Two of my favorite problems

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Problem — Minimizing monoχ k-APs

Minimize the number of monochromatic 3-term arithmetic progressions in a 2-coloring of the numbers $1, \ldots, n$.

Example ($n = 28$)

R R R B B R R R B B B B B B R R R R R R B B B R R B B B

Theorem (Frankl-Graham-Rödl)

There is some $c > 0$ so that *every* coloring of $1, \ldots, n$ has $cn^2 + o(n^2)$ monochromatic 3-term arithmetic progressions.

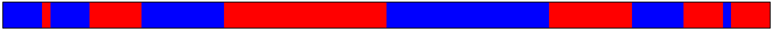
Theorem (Parrilo-Robertson-Saracino; Butler-Costello-Graham)

Expanding the following coloring gives

$$\frac{117}{2192}n^2 + O(n) = \frac{117}{137} \cdot \frac{1}{16}n^2 + O(n)$$

monochromatic 3-APs:

$$\underbrace{R\cdots R}_{28}\underbrace{B\cdots B}_{6}\underbrace{R\cdots R}_{28}\underbrace{B\cdots B}_{37}\underbrace{R\cdots R}_{59}\underbrace{B\cdots B}_{116}\underbrace{R\cdots R}_{116}\underbrace{B\cdots B}_{59}\underbrace{R\cdots R}_{37}\underbrace{B\cdots B}_{28}\underbrace{R\cdots R}_{6}\underbrace{B\cdots B}_{28}$$



Conjecture

This is best possible. (Note at least $\frac{1675}{32768}n^2 + o(n^2)$ monoχ 3-APs.)

A possible approach

50 SHADES OF PURPLE

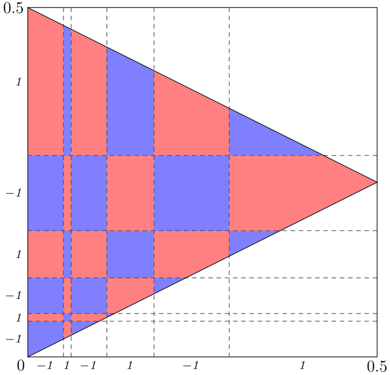
Turn it into a continuous problem. Allow the colors to take a range of values between -1 and 1. Also instead of coloring $1, \ldots, n$ focus on coloring the *interval* $[0, 1]$.

Problem now reduces to finding $f : [0, 1] \to [-1, 1]$ minimizing

$$\left(\int_0^1 f(u)\, du \right)^2 + 4 \int_0^1 \int_{u/2}^{(u+1)/2} f(u) f(v)\, dv\, du.$$

Geometrical variation

- Take a square and subdivide two perpendicular sides using the exact same pattern.
- Form a red/blue checkerboard with blue in the lower left.



- Take the triangle from the lower left corner, to the opposite midpoint to the upper left corner.
- Find the subdivision which maximizes the *red* inside the triangle.

What about 4-APs?

Lu-Peng found the following coloring based on *good* coloring of $\mathbb{Z}_{11}$.

Given $\ell = \sum b_i \cdot 11^i$ and j is the smallest index so that $b_j \neq 0$, then

$$\text{color } \ell \quad \begin{cases} \text{red} & \text{if } b_j = 1, 3, 4, 5, \text{ or } 9; \\ \text{blue} & \text{if } b_j = 2, 6, 7, 8, \text{ or } 10. \end{cases}$$

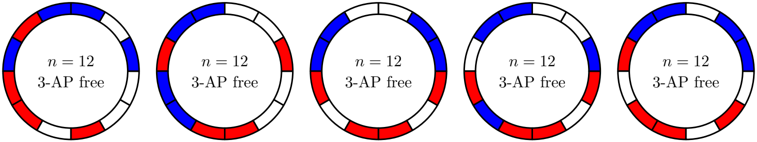
This coloring has $\frac{1}{72}n^2 + O(n)$ monochromatic 4-APs.

Conjecture

This is best possible.

Other problems...

For 3 colors and avoiding 3-APs best known is to repeat one of five different fixed coloring of $\mathbb{Z}_{12}$. Any gives $\frac{1}{48}n^2 + O(n)$. Can we do better?



Conjecture

For fixed k (=AP-length) and r (=colors) we can always *beat* random.

Problem — Induced universal graphs

Given a family $\mathcal{F}$ of graphs, construct a *small* graph F which contains each graph in $\mathcal{F}$ as an induced subgraph.

Examples of possible families $\mathcal{F}$:

- Bipartite graphs on n vertices
- Graphs with n edges
- Hypergraphs on n vertices.
- Hypergraphs on n vertices w/ degrees $\leq d$.
- Graphs on n vertices with bounded chromatic number.
- …

Moon, “On minimal n-universal graphs”

If $\mathcal{F}$ is the family of graphs on n vertices, then there is an induced universal graph with number of vertices N satisfying:

$$2^{(n-1)/2} < N < 2n2^{(n-1)/2}.$$

- Lower bound: Number of graphs is bounded by number of induced subgraphs:

$$\frac{2^{\binom{n}{2}}}{n!} \leq \binom{N}{n} < \frac{N^n}{n!}.$$

- Upper bound: Construction based on starting with a tournament.

Chung, “Universal graphs and induced universal graphs”

Considered several families including trees, planar graphs, graphs with bounded arboricity.

Theorem

Let F be an induced universal graph for $\mathcal{F}$. If every graph in $\mathcal{H}$ can be edge-partitioned into k graphs in $\mathcal{F}$, then there is an induced universal graph H where

$$|V(H)| \leq |V(F)|^k \text{ and } |E(H)| \leq k|E(F)||V(F)|^{2k-2}.$$

Note: This theorem can be easily generalized to multigraphs, directed graphs, hypergraphs, and also can decompose into different families.