A red-blue intersection problem

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Joint work with:

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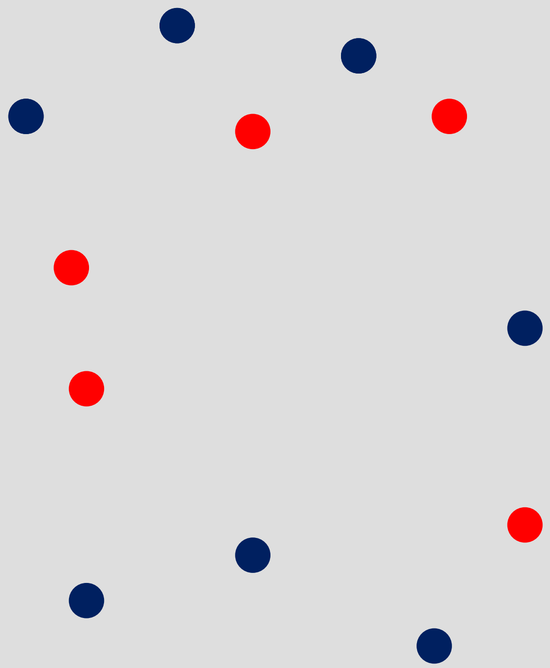
University of Seville, Spain

Midsummer Combinatorial Workshop XIX

The Problem



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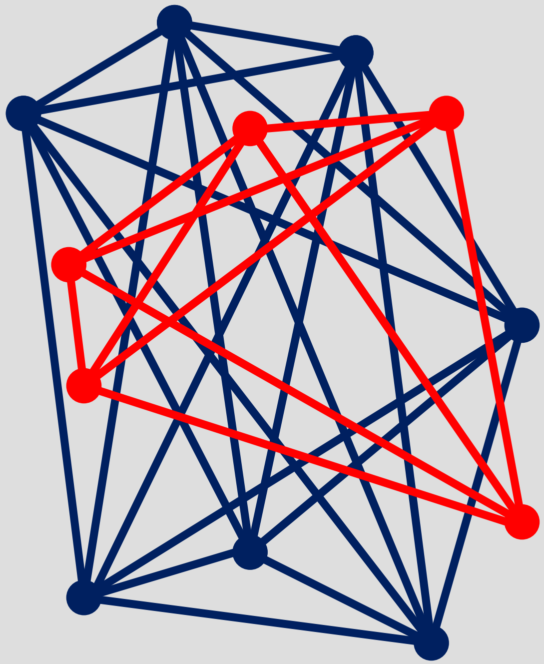

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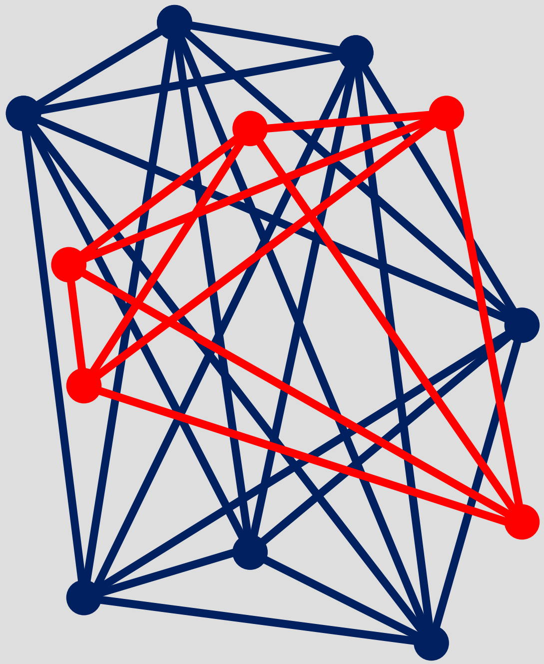
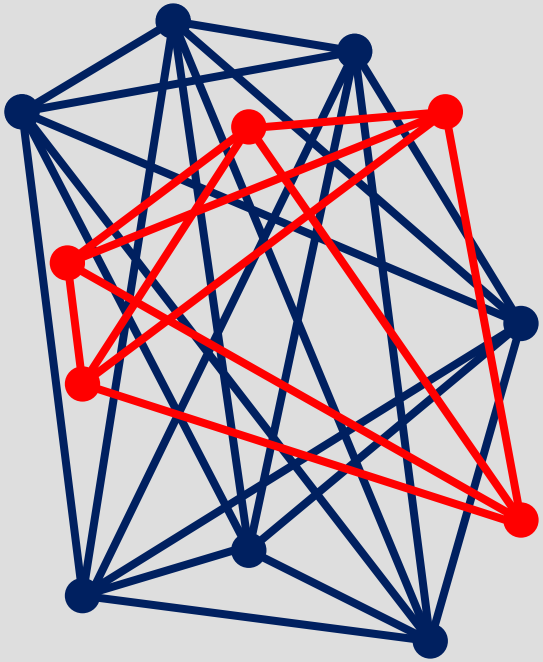


Report the set of segments of each colour intersected by segments of the other colour

The Problem



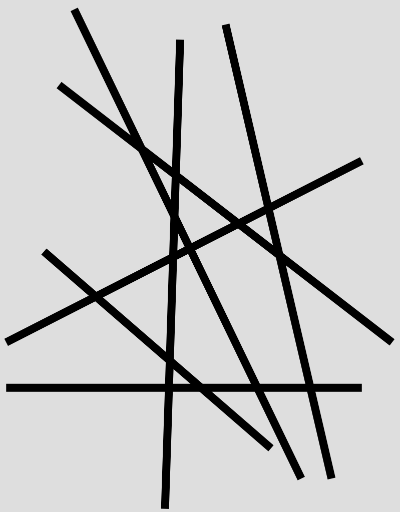
Report the set of segments of each colour intersected by segments of the other colour

Geometric Intersection Problem

Geometric Intersection Problems

Segment Intersection Problem

Report the intersections of
n line segments in the plane



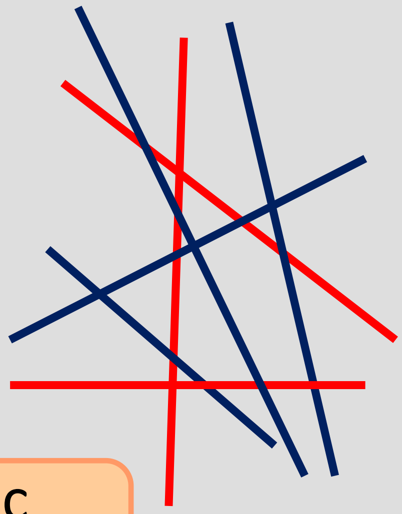
Algorithms for reporting all k intersecting pairs

- Bentley and Ottmann (1979): O((k+n) log n) time and O(n) space
- Chazelle and Edelsbrunner (1992): O(k+n log n) time and O(k+n) space
- Balaban (1995): O(k+n log n) time and O(n) space

Geometric Intersection Problems

Bichromatic Segment Intersection Problem

Report all intersections between
n_r red segments and n_b blue segments



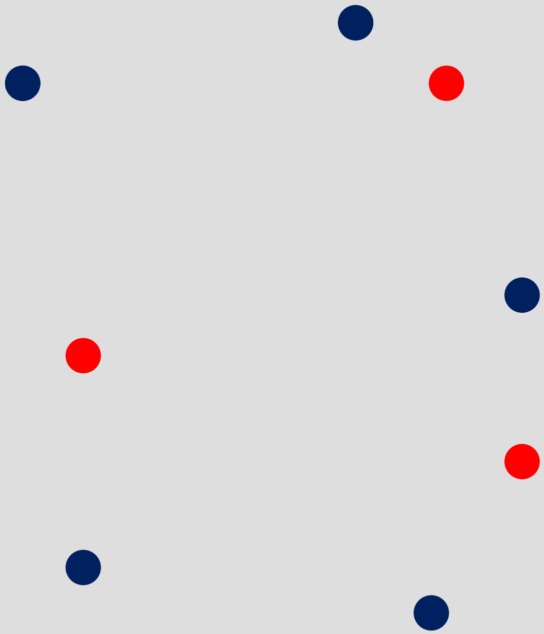
Bichromatic intersections

Algorithms for reporting bichromatic intersections
(monochromatic intersections exist)

- Agarwal and Sharir (1988): $O((n_r\sqrt{n_b} + n_b\sqrt{n_r})\log n)$ where $n = n_r + n_b$
- Agarwal (1990): $O(k + n^{4/3}\log^{o(1)} n)$ where $n = n_r + n_b$

Our Problem

Variation of the bichromatic segment intersection problem



- R=set of n_r red points; B=set of n_b blue points
  $n=n_r+n_b$

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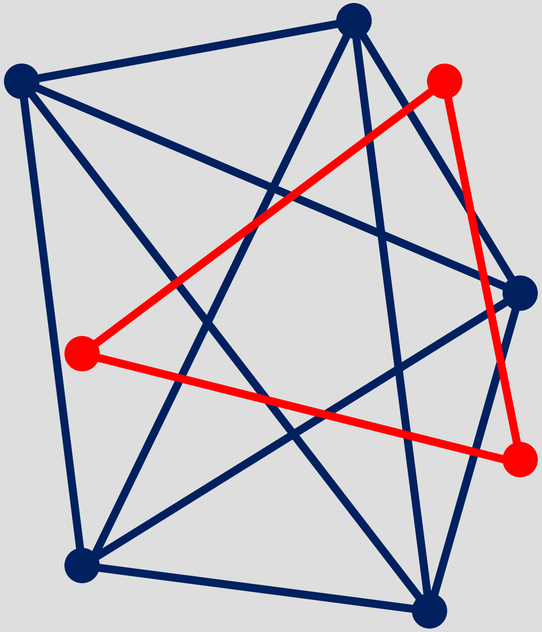

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- Consider all monochromatic segments

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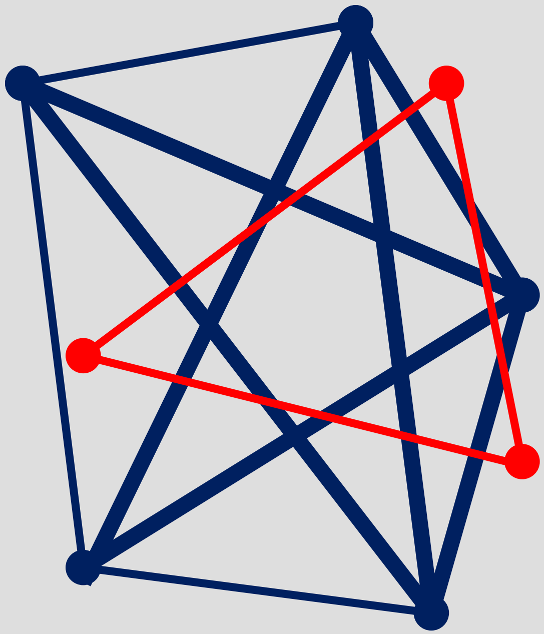

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- S_b=set of blue segments that intersect
  at least one red segment; $s_b=|S_b|$

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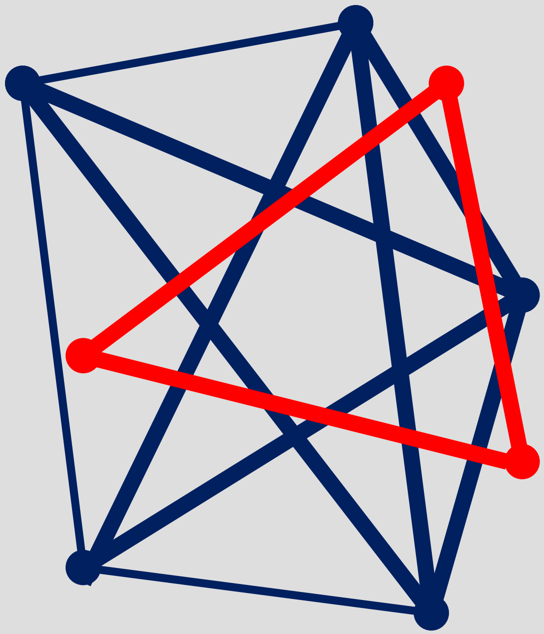

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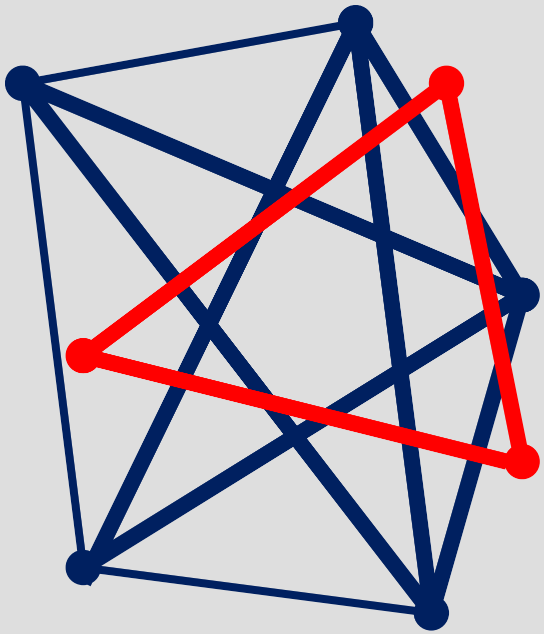

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- Consider all monochromatic segments
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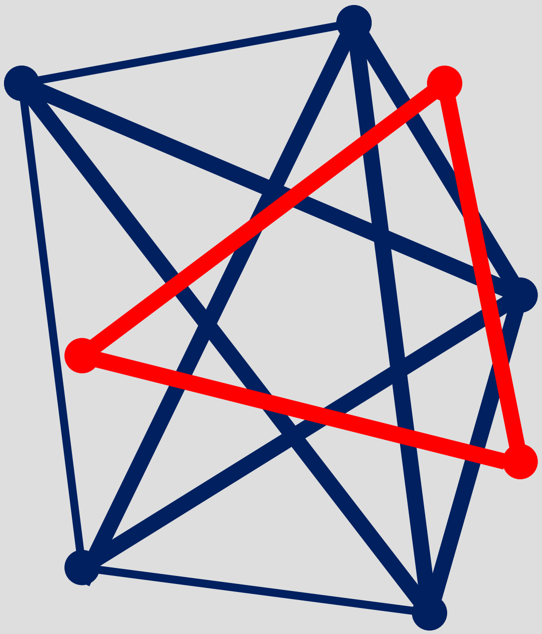
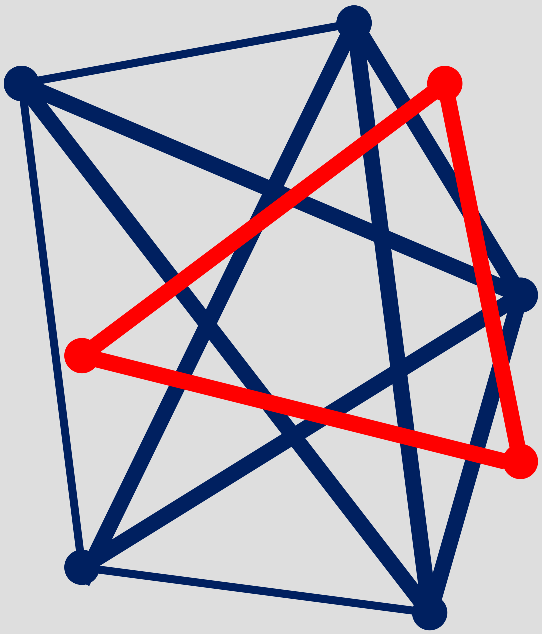
Report S_b and S_r

The Problem: Report S_b and S_r



- We provide an $O(n^2)$ time and space algorithm for reporting S_b and S_r
- We prove that the problem is 3-Sum hard

The Problem: Report S_b and S_r



- We provide an $O(n^2)$ time and space algorithm for reporting S_b and S_r
- We prove that the problem is 3-Sum hard

What is a 3-Sum hard problem?

