List Hadwiger conjecture and extremal K_5-minor-free graphs with fixed girth

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joint work with David R. Wood

Concepts

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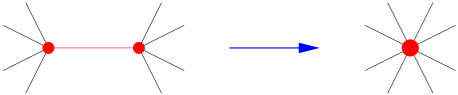
Contraction of an edge



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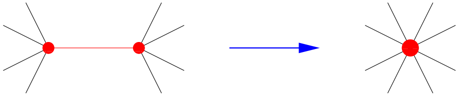
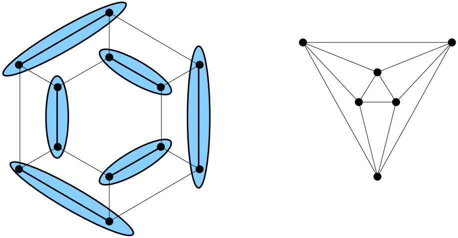


G contains H as a minor



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KM: $c = 3/2$, W: $c = 1$.

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New Conjecture: $c = 4/3$.

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	planar	K_5-mf	K_6-mf	toroidal
general	5	5	6,7,8	6 ex K_7
girth 5	3	3	3,4,5,6,7,8	conj 3
girth 4	4	4	5,6,7,8	4
bipartite	3	4	5,6,7,8	?

Basic facts, planarity

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K_5-minor-free

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Planarity criterion by minors

A graph is planar if and only if it is K_5-minor-free and $K_{3,3}$-minor-free.

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Easy consequence of Euler's formula

Planar graphs can have at most $3n - 6$ edges.

n is the number of vertices, m the number of edges in G

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corollary

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Let G be a K_5-minor-free graph that has girth g. What is the maximum number of edges in G? What are the extremal graphs?

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proposition

For $g = 4$ the answer is $3n - 9$. The extremal graphs are $K_{3,n-3}$.

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Mader's idea

Let $d \geq 3$ and $k \geq 1$. If G is a graph with minimum degree d and girth at least $8k+3$, then G has a minor with minimum degree $d(d-1)^k$.

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If G is K_5-mf and has girth at least 11, then it must have a vertex of degree 2.

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Every K_5-minor free graph of girth 6 has a vertex of degree 2.
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better guess

Every K_5-minor free graph with girth at least 5 has a vertex of degree at most 2 or is planar.

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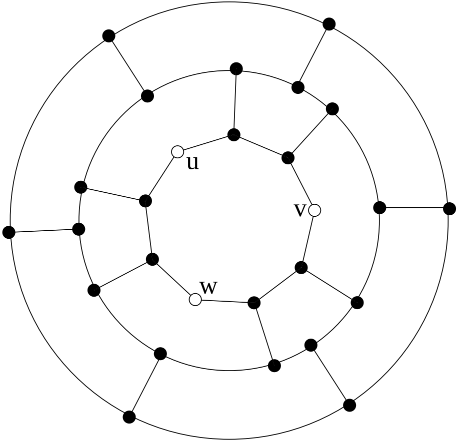

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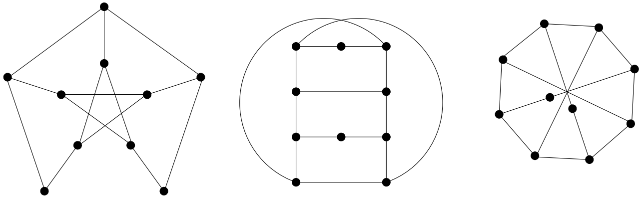
n=	4	5	6	7	8	9	10	11	12	13	14	15
planar	3	5	6	8	10	11	13	15	16	18	20	21
w/conj	3	5	7	9	10	12	14	16	17	19	21	23
K_5-mf	3	5	6	8	10	12	14	15	17	19	21	22

Non-planar constructions



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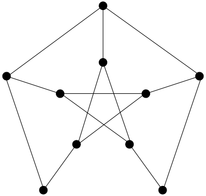

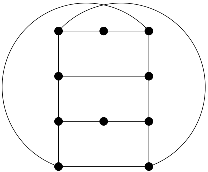

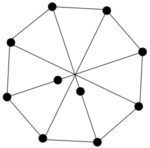

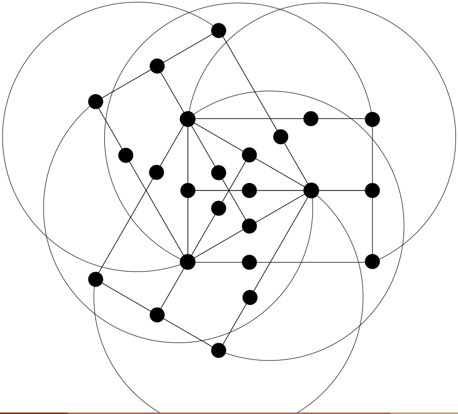

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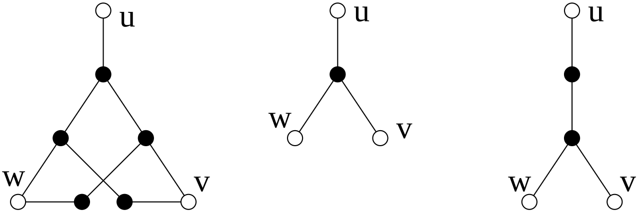

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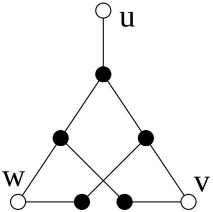

Theorem (JB, David Wood '13+)

If G is a K_5-minor-free graph of girth 5 with n vertices and m edges and $n \geq 4$, then $5m \leq 9n - 21$ except that $5m(G) = 9n(G) - 20$ when G is C_5 or the Petersen graph with one edge deleted.