 + 65536\right) C^8 + 1048576\,C^9 z^6 = 0.
\end{aligned}
$$

The estimates

- The **excess** of a graph ($ex(G)$) is the number of edges minus the number of vertices

$$n![z^n] \overbrace{\frac{U(z)^{n-M+r}}{(n-M+r)!}}^{\text{Trees}, ex=-1} \overbrace{\frac{e^{-T(z)/2-T(z)^2/4}}{\sqrt{1-T(z)}}}^{\text{Unicyclic}, ex=0} \overbrace{\frac{g_r T(z)^{2r}}{(1-T(z))^{3r}}}^{\text{Cubic}, ex=3r-2r=r}$$

- We finally use **saddle point estimates**

Other applications

General families of graphs

Many families of graphs admit an straightforward analysis:

(Noy, Ravelomanana, R.)
Let $\mathcal{G} = \mathrm{Ex}(H_1, \ldots, H_k)$ and assume all the H_i are 3-connected. Let h_r be the number of cubic multigraphs in $\mathcal{G}$ with $2r$ vertices. Then the limiting probability that the random graph $G(n, \frac{n}{2}(1 + \lambda n^{-1/3}))$ is in $\mathcal{G}$ is

$$p_{\mathcal{G}}(\lambda) = \sum_{r \geq 0} \frac{\sqrt{2\pi}}{(2r)!} h_r A\left(3r + \frac{1}{2}, \lambda\right).$$

In particular, the limiting probability that $G(n, \frac{n}{2})$ is in $\mathcal{G}$ is

$$p_{\mathcal{G}}(0) = \sum_{r \geq 0} \sqrt{\frac{2}{3}} \left(\frac{4}{3}\right)^r h_r \frac{r!}{(2r)!^2}.$$

Moreover, for each λ we have

$$0 < p_{\mathcal{G}}(\lambda) < 1.$$

Examples...please

Some interesting families fit in the previous scheme:

- $\mathsf{Ex}(K_4)$: series-parallel graphs: there are not 3-connected elements in the family!
- $\mathsf{Ex}(K_{3,3})$: The same limiting probability as planar... K_5 does not appear as a core!
- Many others: $\mathsf{Ex}(K_{3,3}^+)$, $\mathsf{Ex}(K_5^-)$, $\mathsf{Ex}(K_2 \times K_3) \ldots$

- **PROBLEM:** coloured graphs (in preparation...)
- **PROBLEM:** compute exactly for graphs on surfaces

Gràcies!

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