The Problem: Report S_b and S_r

3-Sum Hard Problems

- Class of problems introduced by Gajentaan and Overmars (1995)
- All problems in the class are at least as hard as *the base problem:*

Given a set S of n integers, are there three elements of S that sum up to zero?

The Problem: Report S_b and S_r

3-Sum Hard Problems

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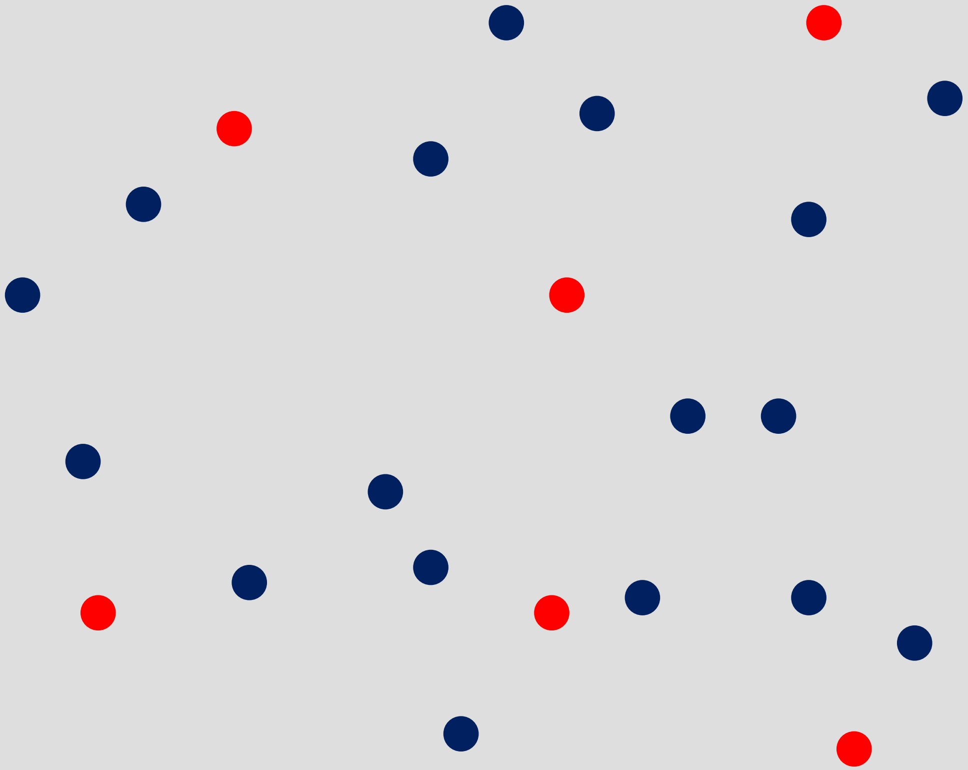
Given a set S of n integers, are there three elements of S that sum up to zero?

There is an $O(n^2)$ algorithm

Conjectured an $\Omega(n^2)$ lower bound

The $O(n^2)$ Algorithm:

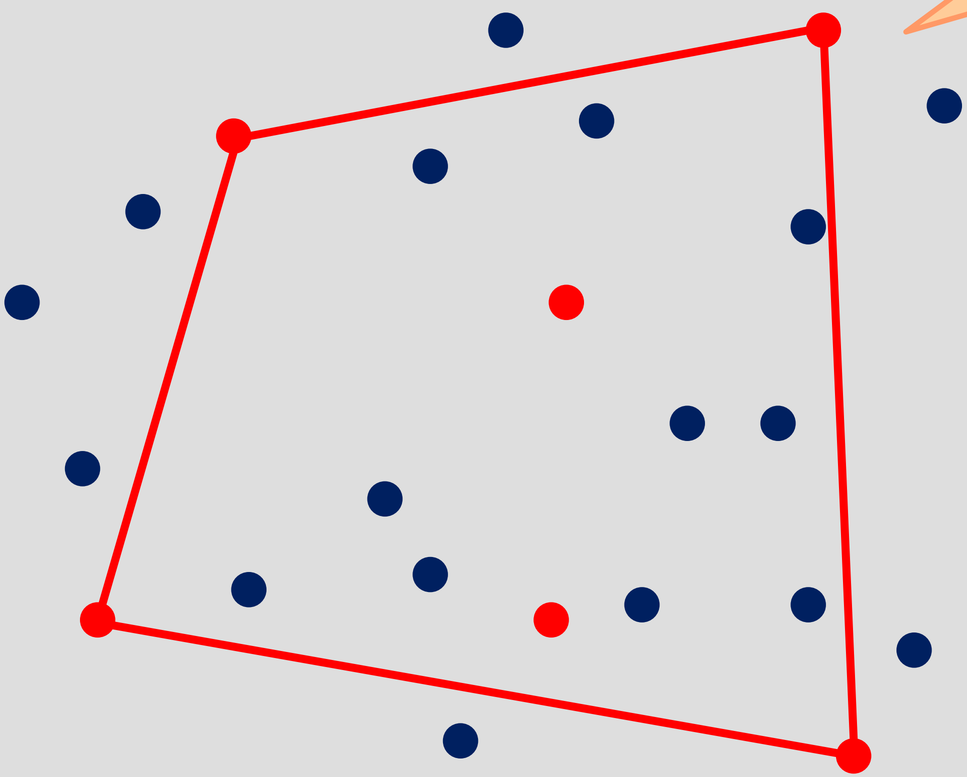
Compute S_b



The $O(n^2)$ Algorithm:

Compute S_b

Be=blue **exterior** points to CH(R)
Bi=blue **interior** points to CH(R)

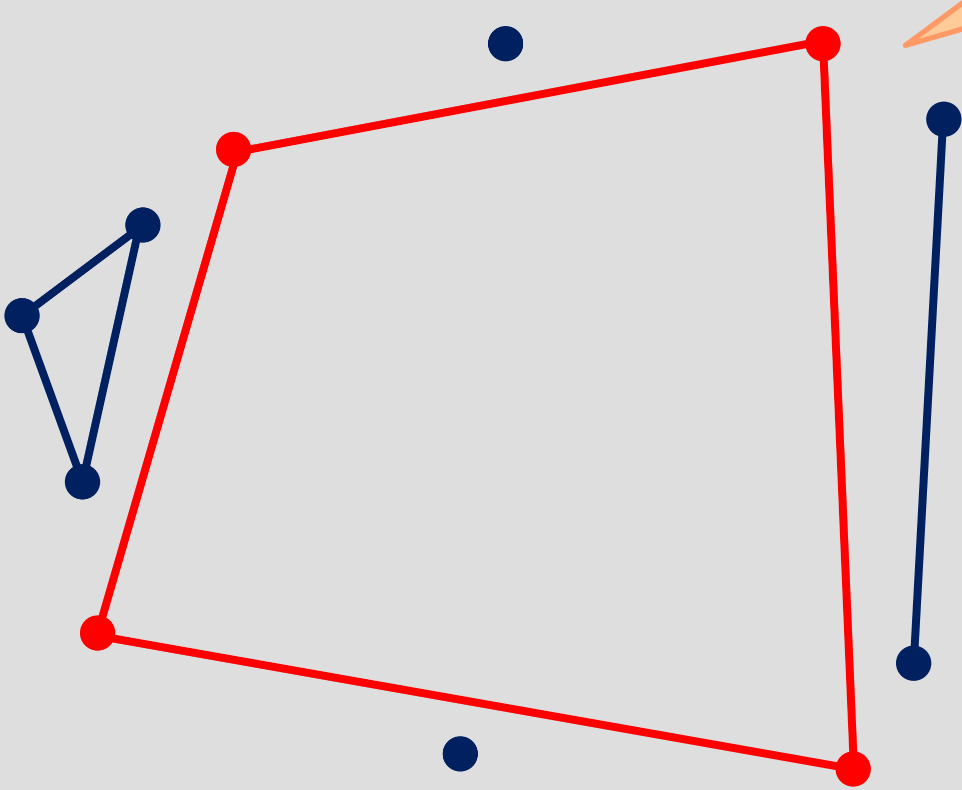


The $O(n^2)$ Algorithm:
Compute S_b

B_e=blue **exterior** points to CH(R)

B_i=blue **interior** points to CH(R)

G_e= exterior graph



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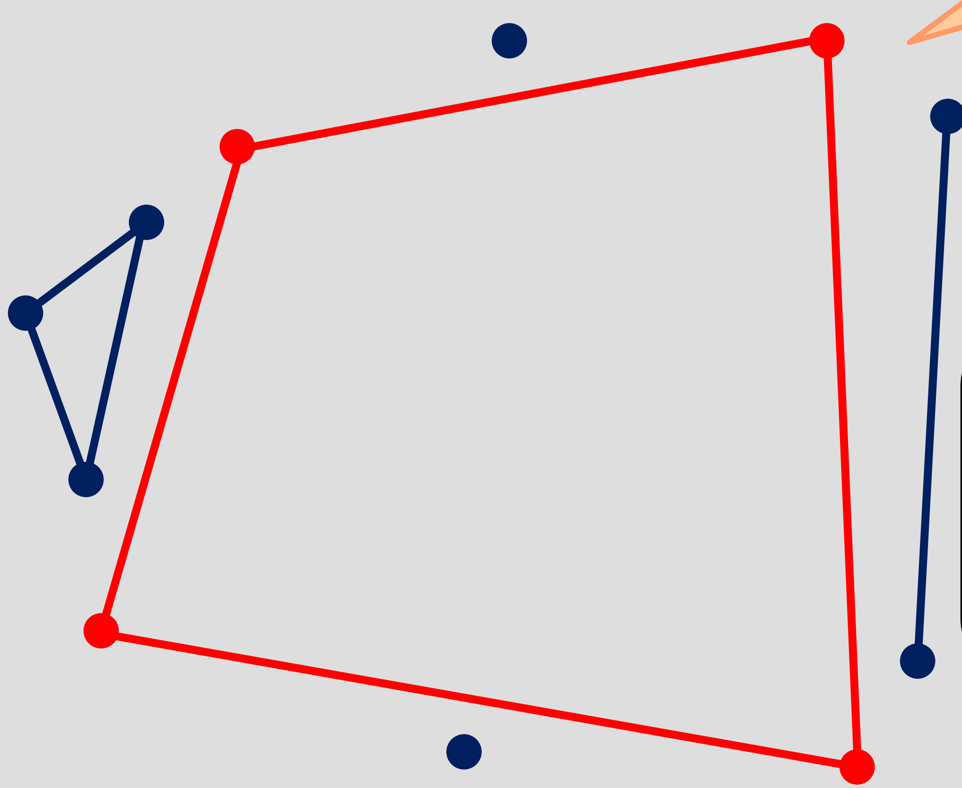

$$S_b = \begin{cases} E(\overline{G}_e) \\ \\ \end{cases}$$

The O(n²) Algorithm:
Compute S_b

Be=blue **exterior** points to CH(R)
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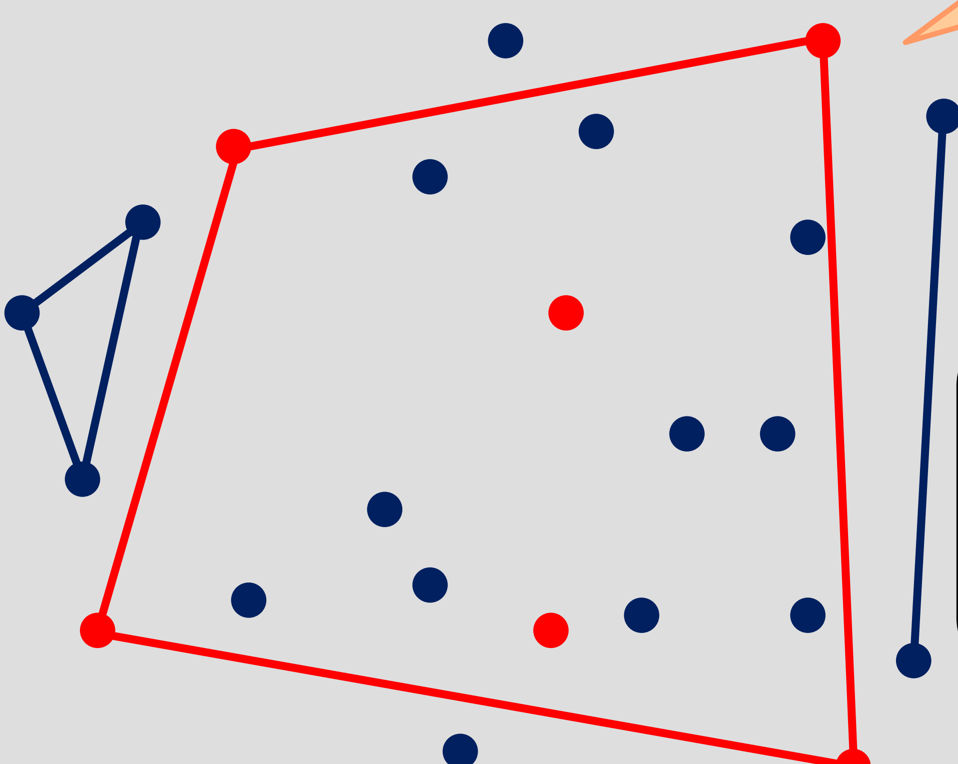
Ge= exterior graph



