Midsummer Combinatorial Workshop 2012

Pavel Rytíř (ed.)

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Introduction

The 18th Prague Midsummer Combinatorial Workshop was held from July 30 to August 3, 2012 in our beautiful building Malostranské náměstí 25. This of course contributed to the comfort of the participants as all the activities (including the lunches) could be taken on the same site. Besides, as it was expressed by several participants, the renovated faculty building surely belongs to the most beautiful math and computer science departments in the world! The workshop was organized by the Department of Applied Mathematics (KAM) of Charles University jointly with DIMATIA and CE ITI. Moreover, the 2013 was the first year of existence of our newly founded Computer Science Institute of Charles University (in Czech Informatický ústav University Karlovy; `http://iuuk.mff.cuni.cz`). Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Maria Axenovich, John Gimbel and András Gyárfás among the participants. The list of speakers is included in this booklet. As it already became a tradition, the workshop benefited from participation of young researchers and PhD students. For example six undergraduate students from the USA and six undergraduate students from Charles University, together with their mentor Kellen Myers from US side and Josef Cibulka from Prague side took part in the workshop, within the DIMATIA-DIMACS program International REU (supported jointly by NSF and Czech Ministry of Education ME 09074). The workshop followed an informal daily routine with morning and early afternoon discussions and presentations. This report reflects some of the presentations during the workshop. On Wednesday August 1 we had an excursion to the world famous Strahov Library where we were give a special tour by Dr. Pařez, especially directed to early mathematical and astronomy manuscripts in the library funds (visit `http://www.360cities.net/gigapixel/strahov-library.html`). Perhaps you can digest some of the atmosphere at the workshop from these proceedings, and you can also see that the fruitful exchange of ideas led directly to some new results and papers. The organization of the workshop from the help of several people. But mainly we benefited from the work of our secretary Mrs. Hana Polišenská and Pavel Rytíř who also edited this volume. Thank to both of them for excellent work. Most of the contributions were supplied by the authors in an electronic form. In a few cases, slight typographical changes were necessary. We apologize for any possible inaccuracies which

might have occurred in the editing process. We gratefully acknowledge financial support of Czech research projects GACR P202/12/G061 and ERC CZ LL1201 CORES.

We hope to meet again in 2013 the same midsummer week!

Jaroslav Nešetřil

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MCW excursion to the baroque library of the Strahov monastery.

The binary paint shop problem

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Joint work with Winfried Hochstättler [1].

1 Introduction

The *binary paint shop problem* introduced by Epping et al. [5] is the following task of combinatorial optimization. We are given an alphabet Σ of n letters and a word $w \in \Sigma^{2n}$ of $2n$ characters in which every letter from Σ occurs exactly twice; such a word is called *double occurrence word.* A *feasible colouring* of w is an assignment $(f_1, \dots, f_{2n}) \in \{\text{red}, \text{blue}\}^{2n}$, such that for any $i \neq j$, if $w_i = w_j$, then $f_i \neq f_j$. For $i = 1, \dots, 2n-1$, there is a *colour change* at position $i + \frac{1}{2}$ if and only if $f_i \neq f_{i+1}$. The goal is to find a feasible colouring with the minimal number of colour changes.

This problem is difficult, even hard to approximate: there is no PTAS unless $\mathcal{P} = \mathcal{NP}$. Bonsma et al. [4] proved this by reducing VERTEX COVER FOR CUBIC GRAPHS to the binary paint shop problem. Meunier and Sebő [6] used a reduction from MAX CUT FOR 3-REGULAR GRAPHS to obtain the same result. Both reductions imply that the binary paint shop decision problem is $\mathcal{NP}$-complete.

Problem 1.1. *Is there a constant factor approximation for the binary paint shop problem?*

2 The greedy heuristic

The *greedy heuristic* colours a given double occurrence word feasibly from left to right, the first character is coloured red, after that a colour is used as long as possible in a feasible colouring. The following is a trivial observation.

Remark 2.1. The greedy heuristic obtains an optimal solution on double occurrence words that do not contain subwords of the form $ABBA$.

Amini et al. [2] resp. Rautenbach and Szigeti [7] improved Remark 2.1.

