The Arithmetic of the Random Poset

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Overview

Next . . .

Understanding structures by finite approximations

What can one say about a structure by looking at its finite substructures?

Understanding structures by finite approximations

Let $\mathcal{A}$ be a countable relational structure.

Let **age**$(\mathcal{A})$ be the class of all finite structures $\mathcal{B}$ such that $\mathcal{B} \hookrightarrow \mathcal{A}$.

Properties of **age**$(\mathcal{A})$:

- there are only countably many pairwise nonisomorphic structures in **age**$(\mathcal{A})$;
- *Hereditary Property (HP):*
  if $\mathcal{B} \in$ **age**$(\mathcal{A})$ and $\mathcal{C} \hookrightarrow \mathcal{B}$ then $\mathcal{C} \in$ **age**$(\mathcal{A})$;
- *Joint Embedding Property (JEP):*
  for all $\mathcal{B}_1, \mathcal{B}_2 \in$ **age**$(\mathcal{A})$ there is a $\mathcal{C} \in$ **age**$(\mathcal{A})$ such that $\mathcal{B}_1 \hookrightarrow \mathcal{C}$ and $\mathcal{B}_2 \hookrightarrow \mathcal{C}$.

Understanding structures by finite approximations

Conversely, let **K** be a class of finite structures such that:

- there are only countably many pairwise nonisomorphic structures in **K**;
- **K** has (HP); and
- **K** has (JEP).

Is there a countable strucutre $\mathcal{A}$ such that **age**$(\mathcal{A}) =$ **K**?

Understanding structures by finite approximations

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Example. It is obvious that **age**$(\mathbb{N}, \leqslant) =$ **age**$(\mathbb{Z}, \leqslant)$.

Understanding structures by finite approximations

What countable structures are uniquely determined by their ages?

What classes of finite structures are ages of such countable structures?

Understanding structures by finite approximations

A countable structure $\mathcal{A}$ is *homogeneous* if every isomorphism $f : \mathcal{B} \to \mathcal{C}$ between finite substructures of $\mathcal{A}$ extends to an automorphism of $\mathcal{A}$.

A class **K** of finite structures is an *amalgamation class* if

- there are only countably many pairwise noniso struct's in **K**;
- **K** has (HP);
- **K** has (JEP); and
- **K** has the *Amalgamation Property (AP):*

  for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in$ **K** and embeddings $f : \mathcal{A} \hookrightarrow \mathcal{B}$ and $g : \mathcal{A} \hookrightarrow \mathcal{C}$, there exist $\mathcal{D} \in$ **K** and embeddings $u : \mathcal{B} \hookrightarrow \mathcal{D}$ and $v : \mathcal{C} \hookrightarrow \mathcal{D}$ such that $u \circ f = v \circ g$.

$$\begin{array}{ccc} \mathcal{C} & \overset{v}{\hookrightarrow} & \mathcal{D} \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle u} \\ \mathcal{A} & \underset{f}{\hookrightarrow} & \mathcal{B} \end{array}$$

Understanding structures by finite approximations

Theorem. *[Fraisse, 1953]*

1. If $\mathcal{A}$ is a countable homogeneous structure, then **age**$(\mathcal{A})$ is an amalgamation class.
2. If **K** is an amalgamation class, then there is a unique (up to isomorphism) countable homogeneous structure $\mathcal{A}$ such that **age**$(\mathcal{A}) =$ **K**.
3. If $\mathcal{B}$ is a countable structure *younger than* $\mathcal{A}$ (that is, **age**$(\mathcal{B}) \subseteq$ **age**$(\mathcal{A})$), then $\mathcal{B} \hookrightarrow \mathcal{A}$.

Definition. If **K** is an amalgamation class and $\mathcal{A}$ is the countable homogeneous structure such that **age**$(\mathcal{A}) =$ **K**, we say that $\mathcal{A}$ is the *Fraïssé limit* of **K**.