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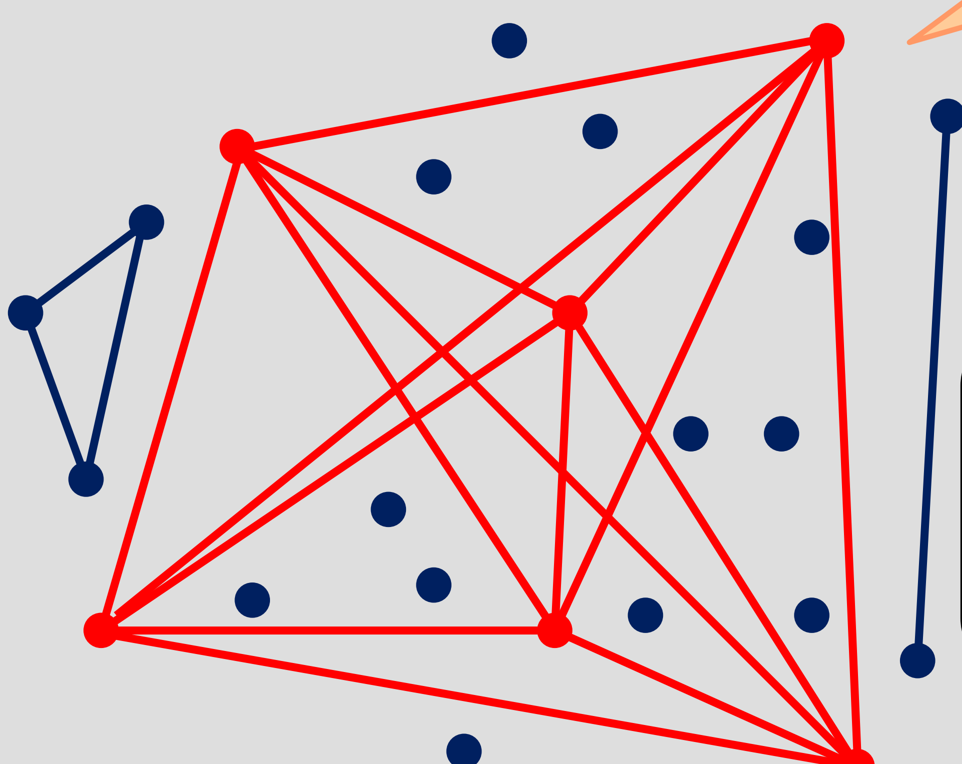

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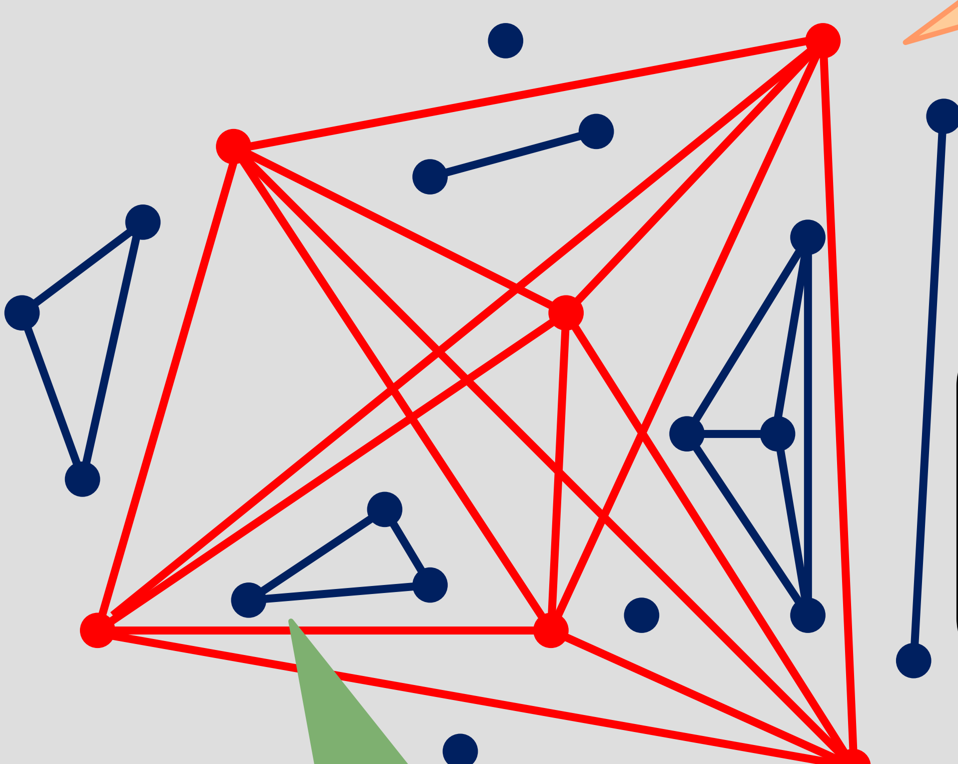

Gi= interior graph

$$S_b = \begin{cases} E(\overline{G_e}) \\ \\ \\ \end{cases}$$

The $O(n^2)$ Algorithm:
Compute S_b

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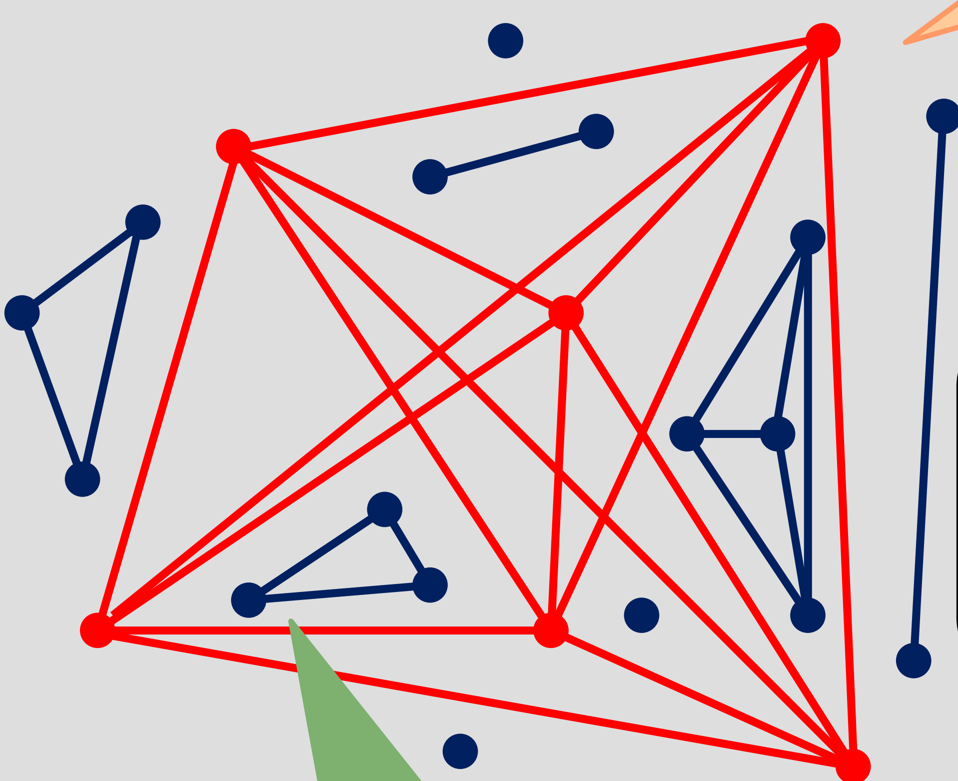

Ge= exterior graph

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$$S_b = \begin{cases} E(\overline{G_e}) \\ E(\overline{G_i}) \\ \end{cases}$$

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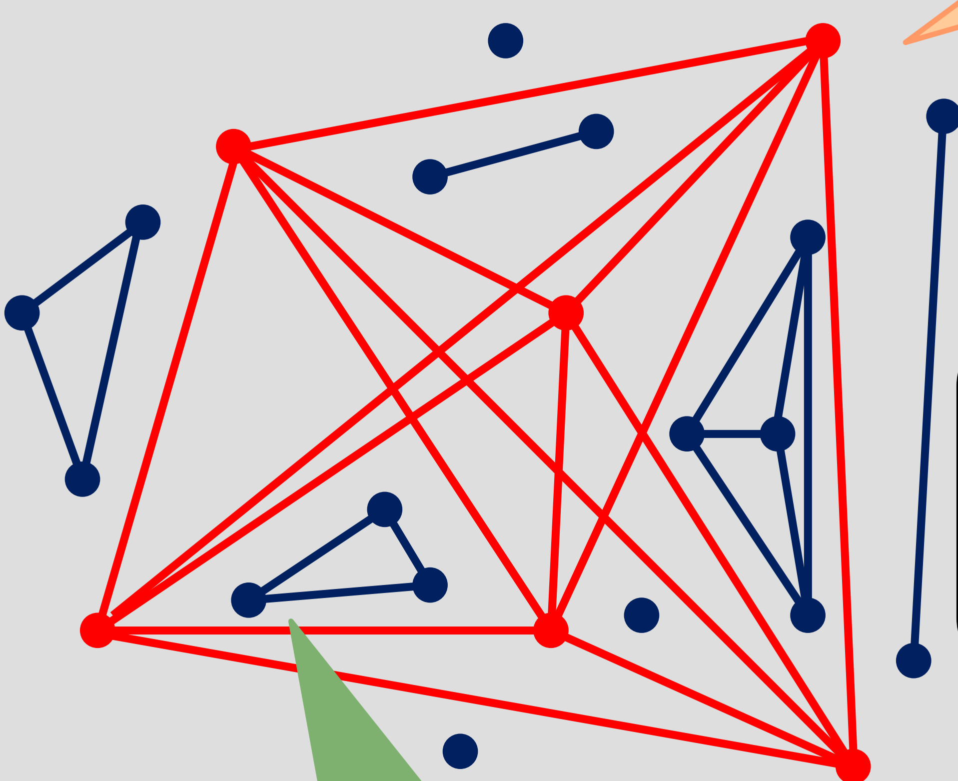

Ge= exterior graph

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$$S_b = \begin{cases} E(\overline{G_e}) \\ E(\overline{G_i}) \\ \{uv : u \in V(G_i), v \in V(G_e)\} \end{cases}$$

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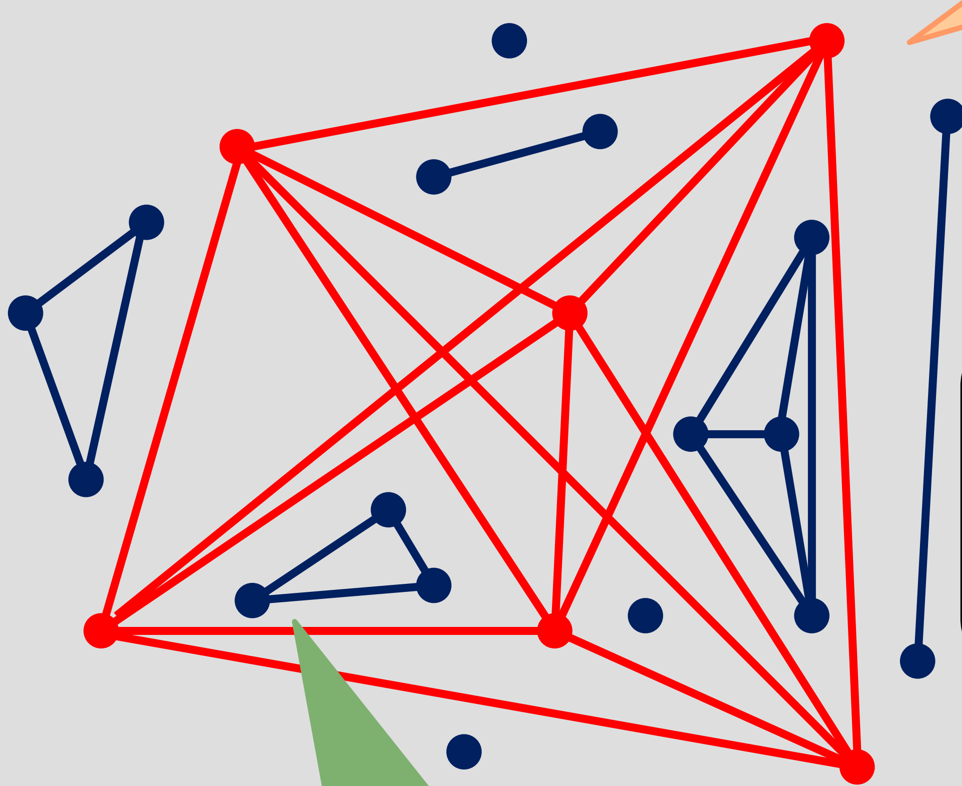

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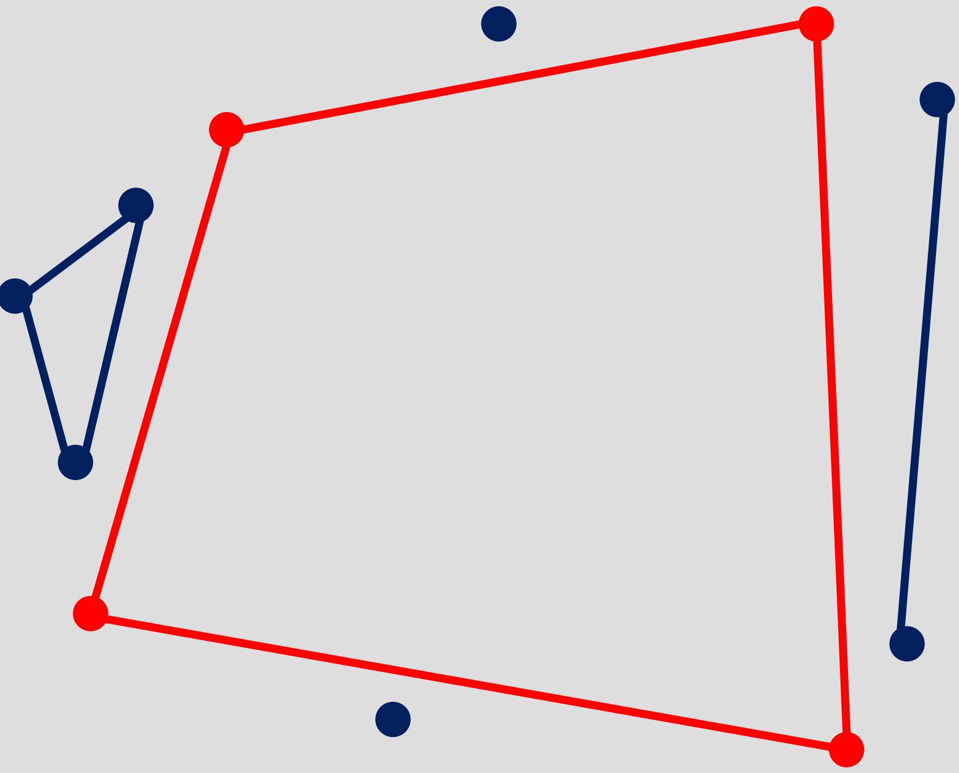
O(nlogn)