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Theorem 2.2 (Amini et al. (2010)). *The greedy heuristic is optimal on instances that do not contain subwords of the form ABACCB or ABBCAC.*

Theorem 2.3 (Rautenbach, Szigeti (2012)). *The greedy heuristic is optimal on every subword of a word w if and only if w does not contain subwords of the form ABACCB or ADDBCCAB or ADDCBCAB.*

Amini et al. [2] also consider the expected performance of the greedy heuristic (assuming uniform distribution).

Theorem 2.4 (Amini et al. (2010)). *The expected number of colour changes for the greedy heuristic on double occurrence words of the length $2n$ is*

$$\mathbb{E}_n(g) \leq \frac{2}{3}n.$$

In [3] we improve this, proving a conjecture of Amini et al. [2].

Theorem 2.5. *The expected number of colour changes for the greedy heuristic on double occurrence words of length $2n$ is*

$$\mathbb{E}_n(g) = \sum_{k=0}^{n-1} \frac{2k^2-1}{4k^2-1}.$$

In [3] two other heuristics were considered, too. The *red-first heuristic* simply colours the first occurrence of every letter red, the second blue. The *recursive greedy heuristic* deletes both occurrences of the last letter, say Z, colours the rest of the word recursively by recursive greedy heuristic, and then proceeds with the two Zs in the following way. If the first occurrence of Z is placed between two differently coloured characters, then it is coloured in such a way that there is no colour change before the last Z, otherwise it is coloured in the colour of its neighbours.

None of the three heuristics mentioned is a constant factor approximation [3]. In [3] analogs of Theorem 2.5 are proved.

Theorem 2.6. *The expected number of colour changes for the red-first heuristic on an instance of length $2n$ is*

$$\mathbb{E}_n(rf) = \frac{2n+1}{3}.$$

Theorem 2.7. *For all $n \geq 1$, the expected number $\mathbb{E}_n(rg)$ of colour changes for the recursive greedy heuristic is bounded by*

$$\frac{2}{5}n + \frac{8}{15} \leq \mathbb{E}_n(rg) \leq \frac{2}{5}n + \frac{7}{10}.$$

3 Problems and a generalization

Problem 3.1. *Find better heuristics (with better expected performance).*

Problem 3.2. *Characterize the instances where the recursive greedy is optimal.*

Problem 3.3. *Determine the expected number of colour changes for optimal colouring.*

The binary paint shop problem is a special case of the optimization version of the so-called necklace splitting problem. In the *necklace splitting problem* q thieves have to divide an open necklace with t types of beads, where each type i occurs exactly $a_i q$ times, in a fair way, i.e. to cut it into pieces, so that every thief receives exactly a_i beads of type i.

Theorem 3.4 (Alon (1987))**.** *There is a solution with at most* $(q-1)t$ *cuts.*

Problem 3.5 (Meunier, Sebő (2009))**.** *Is there a polynomial algorithm to determine the cuts mentioned in Theorem 3.4?*

References

[1] N. Alon, Splitting necklaces, Advances in Math. 63 (1987), 247–253

[2] H. Amini, F. Meunier, H. Michel, A. Mohajeri, Greedy colorings for the binary paintshop problem, J. Discrete Algorithms 8 (2010) 8–14

[3] S.D. Andres, W. Hochstättler, Some heuristics for the binary paint shop problem and their expected number of colour changes, J. Discrete Algorithms 9 (2011), 203–211

[4] P. Bonsma, T. Epping, W. Hochstättler, Complexity results on restricted instances of a paint shop problem for words, Discrete Appl. Math. 154 (2006) 1335–1343

[5] T. Epping, W. Hochstättler, P. Oertel, Complexity result on a paint shop problem, Discrete Appl. Math. 136 (2004) 217–226

[6] F. Meunier, A. Sebő, Paintshop, odd cycles and necklace splitting, Discrete Appl. Math. 157 (2009) 780–793

[7] D. Rautenbach, Z. Szigeti, Greedy colorings of words, Discrete Applied Math. 160 (2012), 1872–1874

Twins in sequences

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Joint work with Yury Person [1] and Svetlana Puzynina [2].

For a word S, let $f(S)$ be the largest integer m such that there are two disjoint identical (scattered) subwords of length m. Let $f(n, \Sigma) = \min\{f(S) : S \text{ is of length } n, \text{ over alphabet } \Sigma\}$. Here, it is shown that

$$2f(n, \{0, 1\}) = n - o(n)$$

using the regularity lemma for words. In other words, any binary word of length n can be split into two identical subwords (referred to as twins) and, perhaps, a remaining subword of length $o(n)$. A similar result is proven for k identical subwords of a word over an alphabet with at most k letters.