Some Fraïssé limits

$(\mathbb{Q}, <)$ = Fraïssé limit of the class of all linear orders

Random graph = Fraïssé limit of the class of all finite graphs

Random poset = Fraïssé limit of the class of all finite posets

Some Fraïssé limits

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What do they look like?

Some Fraïssé limits

Random graph

Vertices: $\mathbb{N} = \{1, 2, 3, \ldots\}$

Edges:

Let $m = \langle a_s a_{s-1} \ldots a_1 \rangle_2$ and $n = \langle b_t b_{t-1} \ldots b_1 \rangle_2$.

Put $m \sim n$ if $a_n = 1$ or $b_m = 1$

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What does the random poset look like? Hm…

Next . . .

Recall: Von Neumann's construction of ω in ZF

$\varnothing$

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$$2 := \{0, 1\}$$

$$3 := \{0, 1, 2\}$$

$$\vdots$$

$$n := \{0, 1, 2, \ldots, n-1\}$$

$$\vdots$$

$$\omega := \{0, 1, \ldots, n, \ldots\}$$

Recall: Construction of $\mathbb{Q}$

Take $\mathbb{Q}^* = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ and let

$$(a, b) \approx (c, d) \quad \text{if} \quad ad = bc$$

For example, $(1, 2) \approx (3, 6) \approx (-1, -2) \approx \ldots =: \dfrac{1}{2}$

Then $\mathbb{Q} = \mathbb{Q}^*/\approx$.

Recall: Dedekind's construction of $\mathbb{R}$

Take any partition $\{L, R\}$ of $\mathbb{Q}$ such that $L < R$

and form a new number $x = (L \mid R)$

with the intuition that $L < x < R$.

Conway’s class of surreal numbers $\mathbb{S}$

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Idea: Start making Dedekind-like cuts immediately!

If L and R are two sets of numbers such that no $x \in L$ is greater or equal to some $y \in R$, then $(L \mid R)$ is a new number.

The intuition is that $(L \mid R)$ is a cut which represents a new number between L and R.

Conway’s construction of $\mathbb{S}$

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$$(\varnothing \mid \{-1\}) \approx (\varnothing \mid \{-1, 0\}) \approx (\varnothing \mid \{-1, 1\}) \approx (\varnothing \mid \{-1, 0, 1\})$$

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$$0 \;\approx\; (\{-1\} \mid \{1\})$$

Conway's construction of $\mathbb{S}$

The following inductive definition formally introduces *names of surreal numbers* and a linear quasiorder $\leqslant_{\mathbb{S}}$ on names:

(1) If L and R are sets of names such that $r \leqslant_{\mathbb{S}} l$ for no $l \in L$ and no $r \in R$ then $(L \mid R)$ is a name.

(2) Let $x = (L \mid R)$ and $y = (U \mid V)$ be names. Then $x \leqslant_{\mathbb{S}} y$ if $y \leqslant_{\mathbb{S}} l$ for no $l \in L$ and $v \leqslant_{\mathbb{S}} x$ for no $v \in V$.

$x \approx y$ if $x \leqslant_{\mathbb{S}} y$ and $y \leqslant_{\mathbb{S}} x$.

$x <_{\mathbb{S}} y$ if $x \leqslant_{\mathbb{S}} y$ and $x \not\approx y$.

Conway's construction of $\mathbb{S}$

Thus we get the following hierarchy of names, indexed by ordinals:

- $\mathbb{S}_0 = \{(\varnothing \mid \varnothing)\}$,
- $\mathbb{S}_{\alpha+1} = \mathbb{S}_\alpha \cup \{(L \mid R) : L, R \subseteq \mathbb{S}_\alpha$ and $L <_{\mathbb{S}} R\}$,
- $\mathbb{S}_\lambda = \bigcup_{\alpha<\lambda} \mathbb{S}_\alpha$, for a limit ordinal λ.

