Katětov expanders

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XIX Midsummer Combinatorial Workshop

July 29 – August 2, 2013, Prague

Joint project with
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Motivation

Definition (Katětov 1986)

Let (X, ϱ) be a metric space. A Katětov function on X is a map $f\colon X \to \mathbb{R}$ satisfying

1. $|f(x) - f(y)| \leqslant \varrho(x, y)$
2. $\varrho(x, y) \leqslant f(x) + f(y)$

for every $x, y \in X$.

Denote by $K(X)$ the space of all Katětov functions on X, endowed with the sup metric.

Claim

The construction above extends to a self-functor on the category of metric spaces with nonexpansive maps.

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The construction above extends to a self-functor on the category of metric spaces with nonexpansive maps.

$$
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & K(X) \arrow[d, "K(f)"] \\
Y \arrow[r, "\eta_Y"'] & K(Y)
\end{tikzcd}
$$

General assumptions:

We fix a class $\mathscr{S}$ of *small* models of a fixed type. We assume:

- $\mathscr{S}$ has the joint embedding property
- $\mathscr{S}$ has the amalgamation property
- $\mathscr{S}$ is closed under isomorphisms

Notation:

$\sigma\mathscr{S}$ will denote the class of all structures isomorphic to unions of countable chains in $\mathscr{S}$.

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Fraïssé limits

Theorem (Fraïssé 1954)

Assume $\mathscr{S}$ is countable. Then there exists a unique $\mathscr{S}$-homogeneous structure $U \in \sigma\mathscr{S}$.

U is often called the Fraïssé limit of $\mathscr{S}$.
$\mathscr{S}$ is called a Fraïssé class.

Problem ((?) Jaligot, 2007)

Let $\mathscr{S}$ be a Fraïssé class with the Fraïssé limit U.
Is it always true that $\mathrm{Aut}(U)$ *is universal for the class* $\{\mathrm{Aut}(X) \colon X \in \sigma\mathscr{S}\}$*?*

Some references

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5. BILGE, D., *Automorphism Groups of Homogeneous Structures*, PhD thesis, Universite Lyon 1, 2012

Expanders

Let $\mathscr{S} \subseteq \sigma\mathscr{S}$ be as before, now treated as categories.

Definition

An expander on $\langle \mathscr{S}, \sigma\mathscr{S} \rangle$ is a pair $\langle F, \eta \rangle$, where $F \colon \mathscr{S} \to \sigma\mathscr{S}$ is a covariant functor and η is a natural transformation from $\mathrm{id}_{\mathscr{S}}$ to F.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & F(A) \arrow[d, "F(f)"] \\ B \arrow[r, "\eta_B"] & F(B) \end{tikzcd}$$

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Lemma

Every expander on $\langle \mathscr{S}, \sigma\mathscr{S} \rangle$ extends to a continuous expander on $\langle \sigma\mathscr{S}, \sigma\mathscr{S} \rangle$.

Continuity means that

$$F(\lim_{n\to\infty} X_n) = \lim_{n\to\infty} F(X_n)$$

whenever $X_0 \subseteq X_1 \subseteq \dots$ is a tower of $\sigma\mathscr{S}$-structures.

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Katětov expanders

Fix a family $\mathscr{F}$ of embeddings between $\mathscr{S}$-objects so that every embedding. $j \colon A \to B$ with $A, B \in \mathscr{S}$ is of the form

$$j = e_1 \circ \ldots \circ e_n \qquad \text{for some} \qquad e_1, \ldots, e_n \in \mathscr{F}.$$

Definition

An expander $\langle K, \eta \rangle$ is Katětov with respect to $\mathscr{F}$ if for every $X \in \mathfrak{L}$, for every embeddings $f \colon A \to X$, $e \colon A \to B$ with $A, B \in \mathscr{S}$ and $e \in \mathscr{F}$, there exists an embedding $\overline{f} \colon B \to K(X)$ such that $\overline{f} \circ e = \eta_X \circ f$. In other words, the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] & K(X) \\ A \arrow[u, "f"] \arrow[r, "e"'] & B \arrow[u, dashed, "\overline{f}"'] \end{tikzcd}$$

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Fact

Let K be a Katětov expander on $\mathscr{S}$. Then $K^\omega(X)$ is the Fraïssé limit of $\mathscr{S}$ for every $X \in \sigma\mathscr{S}$.