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Upper density of quasi-random hypergraphs

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In 1964, Erdős proved that for any $\alpha > 0$, an l-uniform hypergraph G with $n \geq n_0(\alpha, l)$ vertices and $\alpha\binom{n}{l}$ edges contains a large complete l-equipartite subgraph. This implies that any sufficiently large G with density $\alpha > 0$ contains a large subgraph with density at least $l!/l^l$.

In this talk we discuss a similar problem for l-uniform hypergraphs Q with a (weak) quasi-random property. We prove any sufficiently large quasi-random l-uniform hypergraph Q with density $\alpha > 0$ contains a large subgraph with density at least $\frac{(l-1)!}{l^{l-1}-1}$. In particular, for $l = 3$, any sufficiently large such Q contains a large subgraph with density at least $\frac{1}{4}$ which is the best possible lower bound.

We define jumps for quasi-random sequences of l-graphs and our result implies that every number between 0 and $\frac{(l-1)!}{l^{l-1}-1}$ is a jump for quasi-random l-graphs. For $l = 3$ this interval can be improved based on a recent result of Glebov, Král' and Volec. We prove that every number between [0, 0.3192) is a jump for quasi-random 3-graphs.

References

[1] N. Alon, W. de la Vega, R. Kannan, and M. Karpinski. Random sampling and approximation of MAX-CSP problems. In Proceedings of the 34th annual ACM symposium on Theory of computing, pages 232–239. ACM Press, 2002.

[2] P. Erdős. On extremal problems of graphs and generalized graphs. Israel Journal of Mathematics, 2:183–190, 1964.

[3] P. Erdős. Problems and results on graphs and hypergraphs: Similarities and differences. In J. Nešetřil and V. Rödl, editors, Mathematics of Ramsey Theory, pages 223–233. Springer-Verlag, 1990.

[1] Emory University. Research partially supported by NSF grant DMS 0800070 and Emory University Research Committee Grant. `rodl@emory.edu`.

[4] P. Erdős and A. Hajnal. On Ramsey like theorems. Proceedings: Conference in Combinatorial Mathematics, Mathematics Institute, Oxford University, pages 123–140, 1972.

[5] P. Frankl and V. Rödl. Hypergraphs do not jump. Combinatorica, 4(2):149–159, 1984.

[6] D. Gale. On the number of faces of a convex polygon. Canadian Journal of Mathematics, 16:12–17, 1964.

[7] R. Glebov, D. Král', and J. Volec. On a conjecture of Erdős and Sós. In preparation.

[8] J. W. Moon. Topics on Tournaments. Holt, Rhinehart and Winston, 1968.

[9] D. Mubayi and V. Rödl. Uniform edge distribution in hypergraphs is hereditary. Electronic Journal of Combinatorics, 11:1–32, 2004.

[10] A. Razborov. Flag algebras. J. Symbolic Logic, 72(4):1239–1282, 2007.

[11] V. Rödl. On universality of graphs with uniformly distributed edges. Discrete Mathematics, 59(1-2):125–134, 1986.

Tight Hamilton cycles in random hypergraphs

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Joint work with Peter Allen, Yoshiharu Kohayakawa, Yury Person.

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The question of when the random graph $G(n,p)$ becomes hamiltonian is well understood. For random hypergraphs until recently not much was known. The random r-uniform hypergraph $\mathcal{G}^{(r)}(n,p)$ on vertex set $[n]$ is generated by including each hyperedge from $\binom{[n]}{r}$ independently with probability $p = p(n)$. First, Frieze [3] considered loose Hamilton cycles in random 3-uniform hypergraphs. The *loose r-uniform cycle* on vertex set $[n]$ has edges $\{i+1,\dots,i+r\}$ for exactly all $i = k(r-1)$ with $k \in \mathbb{N}$ and $(r-1) \mid n$, where we calculate modulo n. Frieze showed that the threshold for a loose Hamilton cycle in $\mathcal{G}^{(3)}(n,p)$ is $\Theta(\log n/n^2)$. Dudek and Frieze [2] extended this to r-uniform hypergraphs with $r \geq 4$, where the threshold is $\tilde{\Theta}(\log n/n^{r-1})$.