Then $\mathbb{S} = \bigcup_\alpha \mathbb{S}_\alpha$ is the class of names,

and $\mathbb{S}/\approx$ is the class of surreal numbers.

Conway’s class of surreal numbers $\mathbb{S}$

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$\mathbb{S}$ contains infinitesimals e.g. $\epsilon = (\{0\} \mid \{\frac{1}{n} : n \in \mathbb{N}\})$

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$\mathbb{S}$ contains peculiarities e.g. $\omega - 1 = (\{1, 2, 3, \ldots\} \mid \{\omega\})$

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$\mathbb{S}$ contains peculiarities e.g. $\omega - 1 = (\{1, 2, 3, \ldots\} \mid \{\omega\})$

$\ldots$ and much more e.g. $(\omega - 1) + \epsilon$

Conway's class of surreal numbers $\mathbb{S}$

$\mathbb{S}$ is actually a field!

$-x = (-R_x \mid -L_x)$

$x + y = ((L_x + y) \cup (x + L_y) \mid (R_x + y) \cup (x + R_y))$

$x \cdot y = (\mathcal{L}_1 \cup \mathcal{L}_2 \mid \mathcal{R}_1 \cup \mathcal{R}_2)$, where

$$\begin{aligned}
\mathcal{L}_1 &= \{(x^L \cdot y) + (x \cdot y^L) - (x^L \cdot y^L) : x^L \in L_x, y^L \in L_y\}, \\
\mathcal{L}_2 &= \{(x^R \cdot y) + (x \cdot y^R) - (x^R \cdot y^R) : x^R \in R_x, y^R \in R_y\}, \\
\mathcal{R}_1 &= \{(x^L \cdot y) + (x \cdot y^R) - (x^L \cdot y^R) : x^L \in L_x, y^R \in R_y\}, \\
\mathcal{R}_2 &= \{(x^R \cdot y) + (x \cdot y^L) - (x^R \cdot y^L) : x^R \in R_x, y^L \in L_y\}.
\end{aligned}$$

$x^{-1} =$ (something quite awful)

Next . . .

The Hubička-Nešetřil presentation of the random poset

In their paper

J. Hubička, J. Nešetřil. Finite presentation of homogeneous graphs, posets and ramsey classes. Israel Journal of Mathematics 149 (2005), 21–44

J. Hubička and J. Nešetřil introduced an intriguing presentation $\mathcal{P}^{\mathbb{S}}$ of the random poset in $\mathbb{S}_\omega$.

The Hubička-Nešetřil presentation of the random poset

For $a, b \in \mathbb{S}_\omega$ we write $a \preccurlyeq b$ if $(\{a\} \cup R_a) \cap (\{b\} \cup L_b) \neq \varnothing$.

Let $\mathcal{P}^{\mathbb{S}}$ be the set of all $m = (L_m \mid R_m) \in \mathbb{S}_\omega$ such that:

- L_m and R_m are finite subsets of $\mathcal{P}^{\mathbb{S}}$ such that $L_m \cap R_m = \varnothing$;
- $x \preccurlyeq y$ for all $x \in L_m$, $y \in R_m$;
- $L_x \subseteq L_m$ for all $x \in L_m$, and $R_x \subseteq R_m$ for all $x \in R_m$.

Theorem.

1. $\preccurlyeq$ is a partial order on $\mathcal{P}^{\mathbb{S}}$.
2. $(\mathcal{P}^{\mathbb{S}}, \prec)$ is isomorphic to the random poset.
3. For all $a, b \in \mathcal{P}^{\mathbb{S}}$, if $a \prec b$ then $a <_{\mathbb{S}} b$.

The arithmetic of the random poset

Lemma. The following holds for all $a \in \mathcal{P}^{\mathbb{S}}$:

1. $-(-a) = a$;
2. $-a \in \mathcal{P}^{\mathbb{S}}$.

