

Katětov expanders

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Motivation

Definition (Katětov 1986)

Let (X, ϱ) be a metric space. A **Katětov function** on X is a map $f: X \rightarrow \mathbb{R}$ satisfying

- 1 $|f(x) - f(y)| \leq \varrho(x, y)$
- 2 $\varrho(x, y) \leq f(x) + f(y)$

for every $x, y \in X$.

Denote by $K(X)$ the space of all Katětov functions on X , endowed with the sup metric.

Claim

The construction above extends to a self-functor on the category of metric spaces with nonexpansive maps.

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Claim

The construction above extends to a self-functor on the category of metric spaces with nonexpansive maps.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & K(X) \\ \downarrow f & & \downarrow K(f) \\ Y & \xrightarrow{\eta_Y} & K(Y) \end{array}$$

General assumptions:

We fix a class \mathcal{S} of *small* models of a fixed type. We assume:

- \mathcal{S} has the joint embedding property
- \mathcal{S} has the amalgamation property
- \mathcal{S} is closed under isomorphisms

Notation:

$\sigma\mathcal{S}$ will denote the class of all structures isomorphic to unions of countable chains in \mathcal{S} .

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Fraïssé limits

Theorem (Fraïssé 1954)

Assume \mathcal{S} is countable. Then there exists a unique \mathcal{S} -homogeneous structure $U \in \sigma\mathcal{S}$.

U is often called the **Fraïssé limit** of \mathcal{S} .

\mathcal{S} is called a **Fraïssé class**.

Problem ((?) Jaligot, 2007)

*Let \mathcal{S} be a Fraïssé class with the Fraïssé limit U .
Is it always true that $\text{Aut}(U)$ is universal for the class
 $\{\text{Aut}(X) : X \in \sigma\mathcal{S}\}$?*

Some references

- 1 HENSON, C. W., *A family of countable homogeneous graphs*, Pacific J. Math. 38 (1971) 69–83
- 2 JALIGOT, E., *On stabilizers of some moieties of the random tournament*, Combinatorica 27 (2007), no. 1, 129–133
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- 4 DOLINKA, I.; MAŠULOVIĆ, D., *A universality result for endomorphism monoids of some ultrahomogeneous structures*, Proc. Edinb. Math. Soc. (2) 55 (2012), no. 3, 635–656
- 5 BILGE, D., *Automorphism Groups of Homogeneous Structures*, PhD thesis, Université Lyon 1, 2012

Expanders

Let $\mathcal{S} \subseteq \sigma\mathcal{S}$ be as before, now treated as categories.

Definition

An **expander** on $\langle \mathcal{S}, \sigma\mathcal{S} \rangle$ is a pair $\langle F, \eta \rangle$, where $F: \mathcal{S} \rightarrow \sigma\mathcal{S}$ is a covariant functor and η is a natural transformation from $\text{id}_{\mathcal{S}}$ to F .

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & F(A) \\ f \downarrow & & \downarrow F(f) \\ B & \xrightarrow{\eta_B} & F(B) \end{array}$$

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Lemma

Every expander on $\langle \mathcal{S}, \sigma \mathcal{S} \rangle$ extends to a continuous expander on $\langle \sigma \mathcal{S}, \sigma \mathcal{S} \rangle$.

Continuity means that

$$F\left(\lim_{n \rightarrow \infty} X_n\right) = \lim_{n \rightarrow \infty} F(X_n)$$

whenever $X_0 \subseteq X_1 \subseteq \dots$ is a tower of $\sigma \mathcal{S}$ -structures.

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Katětov expanders

Fix a family \mathcal{F} of embeddings between \mathcal{S} -objects so that every embedding $j: A \rightarrow B$ with $A, B \in \mathcal{S}$ is of the form

$$j = e_1 \circ \dots \circ e_n \quad \text{for some} \quad e_1, \dots, e_n \in \mathcal{F}.$$

Definition

An expander $\langle K, \eta \rangle$ is **Katětov** with respect to \mathcal{F} if for every $X \in \mathcal{L}$, for every embeddings $f: A \rightarrow X$, $e: A \rightarrow B$ with $A, B \in \mathcal{S}$ and $e \in \mathcal{F}$, there exists an embedding $\bar{f}: B \rightarrow K(X)$ such that $\bar{f} \circ e = \eta_X \circ f$. In other words, the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & K(X) \\ f \uparrow & & \uparrow \bar{f} \\ A & \xrightarrow{e} & B \end{array}$$

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Fact

Let K be a Katětov expander on \mathcal{S} . Then $K^\omega(X)$ is the Fraïssé limit of \mathcal{S} for every $X \in \sigma\mathcal{S}$.