Tight Hamilton cycles, on the other hand, were considered only later. The *tight r-uniform cycle* on vertex set $[n]$ has edges $\{i+1,\dots,i+r\}$ for all i calculated modulo n. Dudek and Frieze [1] used a second moment argument to show that the threshold for a tight Hamilton cycle in $\mathcal{G}^{(r)}(n,p)$ is sharp and equals e/n for each $r \geq 4$ and for $r = 3$ they showed that $\mathcal{G}^{(3)}(n,p)$ contains a tight Hamilton cycle when $p = \omega(n)/n$ for any $\omega(n)$ that goes to infinity. Since their method is non-constructive they asked for an algorithm to find a tight Hamilton cycle in a random hypergraph. We obtain a randomised algorithm for this problem if p is slightly bigger than in their result.

Theorem 1.1. *For each integer $r \geq 3$ and $0 < \varepsilon < 1/(4r)$ there is a randomised polynomial time algorithm which for any $n^{-1+\varepsilon} < p \leq 1$ a.a.s. finds a tight Hamilton cycle in the random r-uniform hypergraph $\mathcal{G}^{(r)}(n,p)$.*

The probability referred to in Theorem 1.1 is with respect to the random bits used by the algorithm as well as by $\mathcal{G}^{(r)}(n,p)$. The running time of the algorithm in the above theorem is polynomial in n, where the degree of the polynomial depends on ε.

Outline of the proof A simple greedy strategy shows that for $p = n^{\varepsilon-1}$ it is easy to find a tight path (and similarly a tight cycle) in $\mathcal{G}^{(r)}(n,p)$ which covers all but at most $n^{1-\frac{1}{2}\varepsilon}$ of its vertices. Incorporating these few remaining vertices is where the difficulty lies.

To overcome this difficulty we apply the following strategy, which we call the *reservoir method*. We first construct a tight path P of a linear length in n which contains a vertex set W^*, called the *reservoir*, such that for any $W \subseteq W^*$ there is a tight path on $V(P) \setminus W$ whose end $(r-1)$-tuples are the same as that of P. In a second step we use the mentioned greedy strategy to extend P to an almost spanning tight path P', with a leftover set L. The advantage we have gained now is that we are permitted to reuse the vertices in W^*: we will show that, by using a subset W of vertices from W^* to incorporate the vertices from L, we can extend the almost spanning tight path to a spanning tight cycle C. More precisely, we shall delete W from P' (observe that, by construction of P, the hypergraph induced on $V(P) \setminus W$ contains a tight path with the same ends) and use precisely all vertices of W to connect the vertices of L to construct C.

References

[1] A. Dudek and A. Frieze, Tight Hamilton cycles in random uniform hypergraphs, Random Structures Algorithms, to appear.

[2] A. Dudek and A. Frieze, Loose Hamilton cycles in random uniform hypergraphs, Electron. J. Combin. 18 (2011), no. 1, Paper 48, 14.

[3] A. Frieze, Loose Hamilton cycles in random 3-uniform hypergraphs, Electron. J. Combin. 17 (2010), no. 1, Note 28, 4.

On restricted Ramsey numbers

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The general Ramsey-type problem in graph theory involves determining the existence, and consequently the least order, of a graph which guarantees that a certain property holds. One variation of this general problem is to impose certain restrictions on the size of the clique that the graph is allowed to contain. Folkman extended Ramsey theory in this direction. The following problem, originally raised by Erdős and Hajnal [5], asks to construct a graph H that does not contain a copy of K_{n+1} such that in every coloring of its edges with two colors, there is a K_n, all of whose edges have the same color. Folkman [6] proved the existence of such H. The general case, for an arbitrary number of colors, r, instead of just two, was settled affirmatively by Nešetřil and Rödl [10].

An alternate problem is one where we consider coloring the vertices instead of the edges of H. Folkman also proved that there is a K_{n+1}-free graph such that any r-coloring of its vertices yields a monochromatic copy of K_n. Together with Rödl [4] we considered the more general problem of determining $F(r, G)$, the least order of H such that $\omega(H) = \omega(G)$ and any r-coloring of the vertices of H yields a monochromatic and induced copy of G. Note that in addition to the condition that G and H have the same clique number, they also required that the monochromatic copy of G be induced. Some special cases of this function were considered previously by several researchers (see, *e.g.*, [7, 8, 9]). For example, in [4], with Rödl we proved that

$$F(r, K_n) \leq cn^2(\log n)^4$$

for some constant $c = c(r)$. Recently, Ramadurai, Rödl and I [3] obtained a more general result. Conditioning on the clique number of G of order n we showed that

$$F(r, G) \leq \frac{cn^3}{\omega(G)}(\log n)^5 \tag{1}$$

for some $c = c(r)$.