The O(n²) Algorithm:
Compute S_b

Computing $E(\overline{G}_e)$



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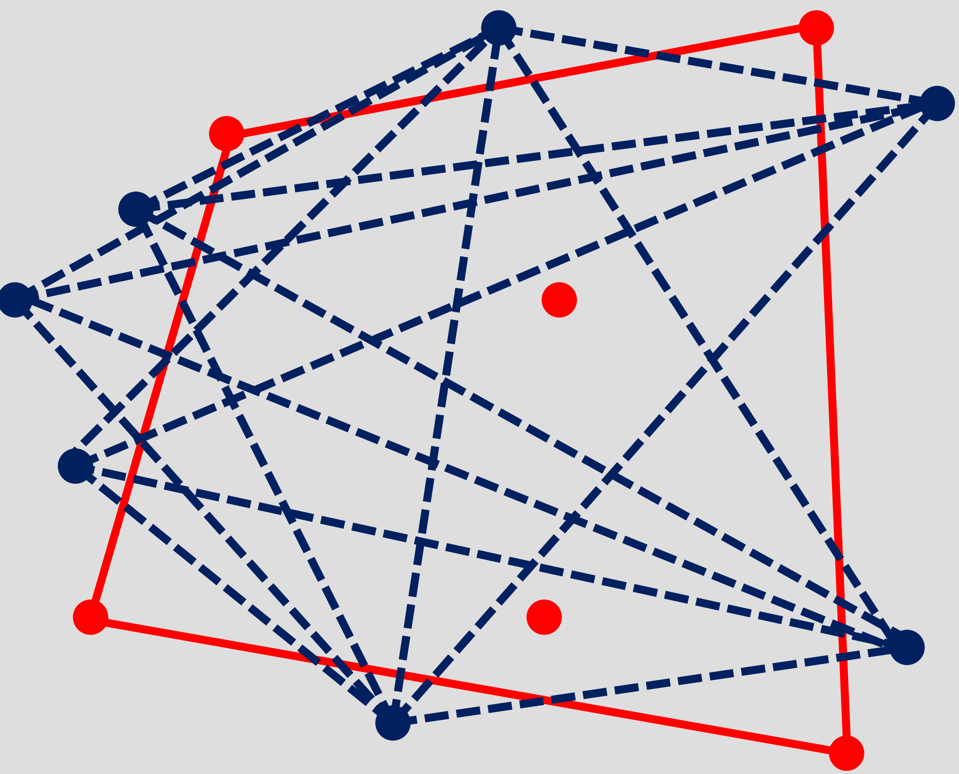

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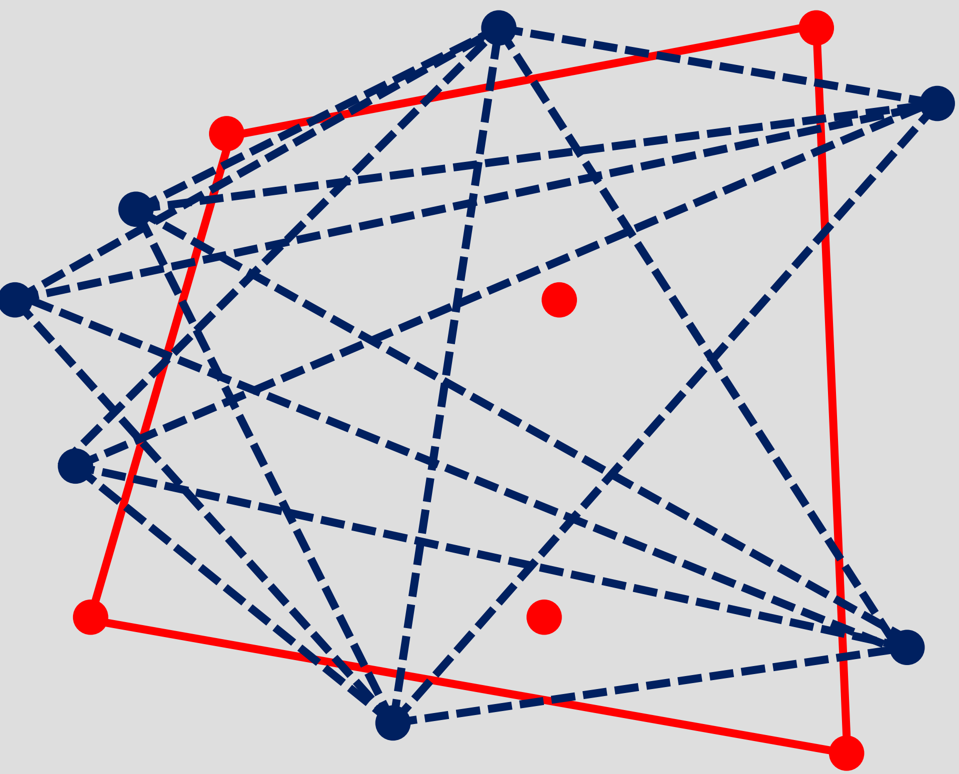

Lemma: P is an n-sided convex polygon, Q a set of n exterior points. To decide whether any of the segments with end points in Q intersects P can be done in:

1) O(nlogn) time and O(n) space.

2) Θ(n) time and space if we know the rotational ordering of the points of Q with respect to P.

The $O(n^2)$ Algorithm:
Compute S_b

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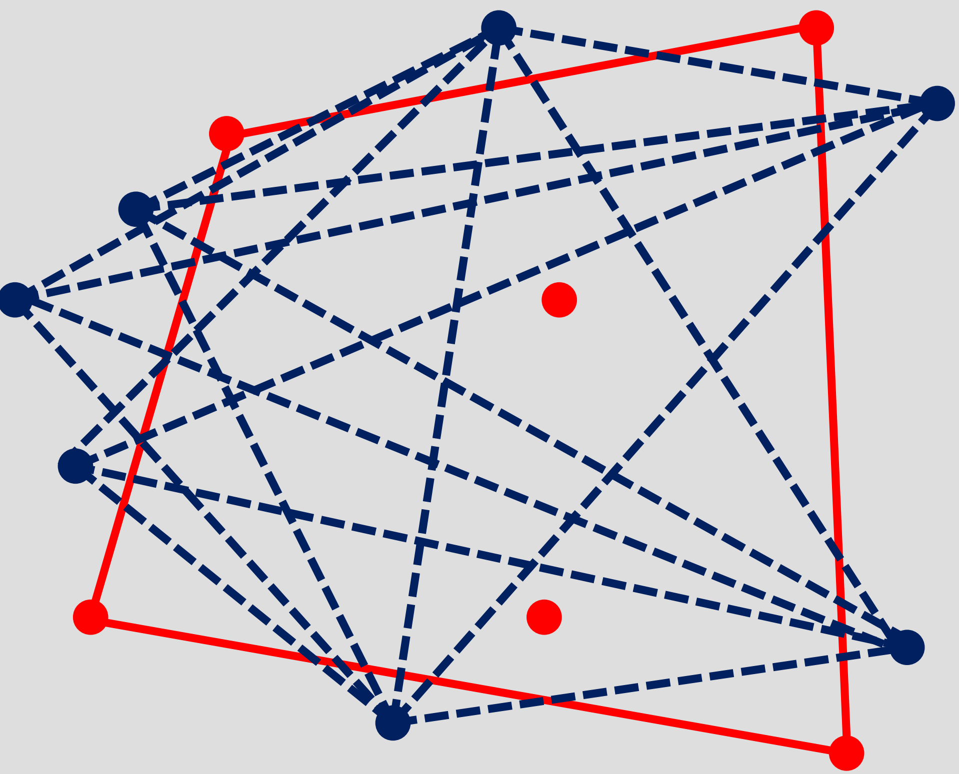

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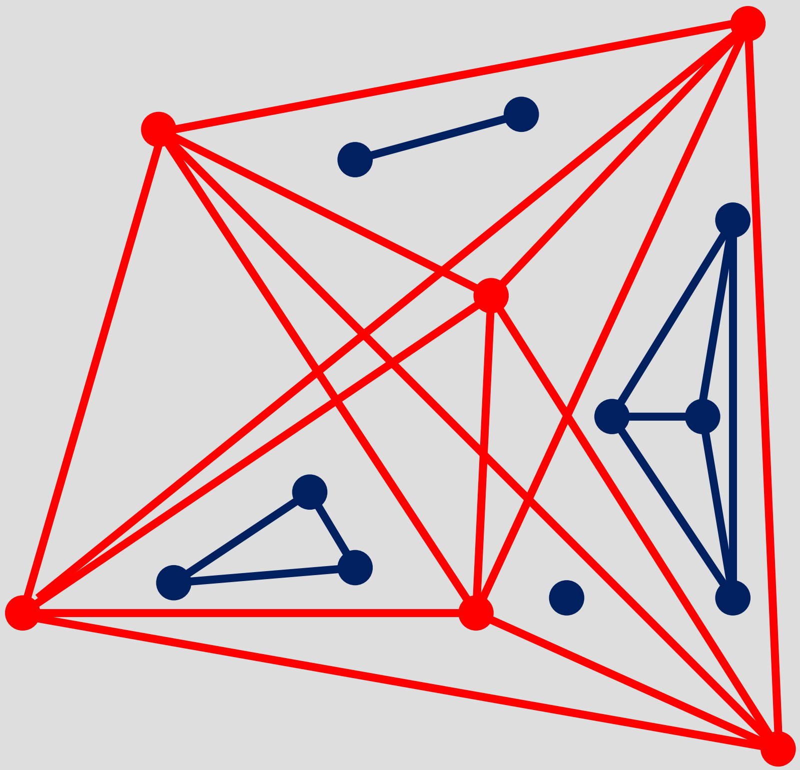
$O(|E(\overline{G}_e)| + n\log n)$

which is at most $O(n^2)$

The $O(n^2)$ Algorithm:
Compute S_b

Computing $E(\overline{G_i})$



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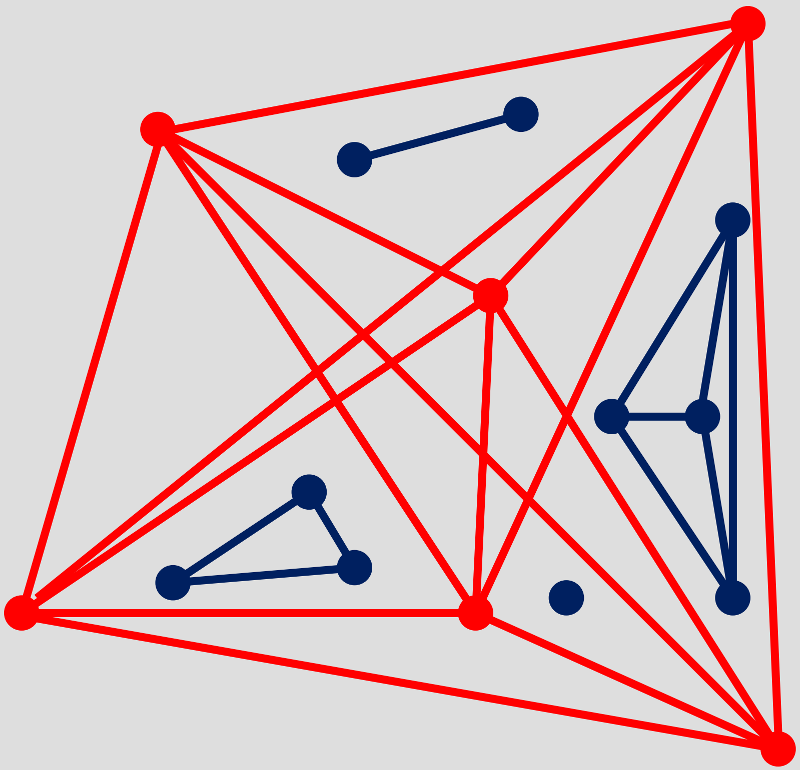


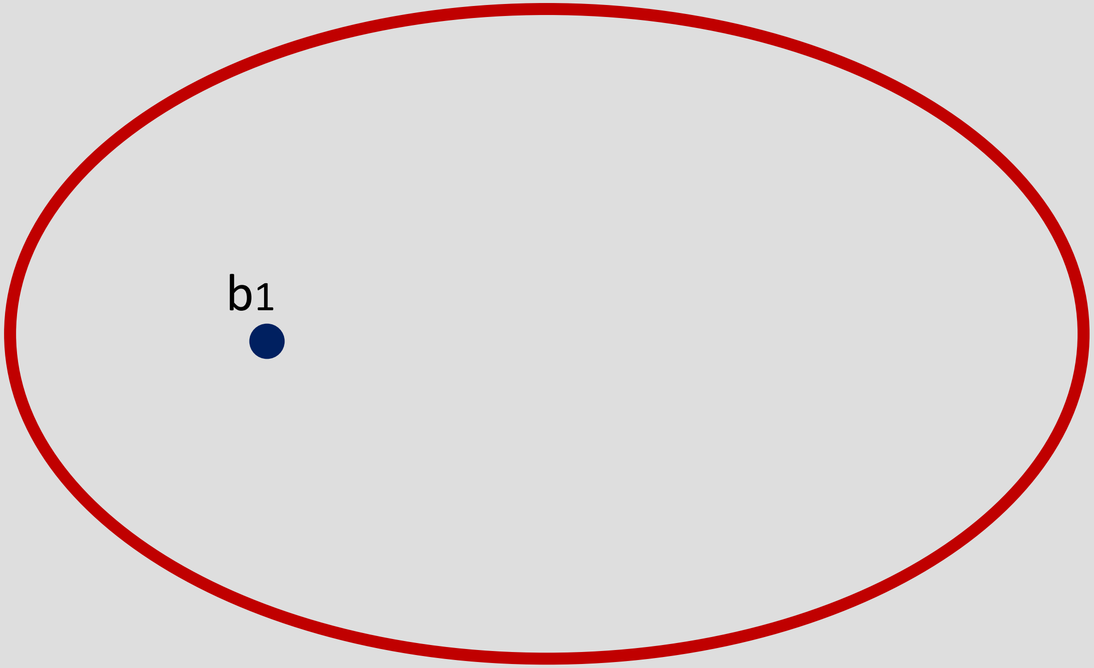
Key tool: a procedure to partition CH(R) into convex regions, each one is either empty or contains only blue points whose segments are not intersected by red segments