Here,

- $K^\omega(X) = \lim_{n\to\infty} K^n(X)$,
- $K^n(X) = K(K^{n-1}(X))$, $K^0(X) = X$.

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Automorphism groups

Theorem

Assume K is a Katětov expander on $\mathscr{S}$ and let U be the Fraïssé limit of $\mathscr{S}$. Then for every $X \in \sigma\mathscr{S}$ the natural embedding

$$X \hookrightarrow K(X)$$

induces an embedding $\mathrm{Aut}(X) \hookrightarrow \mathrm{Aut}(U)$.

If K is an expander on homomorphisms, then the same holds for the endomorphism semigroups.

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Pushouts

Definition

A class of structures $\mathscr{S}$ admits pushouts if for every embeddings $i\colon C \to A$, $j\colon C \to B$ in $\mathscr{S}$, there exist embeddings $i'\colon A \to D$, $j'\colon B \to D$ satisfying $i' \circ i = j' \circ j$ and

- for every homomorphisms $f\colon A \to X$, $g\colon B \to X$ with $f \circ i = g \circ j$, there exists a unique homomorphism $h\colon D \to X$ such that $h \circ i' = f$ and $h \circ j' = g$.

The pushout of $\langle i, j \rangle$

$$
\begin{tikzcd}
B \arrow[r, "j'"'] & D \\
C \arrow[u, "j"] \arrow[r, "i"'] & A \arrow[u, "i'"']
\end{tikzcd}
$$

The pushout of $\langle i, j \rangle$

$$
\begin{tikzcd}
 & & X \\
B \arrow[r, "j'"'] \arrow[urr, bend left, "g"] & D & \\
C \arrow[u, "j"] \arrow[r, "i"'] & A \arrow[u, "i'"] \arrow[uur, bend right, "f"'] &
\end{tikzcd}
$$

The pushout of $\langle i, j \rangle$

$$
\begin{tikzcd}
 & & X \\
B \arrow[r, "j'"'] \arrow[urr, bend left, "g"] & D \arrow[ur, dotted, "h"] & \\
C \arrow[u, "j"] \arrow[r, "i"'] & A \arrow[u, "i'"'] \arrow[uur, bend right, "f"'] &
\end{tikzcd}
$$

Existence result

Theorem

Assume $\mathscr{S}$ admits pushouts. Then there exists a Katětov expander in $\langle \mathscr{S}, \sigma\mathscr{S} \rangle$.

Remark

If, additionally, $\mathscr{S}$ has mixed pushouts then there exists a Katětov expander for all homomorphisms.
Consequently, denoting by U the Fraïssé limit of $\mathscr{S}$, the semigroup $\operatorname{End}(U)$ is universal for $\{\operatorname{End}(X) \colon X \in \sigma\mathscr{S}\}$.

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Corollary

The following Fraïssé classes of finite structures admit Katětov expanders:

- *graphs*
- *directed graphs*
- K_n*-free graphs*
- *posets*
- *semilattices*

Remark

The class of K_n-free graphs ($n > 2$) does not admit a Katětov expander for homomorphisms.
This follows from a result of Mudrinski (2010): The K_n-free Henson graph is retract-rigid (identity is the only retraction).

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Other examples of Katětov expanders

Example

Let $\mathscr{S}$ be the class of finite linearly ordered sets. Given $S \in \mathscr{S}$, define

$$K(S) = S \cup \hom(S, \{0, 1\}),$$

with the natural linear ordering.

Example

Let $\mathscr{S}$ be the class of all finite groups. Given $G \in \mathscr{S}$ let $F(G)$ be the group of all permutations of the set G. Identifying G with the subgroup of $F(G)$, we can extend F to an expander.

Claim

F^{ω} is a Katětov expander.

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Tournaments

Proposition

There exists a Katětov expander in the class of finite tournaments.

Proof.

Given a finite tournament T, let $K(T)$ be the set of one-to-one sequences in T, agreeing that the empty sequence dominates everything.
Identify T with sequences of length 1.
Endow $K(T)$ with the lexicographic tournament structure. □

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Problem

Does there exist a Fraïssé class $\mathscr{F}$ with no Katětov expander?

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When does there exist a Katětov expander for homomorphisms?

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