Very recently, together with Ramadurai [2] we extended those results to hypergraphs.

A *hypergraph* $\mathcal{G}$ is a pair $(V, \mathcal{E})$, where V is a set of vertices, and $\mathcal{E} \subseteq 2^V$ is a set of hyperedges. The *order* of a hypergraph is the size of its vertex set. A hypergraph $\mathcal{G} = (V, \mathcal{E})$ is k*-uniform* if every edge $e \in \mathcal{E}$ has cardinality exactly k. The clique number of $\mathcal{G}$, denoted by $\omega(\mathcal{G})$, is the order of the largest clique contained in $\mathcal{G}$.

Let r be a given number of colors and $\mathcal{G}$ be a k-uniform hypergraph of order n. We define the *(induced) Folkman number* $F(r, \mathcal{G})$ of $\mathcal{G}$ to be the minimum order of a k-uniform hypergraph $\mathcal{H}$ with $\omega(\mathcal{H}) = \omega(\mathcal{G})$ such that every r coloring of the vertices of $\mathcal{H}$ yields a monochromatic and induced copy of $\mathcal{G}$.

With Ramadurai [2] we showed that the Folkman numbers for hypergraphs are almost quadratic. For all natural numbers $r \geq 1$ and $k \geq 3$ there is a constant c such that

$$F(r, \mathcal{G}) \leq cn^2(\log n)^2,$$

for any k-uniform hypergraph $\mathcal{G}$ of order n. Note that this upper bound is always better than the one for graphs (*cf.* (1)). Moreover, there is a $\mathcal{G}_0$ such that the order of magnitude of $F(r, \mathcal{G}_0)$ is at least n^2.

One can also consider a slightly different problem and take the asymptotic in number r of colors. In [2] we complemented the previous results as follows. For every k and n there is a constant c such that for any k-uniform hypergraph $\mathcal{G}$ of order n and any number r of colors

$$F(r, \mathcal{G}) \leq cr^2.$$

Recently, with Mubayi [1] we showed that in the special case when $\mathcal{G}$ is the complete k-uniform hypergraph

$$F(r, \mathcal{K}_n^k) \leq cr(\log r)^{\frac{1}{k-2}},$$

where $c = c(k, n)$.

In view of the current bounds on $F(r, \mathcal{G})$, the following questions might be of some interest:

(P1) Is there a family of hypergraphs $\{\mathcal{G}_n\}$ for which $F(r, \mathcal{G}_n)$ is asymptotically larger than n^2?

(P2) Is there a hypergraph $\mathcal{G}$ of a fixed order n such that $F(r, \mathcal{G}) = \Omega(r^2)$?

References

[1] A. Dudek and D. Mubayi, On generalized Ramsey numbers for 3-uniform hypergraphs, submitted.

[2] A. Dudek and R. Ramadurai, Some remarks on vertex Folkman numbers for hypergraphs, Discrete Math. 312 (2012), 2952–2957.

[3] A. Dudek, R. Ramadurai and V. Rödl, On induced Folkman numbers, Random Structures and Algorithms 40 (2012), no. 4, 493–500.

[4] A. Dudek and V. Rödl, An almost quadratic bound on vertex Folkman numbers, J. Combin. Theory, Ser. B 100 (2010), no. 2, 132–140.

[5] P. Erdős, Problems and results in finite and infinite graphs, Recent advances in graph theory (Proceedings of the Symposium held in Prague), edited by M. Fiedler, Academia Praha (1975), 183–192.

[6] J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring, SIAM J. Appl. Math. 18 (1970), 19–24.

[7] N. Kolev and N. Nenov, New upper bound for a class of vertex Folkman numbers, Electron. J. Combin. 13 (2006), #R14.

[8] T. Łuczak, A. Ruciński, and S. Urbański, On minimal vertex Folkman graphs, Discrete Math. 236 (2001), 245–262.