The $O(n^2)$ Algorithm:
Compute S_b

Computing $E(\overline{G}_i)$

Equivalence relation

$b_j \sim b_k$ if and only if the segment b_jb_k crosses no red segment



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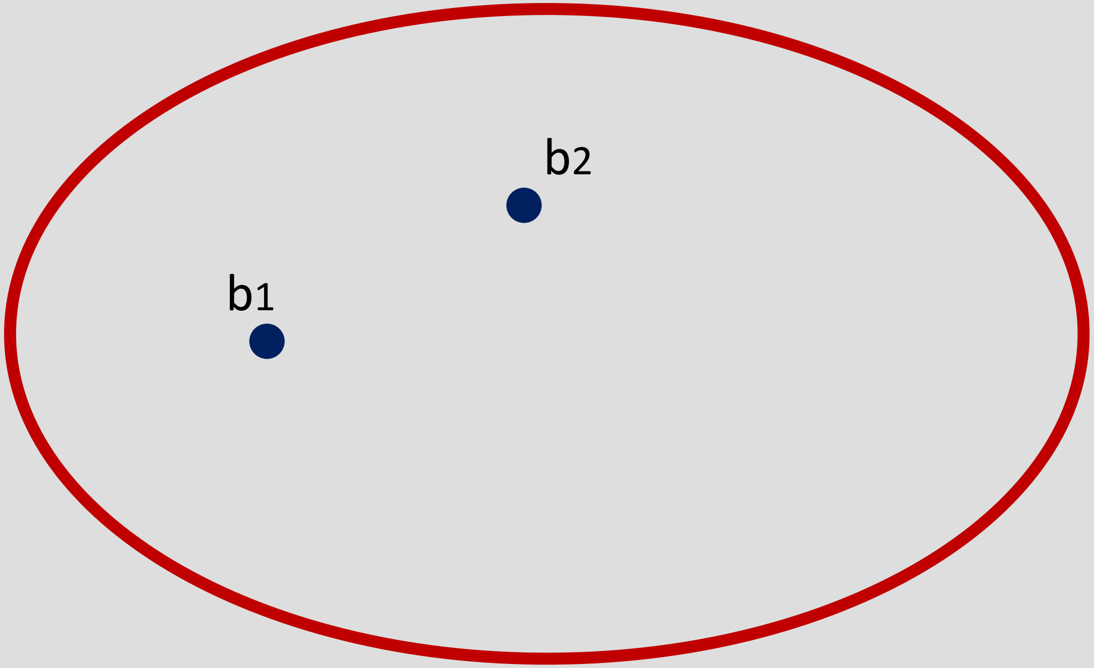

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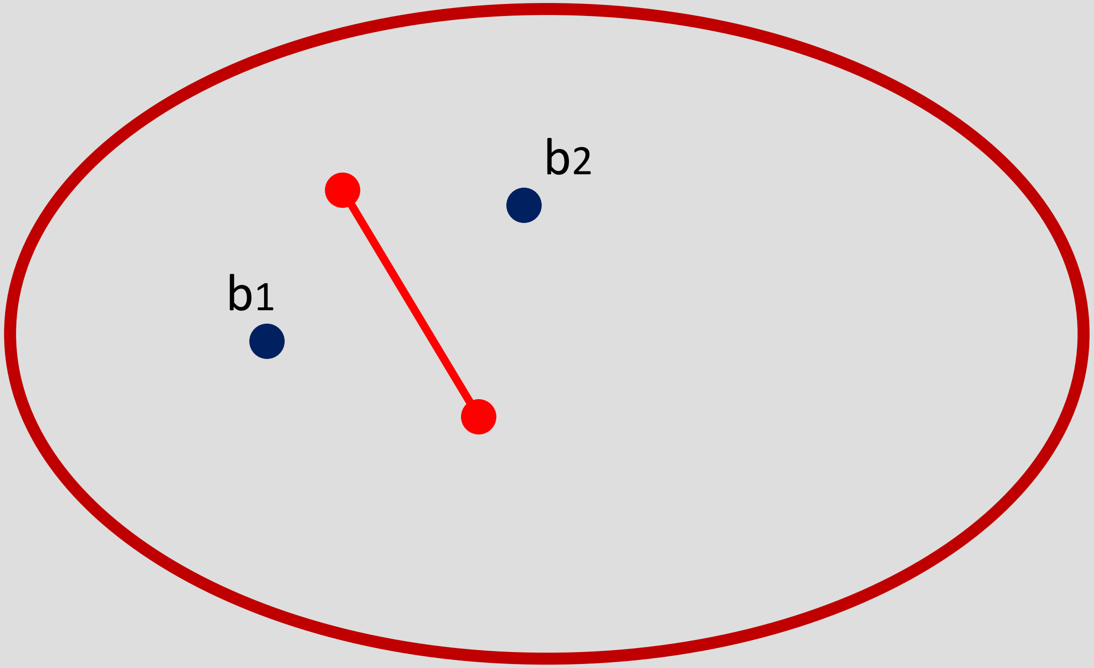

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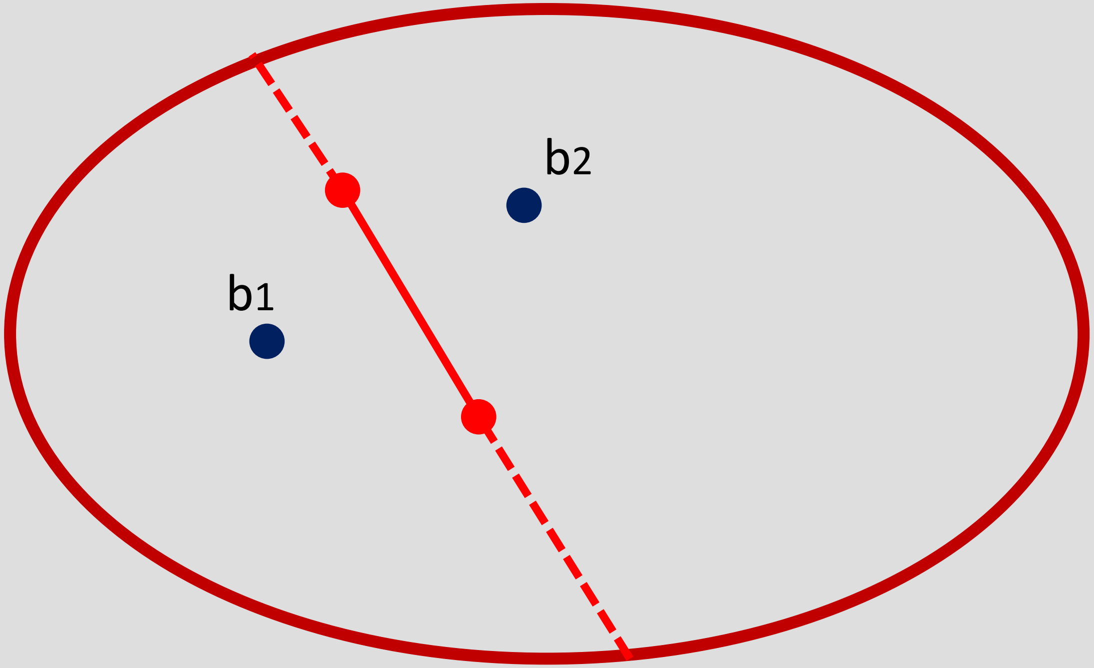

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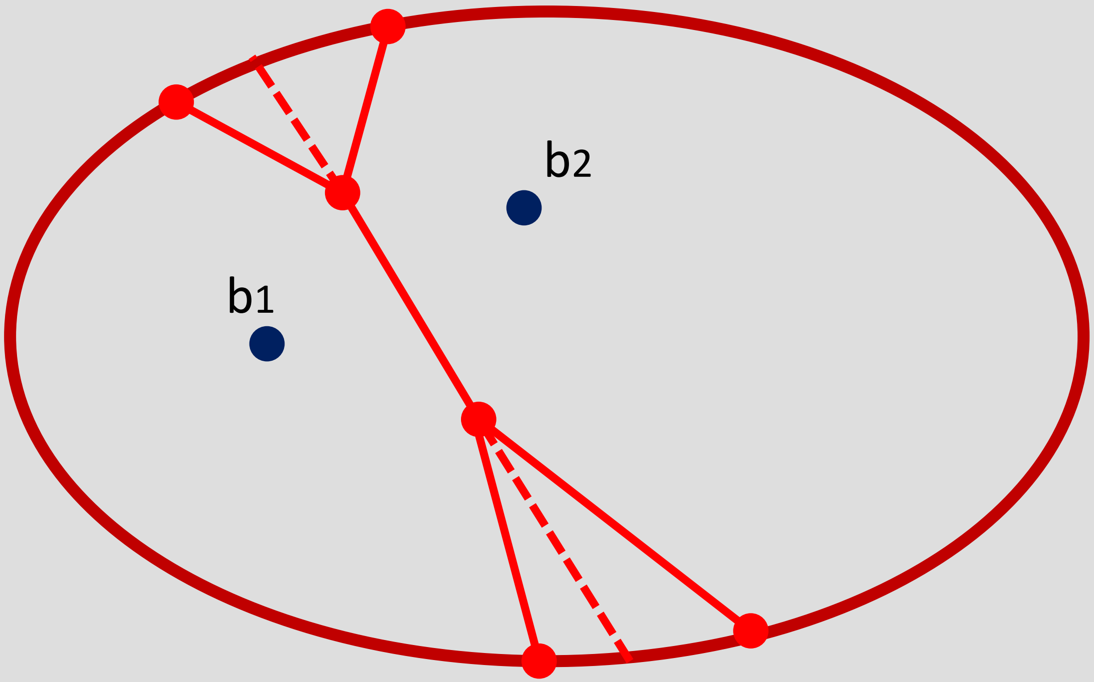

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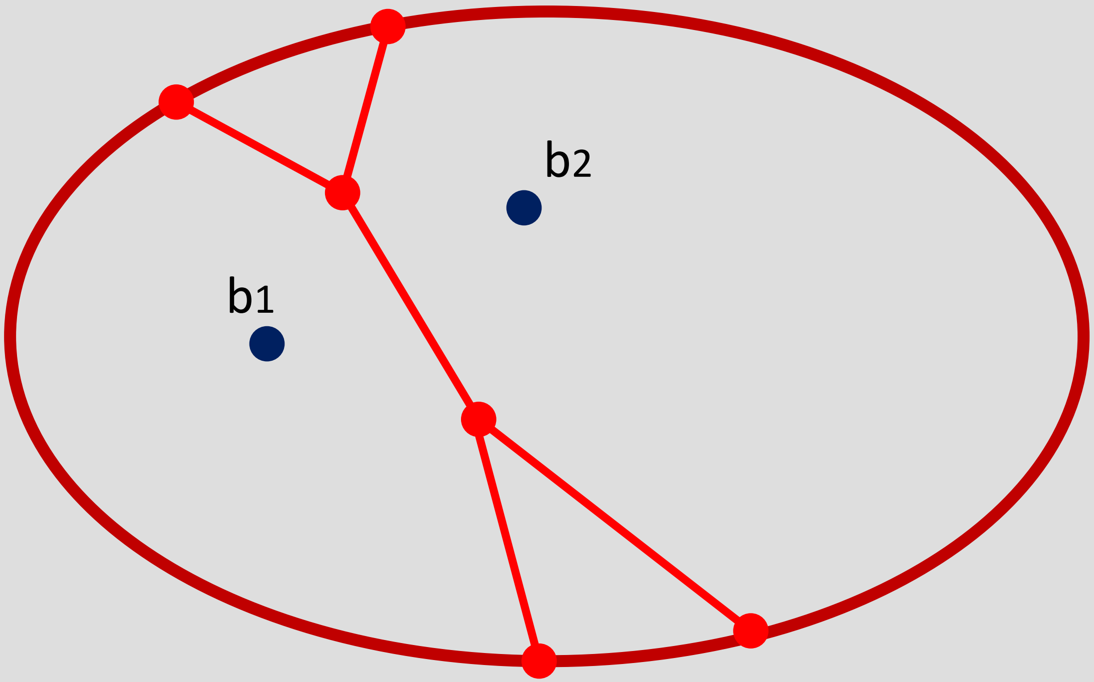

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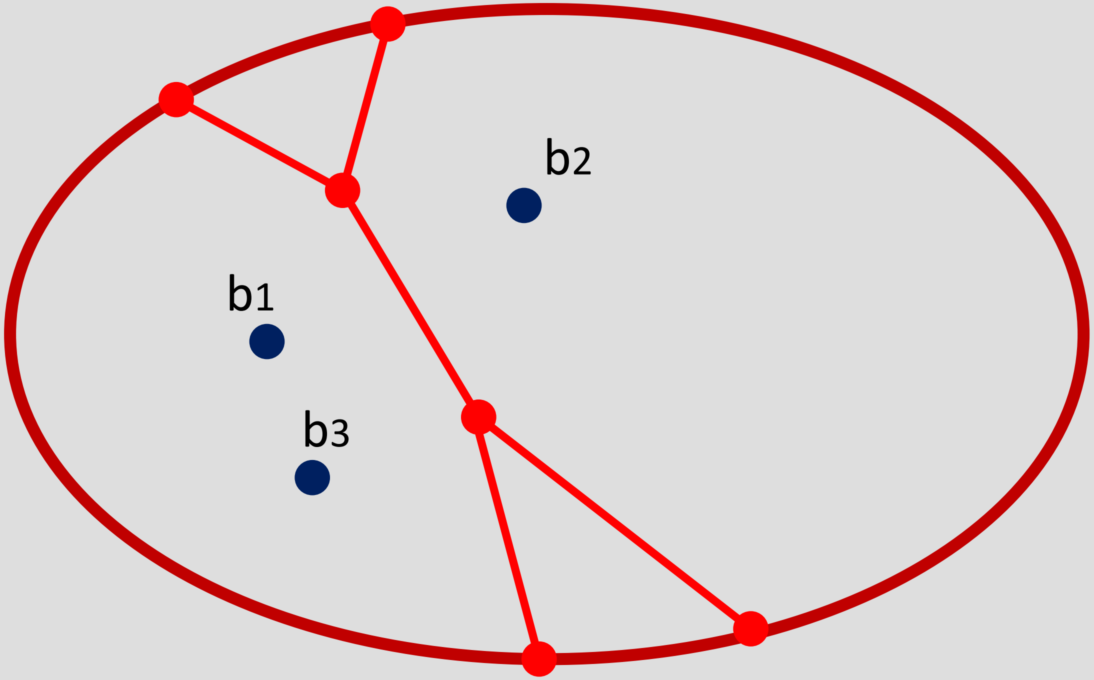