Here,

- $K^\omega(X) = \lim_{n \rightarrow \infty} K^n(X)$,
- $K^n(X) = K(K^{n-1}(X))$, $K^0(X) = X$.

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Automorphism groups

Theorem

Assume K is a Katětov expander on \mathcal{S} and let U be the Fraïssé limit of \mathcal{S} . Then for every $X \in \sigma\mathcal{S}$ the natural embedding

$$X \hookrightarrow K(X)$$

induces an embedding $\text{Aut}(X) \hookrightarrow \text{Aut}(U)$.

If K is an expander on homomorphisms, then the same holds for the endomorphism semigroups.

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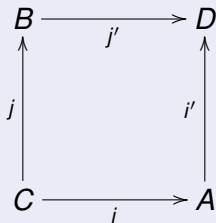
Pushouts

Definition

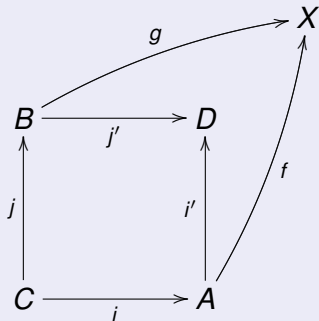
A class of structures \mathcal{S} admits **pushouts** if for every embeddings $i: C \rightarrow A$, $j: C \rightarrow B$ in \mathcal{S} , there exist embeddings $i': A \rightarrow D$, $j': B \rightarrow D$ satisfying $i' \circ i = j' \circ j$ and

- for every homomorphisms $f: A \rightarrow X$, $g: B \rightarrow X$ with $f \circ i = g \circ j$, there exists a **unique** homomorphism $h: D \rightarrow X$ such that $h \circ i' = f$ and $h \circ j' = g$.

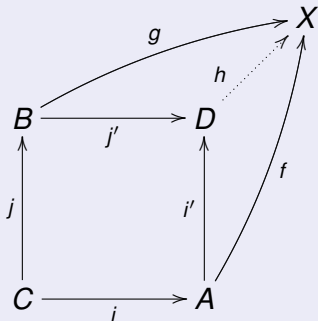
The pushout of $\langle i, j \rangle$



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Existence result

Theorem

Assume \mathcal{S} admits pushouts. Then there exists a Katětov expander in $\langle \mathcal{S}, \sigma\mathcal{S} \rangle$.

Remark

If, additionally, \mathcal{S} has **mixed pushouts** then there exists a Katětov expander for all homomorphisms.

Consequently, denoting by U the Fraïssé limit of \mathcal{S} , the semigroup $\text{End}(U)$ is universal for $\{\text{End}(X) : X \in \sigma\mathcal{S}\}$.

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Corollary

The following Fraïssé classes of finite structures admit Katětov expanders:

- *graphs*
- *directed graphs*
- *K_n -free graphs*
- *posets*
- *semilattices*

Remark

The class of K_n -free graphs ($n > 2$) does not admit a Katětov expander for homomorphisms.

This follows from a result of Mudrinski (2010): The K_n -free Henson graph is retract-rigid (identity is the only retraction).

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Other examples of Katětov expanders

Example

Let \mathcal{S} be the class of finite linearly ordered sets. Given $S \in \mathcal{S}$, define

$$K(S) = S \cup \text{hom}(S, \{0, 1\}),$$

with the natural linear ordering.

Example

Let \mathcal{S} be the class of all finite groups. Given $G \in \mathcal{S}$ let $F(G)$ be the group of all permutations of the set G . Identifying G with the subgroup of $F(G)$, we can extend F to an expander.

Claim

F^ω is a Katětov expander.

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Tournaments

Proposition

There exists a Katětov expander in the class of finite tournaments.

Proof.

Given a finite tournament T , let $K(T)$ be the set of one-to-one sequences in T , agreeing that the empty sequence dominates everything.

Identify T with sequences of length 1.

Endow $K(T)$ with the lexicographic tournament structure. □

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Problem

Does there exist a Fraïssé class \mathcal{F} with no Katětov expander?

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When does there exist a Katětov expander for homomorphisms?

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