[9] N. Nenov, Application of the corona-product of two graphs in Ramsey theory, Annuaire Univ. Sofia Fac. Math. Inform. 79 (1985), 349–355 (in Russian).

[10] J. Nešetřil and V. Rödl, The Ramsey property for graphs with forbidden complete subgraphs, J. Combin. Theory Ser. B 20 (1976), no. 3, 243–249.

Prague problems for complete triple systems

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1. Ramsey number of loose triangles. Let C_3^3 denote the loose 3-uniform triangle, three triples abc, cde, efa. The k-color Ramsey number $R_k(C_3^3)$ is the smallest n for which in every k-coloring of the triples of an n-element set V, there is a monochromatic C_3^3. It was proved in [1] that for $r \geq 3$, $k + 5 \leq R_k(C_3^3) \leq 3k$. Can one improve by 1 the lower or upper bound? ($R_2(C_3^3) = 7, R_3(C_3^3) = 8$) and we suspect that the lower bound is the truth for every k.)

2. Domination in rooted triple systems. A *rooted triple system* is the set of all triples of V such that each triple T has a root vertex $v \in T$. A set $X \subset V$ is a *dominating set* if for every $u \notin X$ there exists a $v \in X$ and a $w \in V$ such that v is the root of the triple uvw. I have two conjectures, **1:** for any k there are k-paradoxical rooted triple systems, i.e. which cannot be dominated by k vertices. **2:** if a rooted triple system has property H then it can be dominated by at most 2012 vertices, H: in any four vertices of V at least two triples have the same root.

References

[1] A. Gyárfás, G. Raeisi, The Ramsey number of loose triangles and quadrangles in hypergraphs, Electronic J. of Combinatorics, 19(2) (2012) N30.

Simple Treewidth

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Joint work with Torsten Ueckerdt [2].

1 Introduction

A *k-tree* is a graph that can be constructed starting with a $(k+1)$-clique and in every step attaching a new vertex to a k-clique of the already constructed graph. The *treewidth* $\mathrm{tw}(G)$ of a graph G is the minimum k such that G is a *partial k-tree*, i.e., G is a subgraph of some k-tree [7].

We consider a variation of treewidth, called *simple treewidth*. A simple k-tree is a k-tree with the extra requirement that there is a construction sequence in which no two vertices are attached to the same k-clique. (Equivalently, a k-tree is simple if it has a tree representation of width k in which every $(k-1)$-set of subtrees intersects at most 2 tree-vertices.) Now, the *simple treewidth* $\mathrm{stw}(G)$ of G is the minimum k such that G is a partial simple k-tree, i.e., G is a subgraph of some simple k-tree.

We have encountered simple treewidth as a natural parameter in questions concerning geometric representations of graphs, i.e., representing graphs as intersection graphs of geometrical objects where the quality of the representation is measured by the complexity of the objects. E.g., we have shown that the maximal *interval-number*([3]) of the class of treewidth k graphs is $k+1$, whereas for the class of simple treewidth k graphs it is k, see [6]. Another example is the *bend-number*([4]), which for treewidth 3 graphs is 4 and for simple treewidth 3 graphs is 3, see [5] and a corresponding statement for higher values is conjectured in [4].

The aim of this note is to compare these two parameters and to motivate simple treewidth by indicating that it endows treewidth with a topological flavor. We pose several interesting open problems.

[1] The research was supported by DFG grant FE-340/8-1 as part of ESF project GraDR EUROGIGA

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2 Properties of simple treewidth

Let us first observe, that both parameters cannot differ too much.

Observation 2.1. *For every G we have* $\mathrm{tw}(G) \leq \mathrm{stw}(G) \leq \mathrm{tw}(G) + 1$.

Proof. The first inequality is clear. For the second inequality we show that every k-tree G is a subgraph of a simple $(k+1)$-tree H. Whenever in the construction sequence of G several vertices $\{v_1, \dots, v_n\}$ are attached to the same k-clique C we consider $C \cup \{v_1\}$ as a $(k+1)$-clique in the construction sequence for H. Attaching v_i to C can be interpreted as attaching v_i to $C \cup \{v_{i-1}\}$ and omitting the edge $\{v_{i-1}, v_i\}$. This way we avoid that several vertices are attached to the same k-clique by considering $(k+1)$-cliques. □

Simple treewidth endows the notion of treewidth with a more topological flavor, as indicated for small k in the table below:

	≤ 1	≤ 2	≤ 3
stw	paths	outerplanar	planar & tw ≤ 3, [2]
tw	forest	series-parallel	tw ≤ 3

A *linkless embedding* of G is an embedding into $\mathbb{R}^3$ with the property that no two cycles of G form a *link*, see [9].