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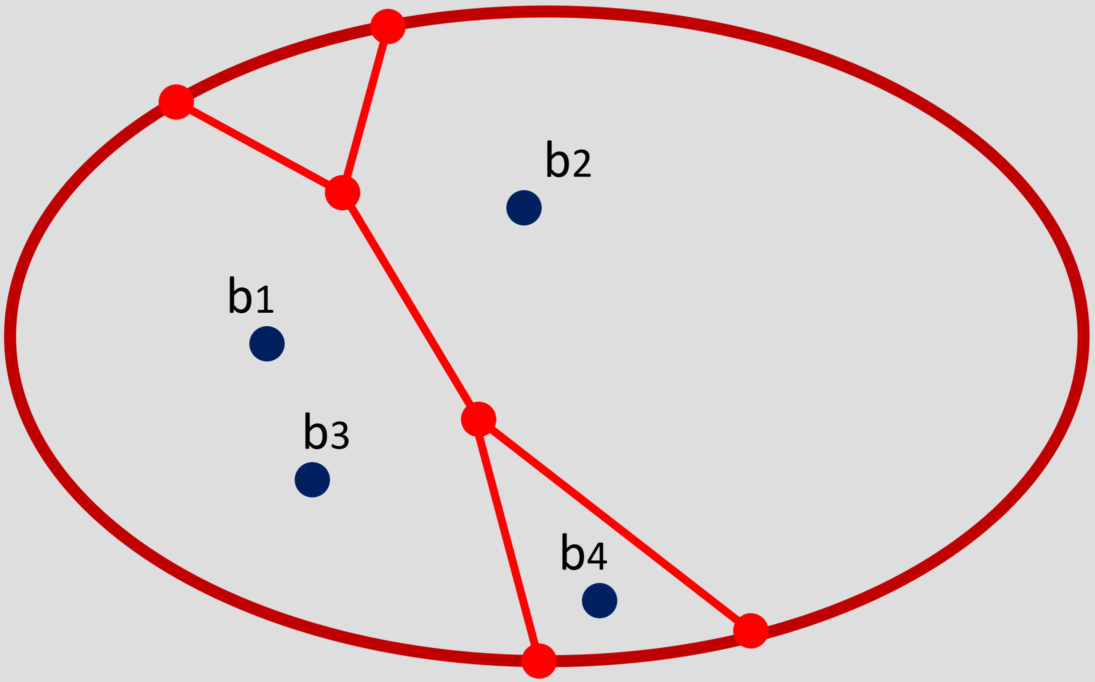

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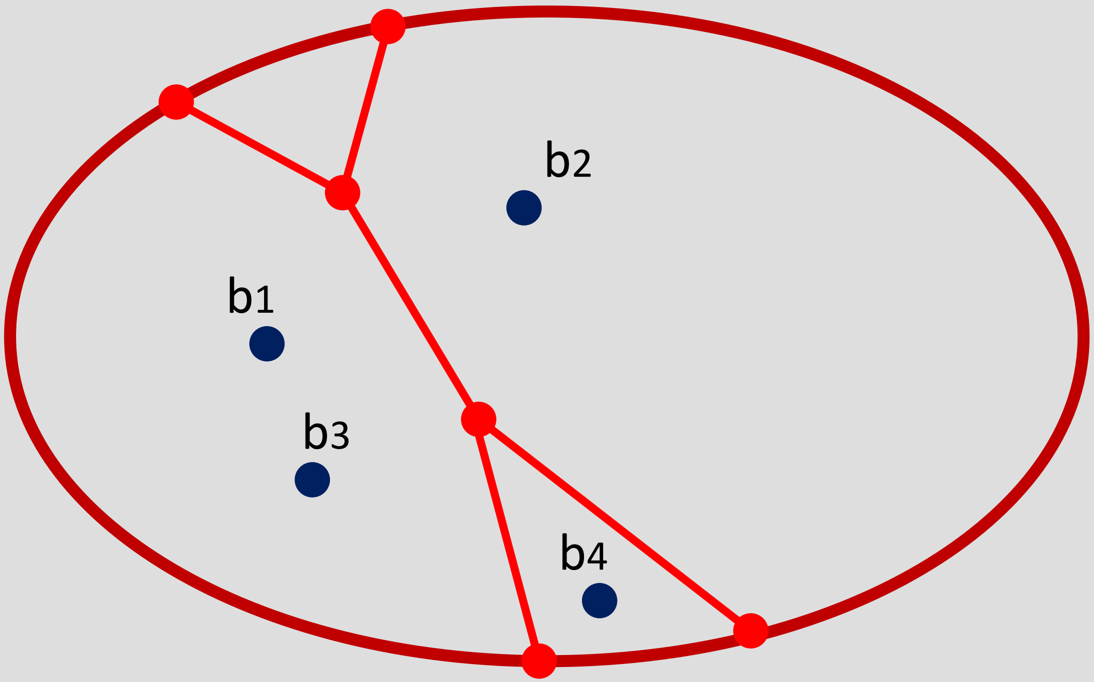


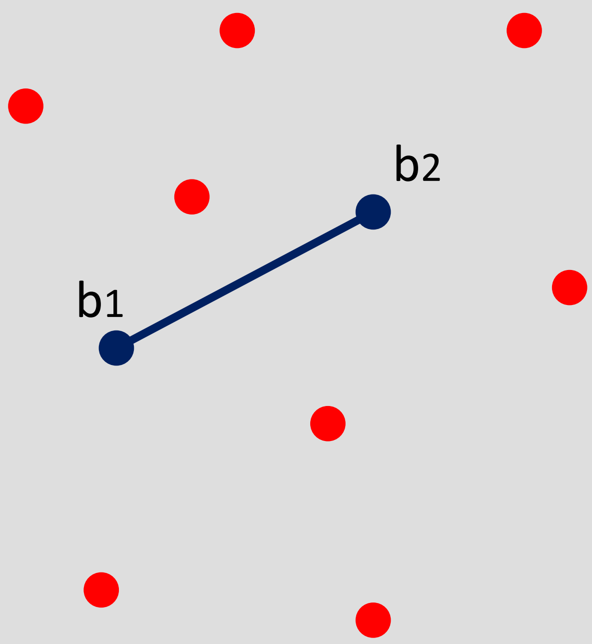
Equivalence relation

$b_j \sim b_k$ if and only if the segment $b_j b_k$ crosses no red segment

Theorem: Computing the planar subdivision formed by the convex regions partitioning CH(R) can be done in O(nlogn) time and O(n) space.

The O(n²) Algorithm:
Compute S_b

Computing $E(\overline{G}_i)$



Lemma: P is an n-sided convex polygon, Q a set of n exterior points. To decide whether any of the segments with endpoints in Q intersects P can be done in:

1) O(nlogn) time and O(n) space.

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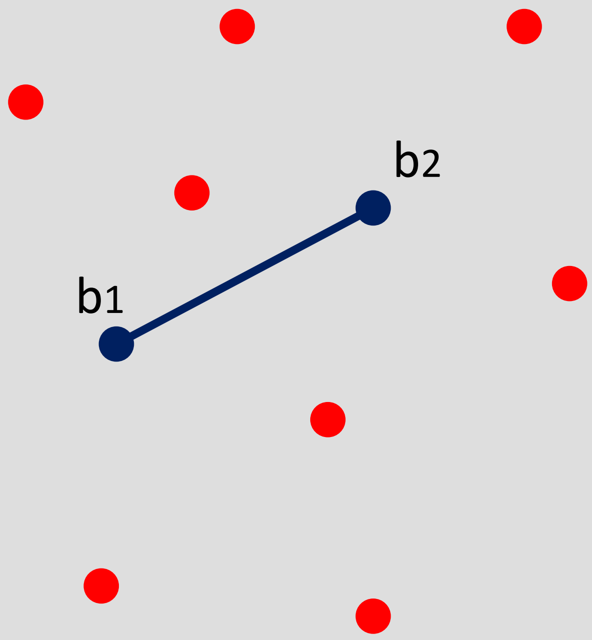
The O(n²) Algorithm: Compute S_b

Computing $E(\overline{G}_i)$



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1) O(nlogn) time and O(n) space.

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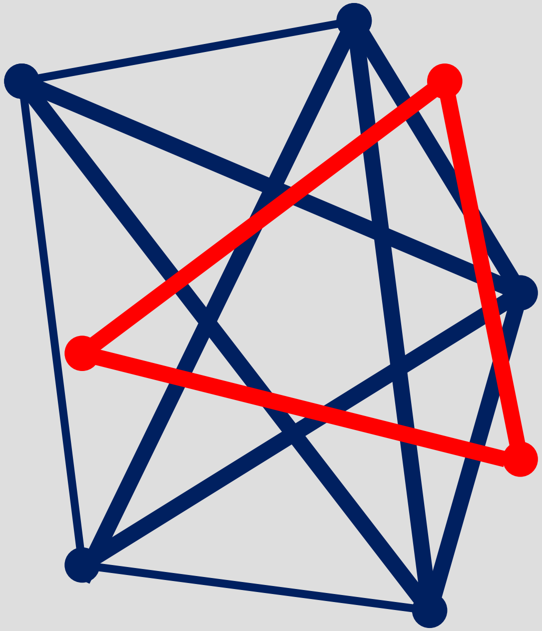
<u>Go from $O(n^2 \log n)$ to $O(n^2)$</u>

- In $O(n^2)$, construct the dual arrangement of lines from the points of RUB_i
- In O(n) one can read the rotational ordering of the red points with respect to a blue point in the dual.

The Problem: Report S_b and S_r



Theorem: The sets S_b and S_r can be computed in $O(n^2)$ time and space.

Is the problem 3-Sum hard?

Hardness

Number of blue segments crossed by at least one red segment

Theorem: Computing s_r and s_b is 3-Sum hard

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Number of blue segments crossed by at least one red segment

Theorem: Computing s_r and s_b is 3-Sum hard

Corollary: Computing S_b and S_r is 3-Sum hard

Hardness

Number of blue segments crossed by at least one red segment

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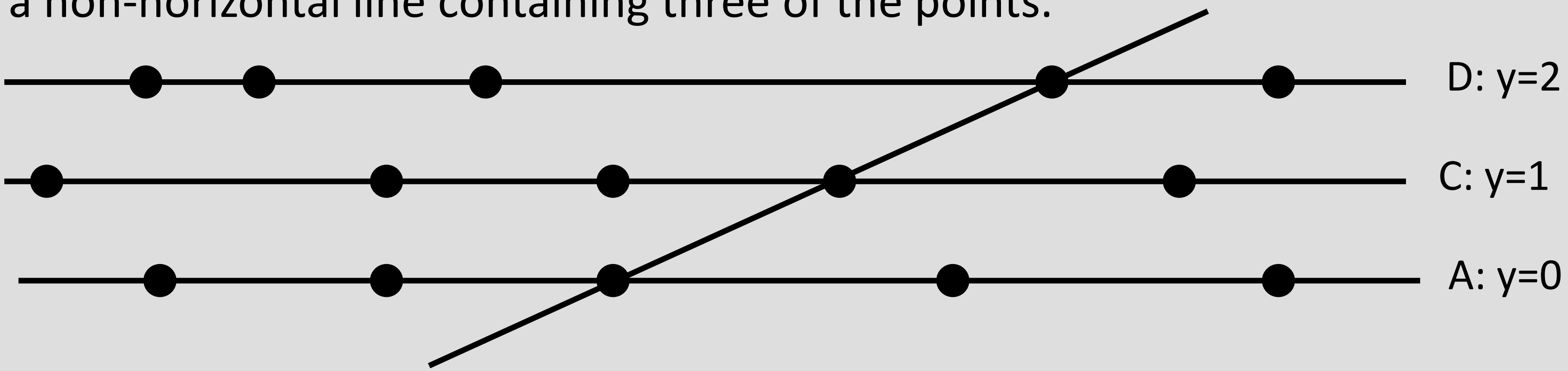
Corollary: Computing S_b and S_r is 3-Sum hard

Proof of the theorem

3-Sum Hard problem: Given a set of n points with integer coordinates on three horizontal lines y=0, y=1 and y=2, determine whether there exists a non-horizontal line containing three of the points.



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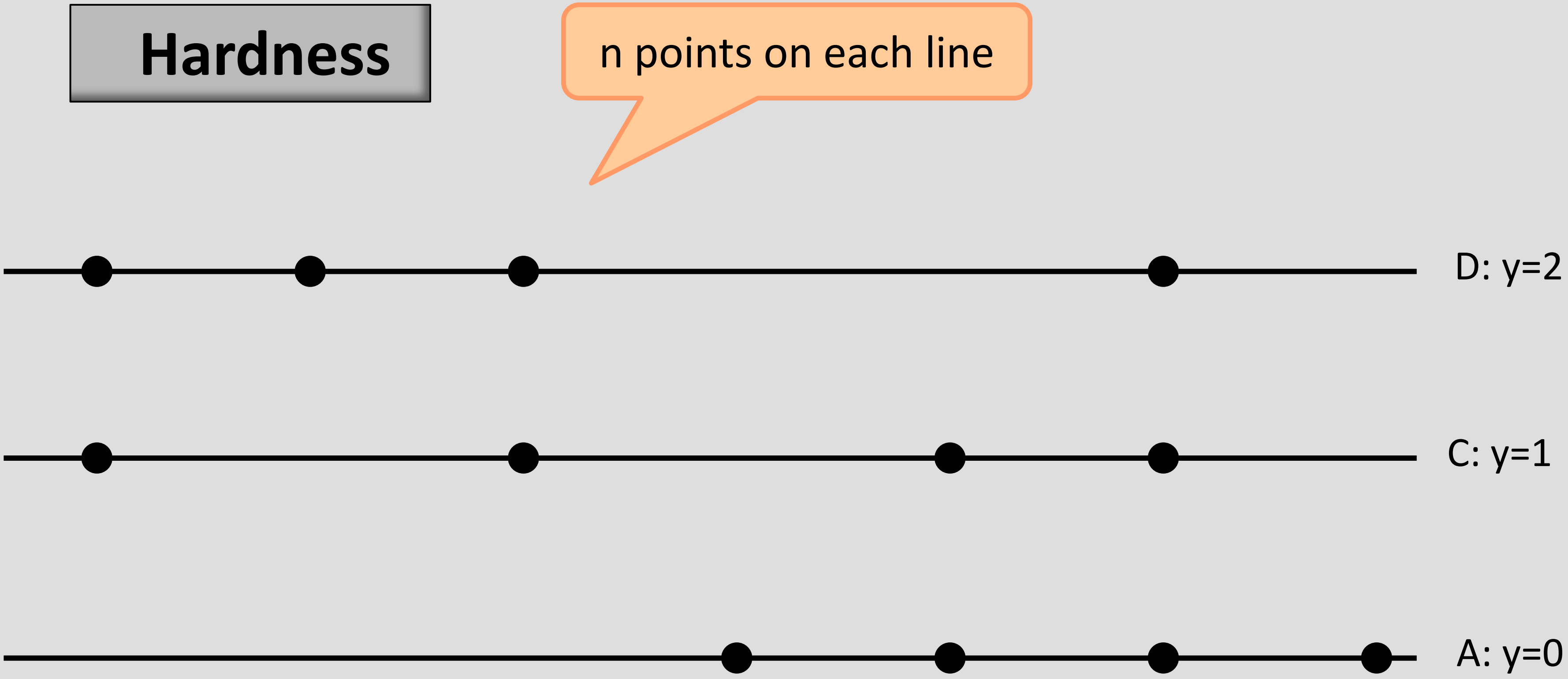