For $a, b \in \mathcal{P}^{\mathbb{S}}$ define $a + b$ as follows:

$$a + b = \big((L_a + b) \cup (a + L_b) \cup (L_a + L_b) \mid (R_a + b) \cup (a + R_b) \cup (R_a + R_b)\big).$$

The arithmetic of the random poset

Theorem. $(\mathcal{P}^{\mathbb{S}}, +, -, 0, \preccurlyeq)$ is an ordered commutative monoid with involution. In other words, the following holds for all $a, b, c, d \in \mathcal{P}^{\mathbb{S}}$:

1. $a + b \in \mathcal{P}^{\mathbb{S}}$;
2. $0 + a = a + 0 = a$;
3. $a + b = b + a$;
4. $-(a + b) = (-a) + (-b)$;
5. $(a + b) + c = a + (b + c)$;
6. if $a \preccurlyeq b$ and $c \preccurlyeq d$ then $a + c \preccurlyeq b + d$;
7. if $a \prec b$ and $c \preccurlyeq d$ then $a + c \prec b + d$.

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5. $(a + b) + c = a + (b + c)$;
6. if $a \preccurlyeq b$ and $c \preccurlyeq d$ then $a + c \preccurlyeq b + d$;
7. if $a \prec b$ and $c \preccurlyeq d$ then $a + c \prec b + d$.

Why not more than that?

The arithmetic of the random poset

Why not more than that?

Recall: $0 = (\varnothing \mid \varnothing)$, $-1 = (\varnothing \mid \{0\})$, $1 = (\{0\} \mid \varnothing)$.

An easy computation shows that $1 + (-1) = (\{-1\} \mid \{1\})$.

Clearly, $0 \neq (\{-1\} \mid \{1\}) \in \mathcal{P}^{\mathbb{S}}$.

Therefore, $x + (-x) \neq 0$ in general.

The arithmetic of the random poset

Let $\mathcal{N} = \{x \in \mathcal{P}^{\mathbb{S}} : x \approx 0\}$.

Lemma.

1. $0 \in \mathcal{N}$;
2. $-\mathcal{N} = \mathcal{N}$;
3. $\mathcal{N} + \mathcal{N} = \mathcal{N}$;
4. $a + \mathcal{N} = b + \mathcal{N}$ if and only if $a + (-b) \in \mathcal{N}$, for all $a, b \in \mathcal{P}^{\mathbb{S}}$.

Theorem. $(\mathcal{P}^{\mathbb{S}}/\mathcal{N}, +, -, 0, \preccurlyeq)$ is an ordered abelian group.

Multiplication?

Bad luck!

Conway's multiplication can be adapted, but it applies only to "integers" in $\mathbb{S}$:

Lemma. Assume that for some $x, y \in \mathcal{P}^{\mathbb{S}}$ we have $L_x = \varnothing$ or $R_x = \varnothing$, and $L_y = \varnothing$ or $R_y = \varnothing$. Then $x \cdot y \in \mathcal{P}^{\mathbb{S}}$.

Example. Recall: $\frac{1}{2} = (\{0\} \mid \{1\}) \in \mathcal{P}^{\mathbb{S}}$. One can show that $\frac{1}{2} \cdot \frac{1}{2} \notin \mathcal{P}^{\mathbb{S}}$.

Open Problems

1. Is it possible to adapt Conway's multiplication so that it applies to the whole of $\mathcal{P}^{\mathbb{S}}$?
2. Is it possible to adapt Conway's inverse (x^{-1}) so that it applies to $\mathcal{P}^{\mathbb{S}}$?
3. Is it possible to turn $\mathcal{P}^{\mathbb{S}}$ into a field (so that it corresponds to the fact that $\mathbb{Q}$ is a field)?
4. There is a Hubička-Nešetřil presentation $\mathcal{R}^{\mathbb{S}}$ of the random graph in terms of surreal numbers. Is it possible to adapt Conway's arithmetic operations to $\mathcal{R}^{\mathbb{S}}$?