Observation 2.2. *If* $\mathrm{stw}(G) \leq 4$ *then G has a linkless embedding.*

Proof. It suffices to show that simple 4-trees have linkless embeddings, since edge-deletion does not destroy the linkless embedding. Therefore consider K_5 embedded in $\mathbb{R}^3$ as a tetrahedron plus a vertex in its interior. In each step of the construction sequence every available 4-clique is represented by a tetrahedron with empty interior, where we insert the new vertex. It is easy to see that the resulting embedding is linkless. □

Non-simple 4-trees do not have linkless embeddings, which is easy to see using the forbidden-minor chracterization of linkless embeddable graphs [8].

Problem 2.3. $\mathrm{stw}(G) \leq 4 \Leftrightarrow G$ *is linkless embeddable and* $\mathrm{tw}(G) \leq 4$.

Simple treewidth also has connections to discrete geometry. In [1] a *stacked polytope* is defined to be a polytope admitting a triangulation whose dual graph is a tree. In that paper it is proved that a full-dimensional polytope $P \subset \mathbb{R}^d$ is stacked if and only if $\mathrm{tw}(G(P)) \leq d$, were $G(P)$ denotes the 1-skeleton of P. Indeed, we strongly suspect:

Problem 2.4. *A graph G is the 1-skeleton of a stacked d-polytope if and only if* $\mathrm{stw}(G) = d$ *and G is d-connected.*

One can show that the class of simple treewidth at most k graphs is minor-closed. A proof of the following statement would imply that for planar graphs with treewidth at least 3 treewidth and simple treewidth coincide.

Problem 2.5. *If G has no $K_{3,k}$-minor and* $\mathrm{tw}(G) = k$ *then* $\mathrm{stw}(G) = k$.

References

[1] A. Below, J. A. De Loera, and J. Richter-Gebert. The complexity of finding small triangulations of convex 3-polytopes. J. Algorithms, 50(2):134–167, 2004. SODA 2000 special issue.

[2] E. S. El-Mallah and Ch. J. Colbourn. On two dual classes of planar graphs. Discrete Math., 80(1):21–40, 1990.

[3] F. Harary and W. T. Trotter, Jr. On double and multiple interval graphs. J. Graph Theory, 3(3):205–211, 1979.

[4] D. Heldt, K. Knauer, and T. Ueckerdt. Edge-intersection graphs of grid paths: the bend-number. ArXiv e-prints, September 2010.

[5] D. Heldt, K. Knauer, and T. Ueckerdt. On the bend-number of planar and outerplanar graphs. In LATIN, pages 458–469, 2012.

[6] K. Knauer and T. Ueckerdt. Three ways to cover a graph. ArXiv e-prints, May 2012.

[7] N. Robertson and P. D. Seymour. Graph minors—a survey. In Surveys in combinatorics 1985 (Glasgow, 1985), volume 103 of *London Math. Soc. Lecture Note Ser.*, pages 153–171. Cambridge Univ. Press, Cambridge, 1985.

[8] N. Robertson, P. D. Seymour, and R. Thomas. Sachs' linkless embedding conjecture. J. Combin. Theory Ser. B, 64(2):185–227, 1995.

[9] H. Sachs. On a spatial analogue of Kuratowski's theorem on planar graphs—an open problem. In Graph theory (Łagów, 1981), volume 1018 of Lecture Notes in Math., pages 230–241. Springer, Berlin, 1983.

Casinos with multiple types of coins.

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For a countable relational structure $\mathcal{A}$, the class of all finite structures that embed into $\mathcal{A}$ is called the *age* of $\mathcal{A}$ and we denote it by $\text{age}(\mathcal{A})$. A class $\mathbf{K}$ of finite structures is an *age* if there is countable structure $\mathcal{A}$ such that $\mathbf{K} = \text{age}(\mathcal{A})$. It is easy to see that a class $\mathbf{K}$ of finite structures is an age if and only if $\mathbf{K}$ is an abstract class (that is, closed for isomorphisms), there are at most countably many pairwise nonisomorphic structures in $\mathbf{K}$, $\mathbf{K}$ has the hereditary property (HP), and $\mathbf{K}$ has the joint embedding property (JEP). An age $\mathbf{K}$ is a *Fraïssé age* (= Fraïssé class = amalgamation class) if $\mathbf{K}$ satisfies the amalgamation property (AP).