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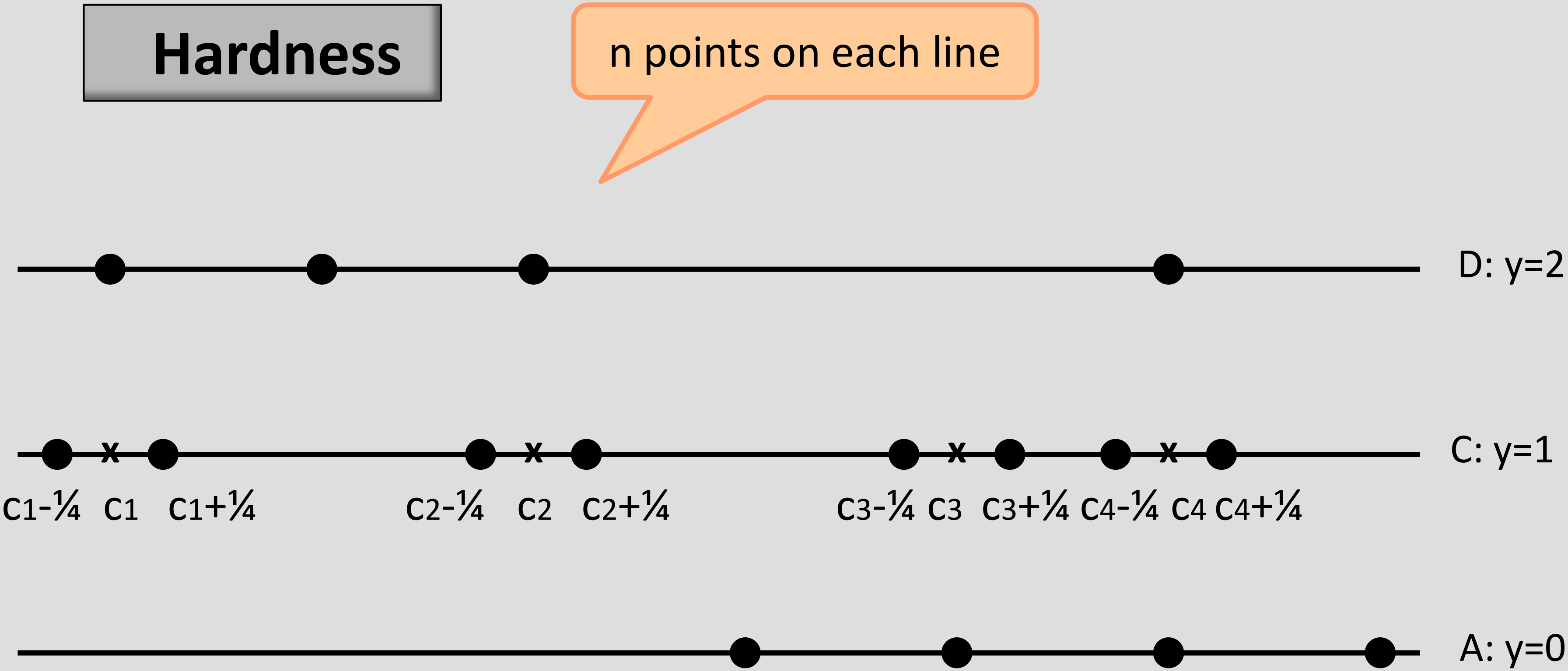
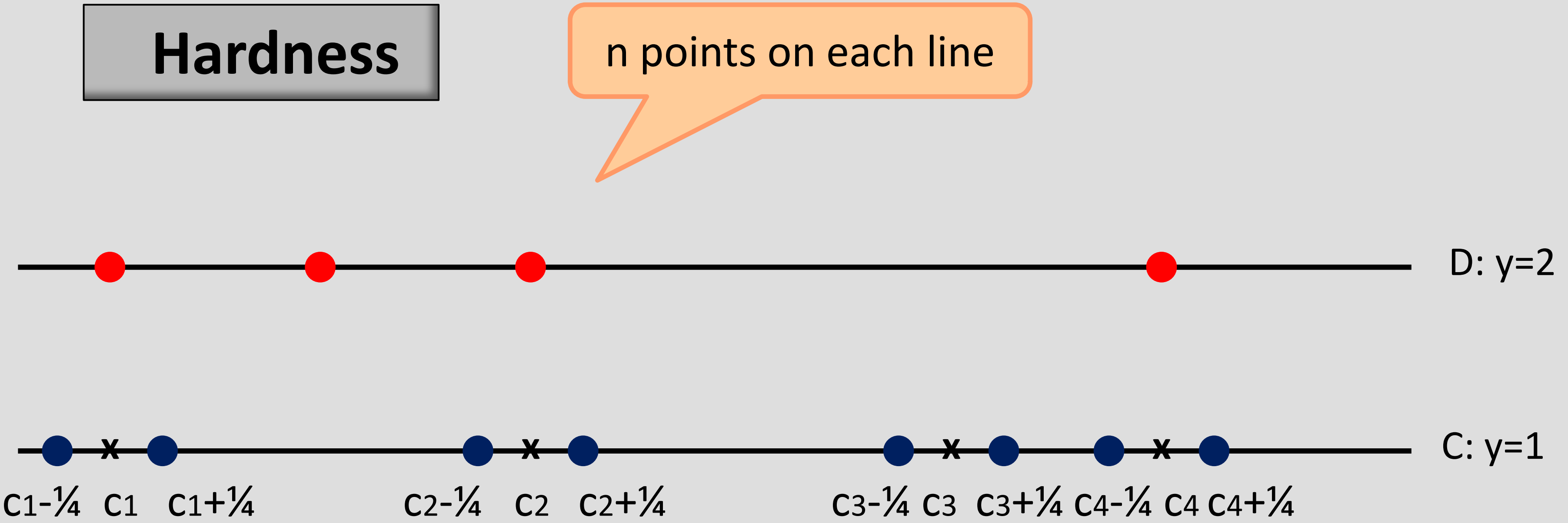
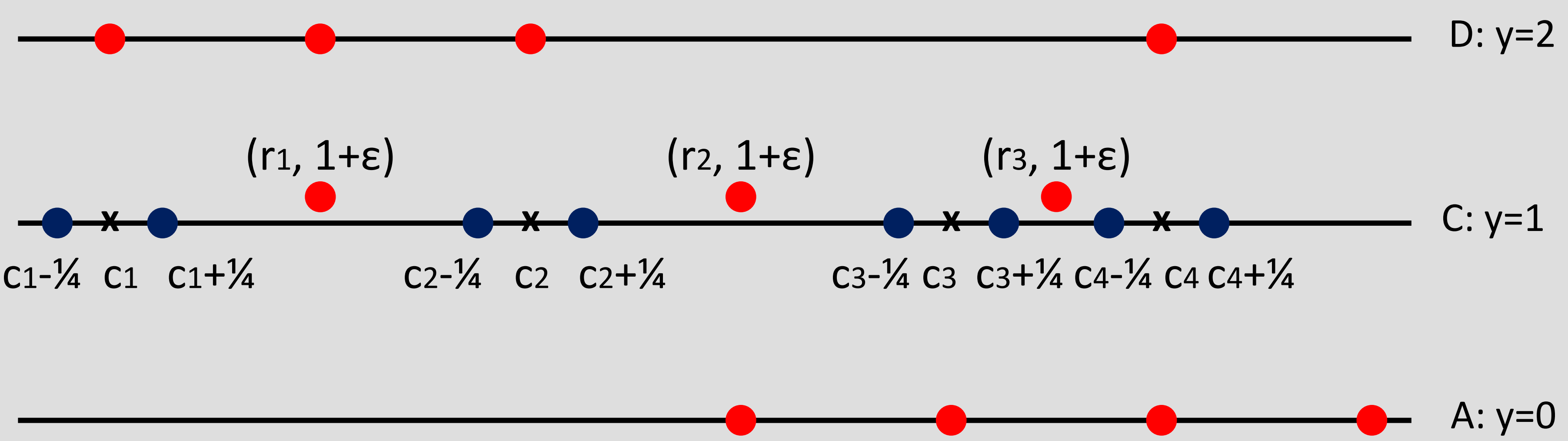

n points on each line
D: y=2
x
C: y=1
C1-¼
C1
C1+¼
C2-¼
C2
C2+¼
C3-¼
C3
C3+¼
C4-¼
C4
C4+¼
A: y=0

Hardness

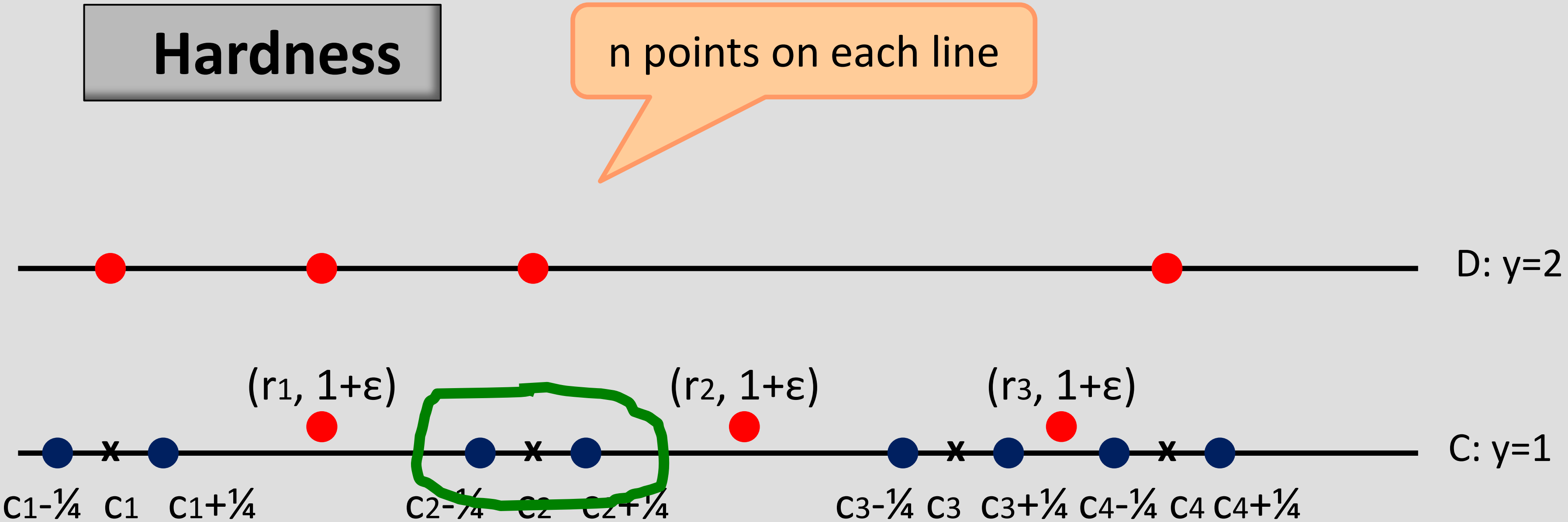

n points on each line
D: y=2
C: y=1
x
C1-¼ C1 C1+¼
C2-¼ C2 C2+¼
C3-¼ C3 C3+¼ C4-¼ C4 C4+¼
A: y=0

Hardness

n points on each line



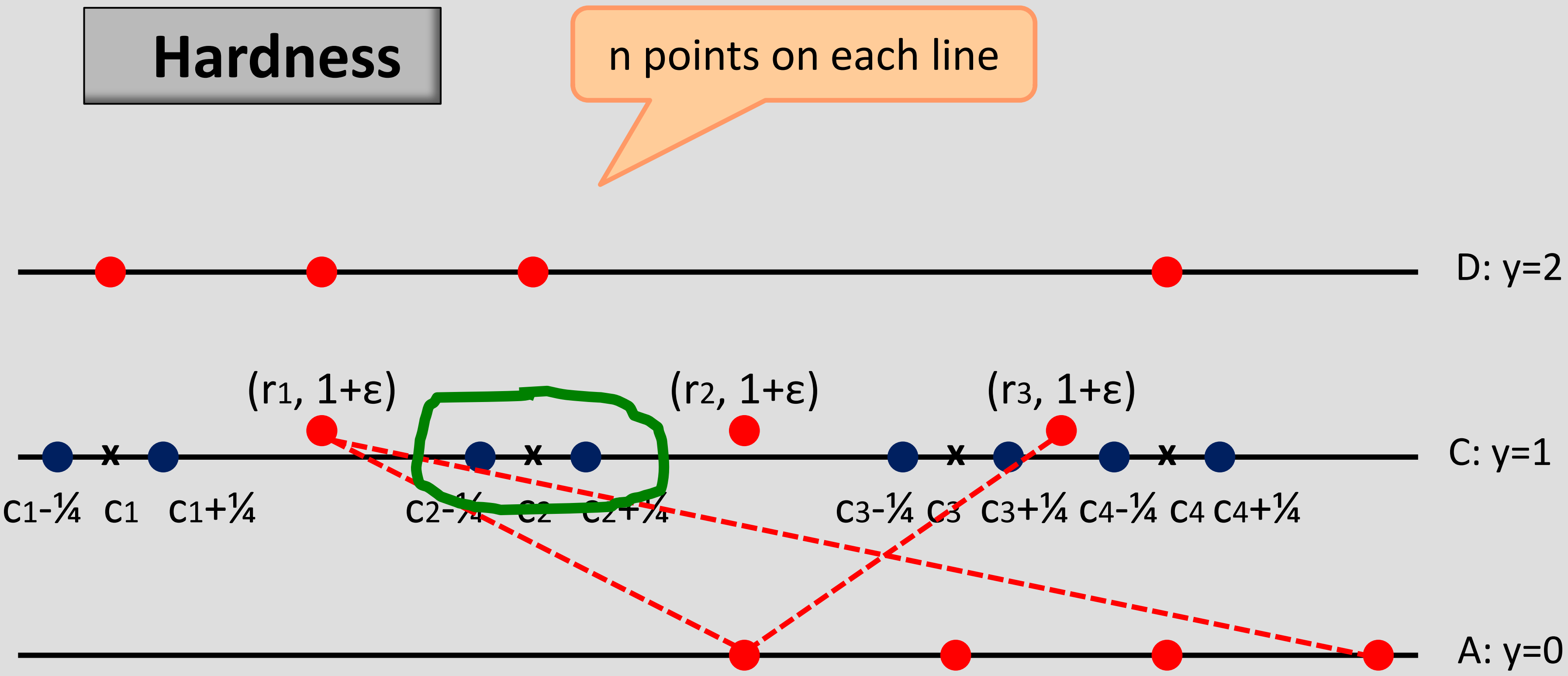
Hardness



$$r_i = \frac{c_i + c_{i+1}}{2}$$

ε such that no blue segment with endpoints $c - \frac{1}{4}, c + \frac{1}{4}$ is intersected by a red segment with one endpoint of the form $(r_i, 1+\varepsilon)$

Hardness

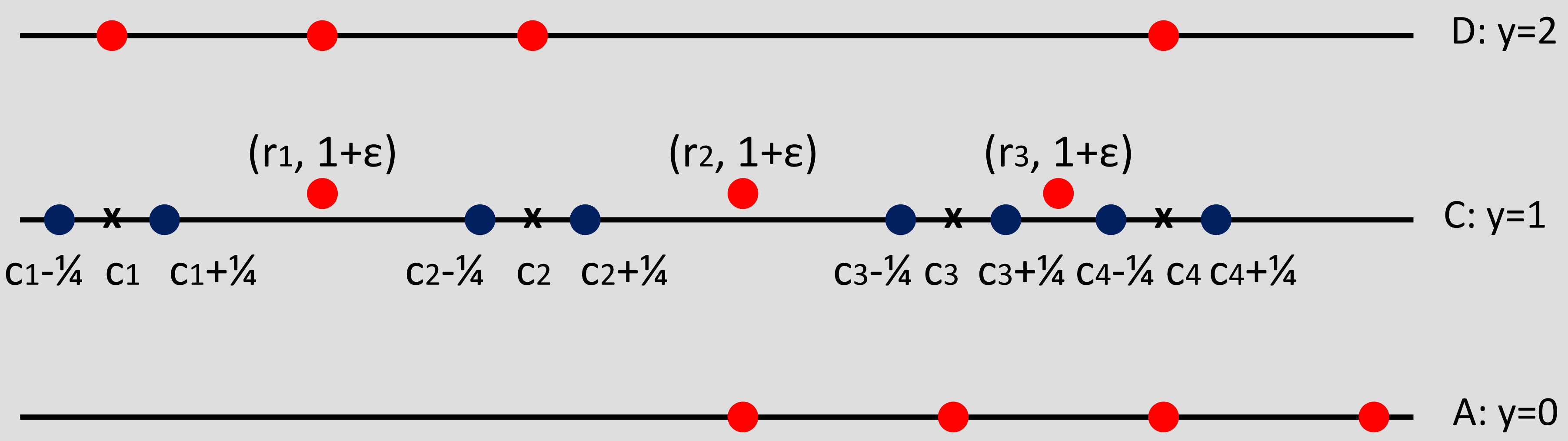


$$r_i = \frac{c_i + c_{i+1}}{2}$$

ε such that no blue segment with endpoints $c - \frac{1}{4}, c + \frac{1}{4}$ is intersected by a red segment with one endpoint of the form $(r_i, 1+\varepsilon)$

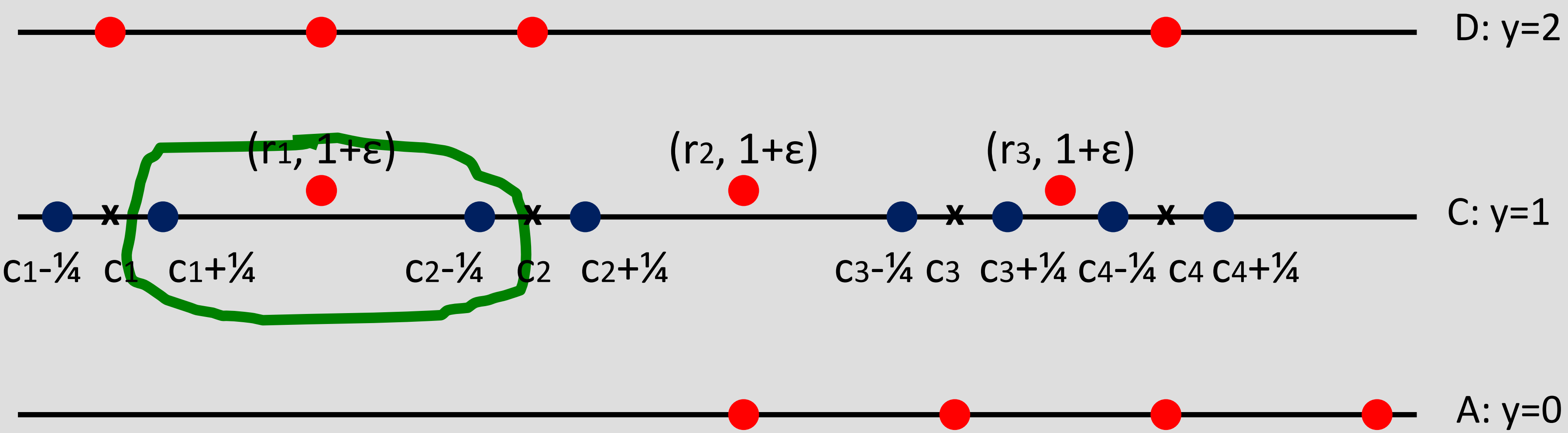
Hardness

$\exists$ line through $a \in A$, $c \in C$ and $d \in D \Leftrightarrow s_b > 2n(n-1)$



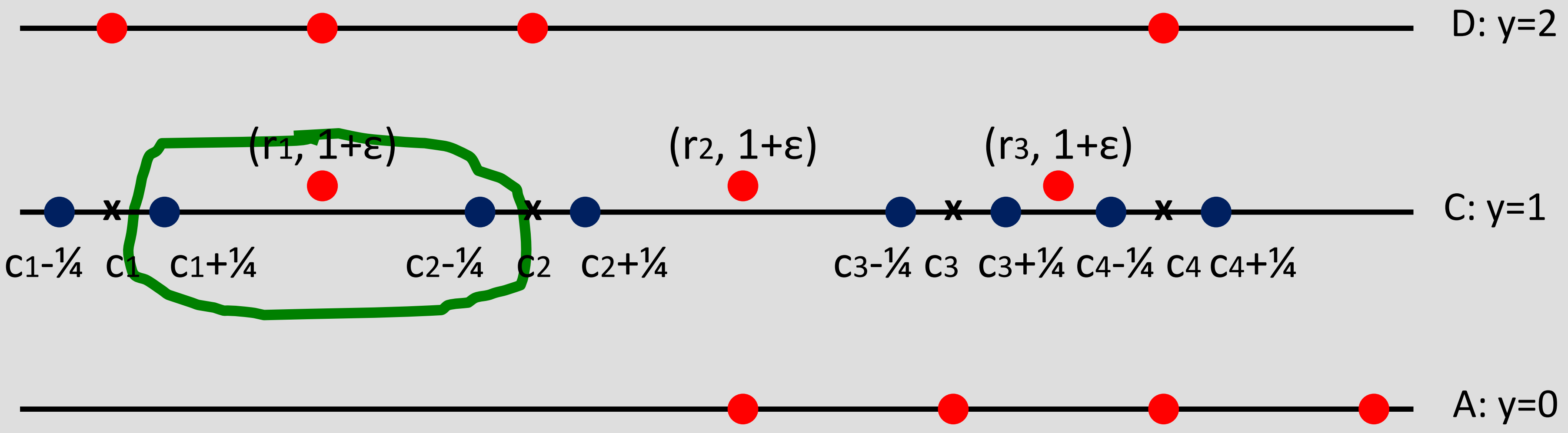
Hardness

$$\exists \text{ line through } a \in A,\ c \in C \text{ and } d \in D \Leftrightarrow s_b > 2n(n-1)$$



Hardness

$\exists$ line through $a \in A$, $c \in C$ and $d \in D \Leftrightarrow \; s_b > 2n(n-1)$

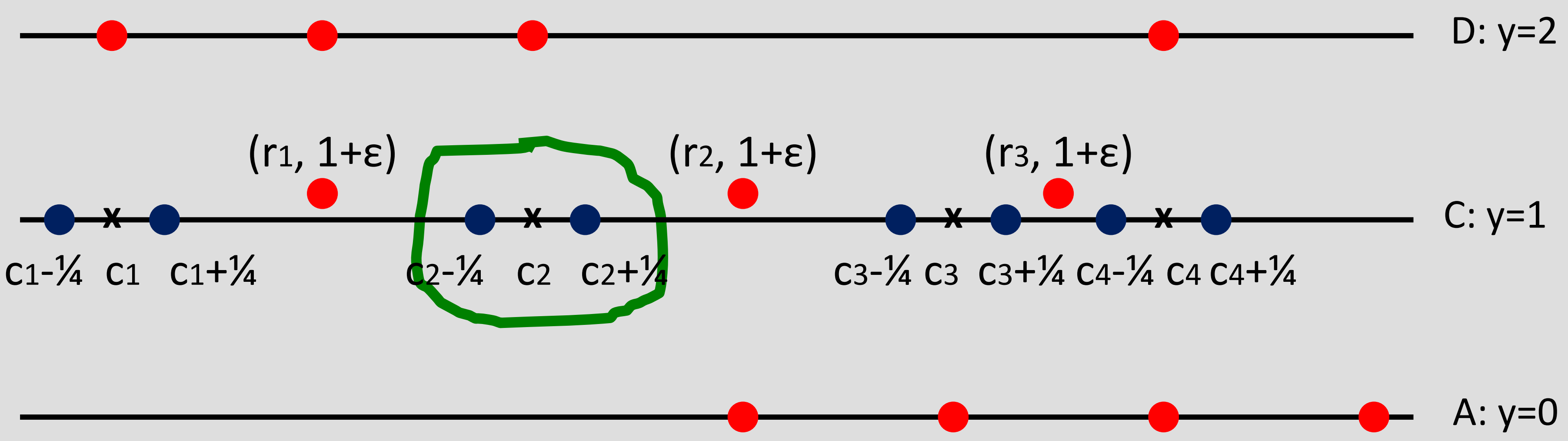


$$s_b \geq \frac{2n(2n-1)}{2} - n = 2n(n-1)$$

number of blue segments with endpoints $c_j \pm ¼$ and $c_k \pm ¼$ with $j \neq k$

Hardness

$\exists$ line through $a \in A$, $c \in C$ and $d \in D \Leftrightarrow s_b > 2n(n-1)$

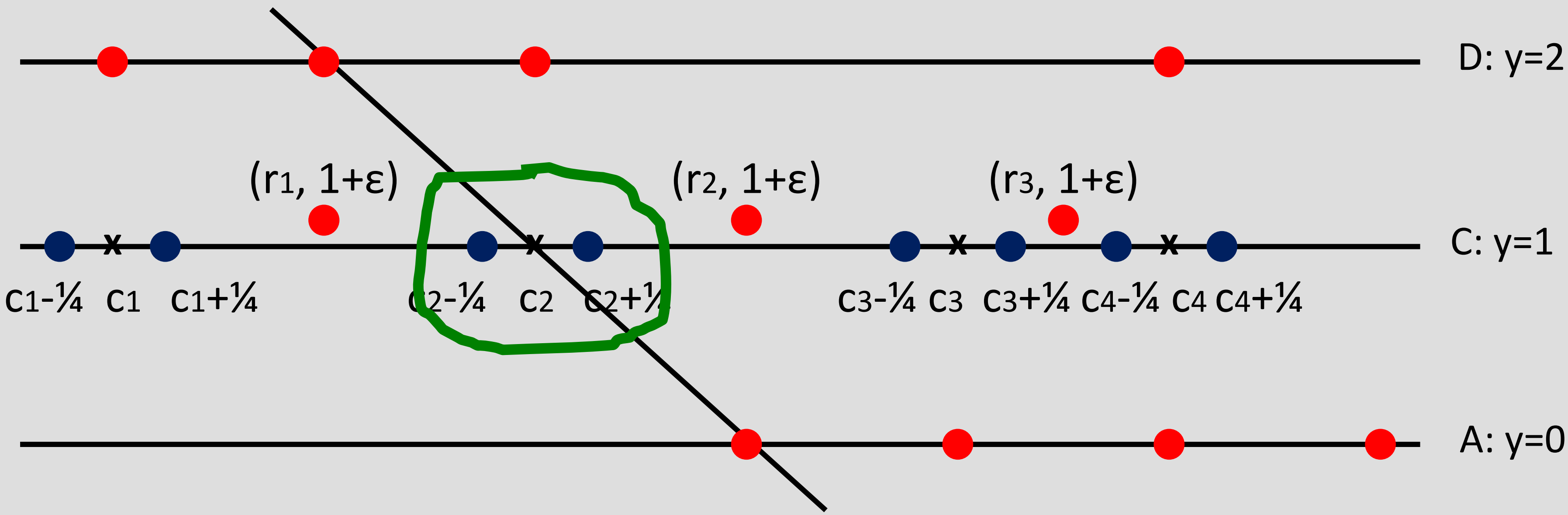


$$s_b \geq \frac{2n(2n-1)}{2} - n = 2n(n-1)$$

number of blue segments with endpoints $c_j \pm ¼$ and $c_k \pm ¼$ with $j \neq k$

Hardness

$\exists$ line through $a \in A$, $c \in C$ and $d \in D \Leftrightarrow s_b > 2n(n-1)$



$$s_b \geq \frac{2n(2n-1)}{2} - n = 2n(n-1)$$

number of blue segments with endpoints $c_j \pm 1/4$ and $c_k \pm 1/4$ with $j \neq k$

Open Problem:

Extend the problem to 3D using monochromatic triangles

Thank you