For every Fraïssé age $\mathbf{K}$ there is a unique (up to isomorphism) countable homogeneous structure $\mathcal{A}$ such that $\mathbf{K} = \text{age}(\mathcal{A})$. We say that $\mathcal{A}$ is the *Fraïssé limit* of $\mathbf{K}$.

Let L be a relational language and let $\propto \notin L$ be a new binary relational symbol. Let $L_\propto = L \cup \{\propto\}$. We say that an $L_\propto$-relational structure $\mathcal{A}$ is *ordered* if the interpretation $\propto^\mathcal{A}$ of $\propto$ in $\mathcal{A}$ is a linear order on A, the base set of $\mathcal{A}$. Given an L-structure $\mathcal{A}$, in this paper we consider various interpretations of $\propto$ in $\mathcal{A}$. In order to make it easier to follow which particular interpretation of $\propto$ we have in mind, we write $\mathcal{A}_<$ to indicate that $<$ is an interpretation of $\propto$ we are currently interested in. Therefore, it makes sense to write a statement like $\mathcal{A}_< \cong \mathcal{A}_\sqsubset$ because both $\mathcal{A}_<$ and $\mathcal{A}_\sqsubset$ are $L_\propto$-structures (with possibly different interpretations of $\propto$). Moreover, we shall always write $\mathcal{A}$ to denote the L-reduct of $\mathcal{A}_<$. Conversely, if $\mathcal{A}$ is an L-structure and $<$ is a liner order on A, then $\mathcal{A}_<$ denotes the corresponding $L_\propto$-structure, while $\mathcal{A}_>$ denotes the same structure endowed with the dual of $<$.

If $\mathbf{K}$ is a class of L-structures, by $\mathbf{K}_*$ we denote the class of ordered $L_\propto$-structures obtained by expanding structures from $\mathbf{K}$ by linear orders in some unspecified way, but so that

- for every $\mathcal{A}_< \in \mathbf{K}_*$ we have $\mathcal{A} \in \mathbf{K}$, and
- for every $\mathcal{A} \in \mathbf{K}$ there is at least one linear order $<$ on A such that $\mathcal{A}_< \in \mathbf{K}_*$.

A class $\mathbf{K}_*$ of ordered $L_\propto$ structures is a *Ramsey class* if for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}_*$ and every $k \geqslant 1$ there is a $\mathcal{C} \in \mathbf{K}_*$ such that $\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$. If $\mathbf{K}_*$ is a Ramsey class of finite ordered $L_\propto$-structures which is closed under isomorphisms and taking substructures, and has the joint embedding property, then $\mathbf{K}_*$ is a Fraïssé age [2]. In that case we say that $\mathbf{K}_*$ is a *Ramsey age*. So, every Ramsey age is a Fraïssé age.

We say that $\mathbf{K}_*$ has the *ordering property over* $\mathbf{K}$ if the following holds: for every $\mathcal{A} \in \mathbf{K}$ there is a $\mathcal{B} \in \mathbf{K}$ such that $\mathcal{A}_< \hookrightarrow \mathcal{B}_\sqsubset$ for every linear order $<$ on A such that $\mathcal{A}_< \in \mathbf{K}_*$, and every linear order $\sqsubset$ on B such that $\mathcal{B}_\sqsubset \in \mathbf{K}_*$. We say that $\mathcal{B}$ is a *witness for the ordering property for* $\mathcal{A}$.

Definition 0.1. An L-structure $\mathcal{A}$ has the *expansion-by-linear-orders property (ELOP)* if $\mathrm{age}(\mathcal{A}_<) = \mathrm{age}(\mathcal{A}_\sqsubset)$ for arbitrary linear orders $<, \sqsubset$ on A.

In this talk we exibit examples of structures which have ELOP.

To start with, note that $(\mathbb{Q}, <)$ does not have ELOP. Moreover, we by endowing $(\mathbb{Q}, <)$ with an additional linear order one can get $\aleph_0$ distinct ages (J. Hubička).

By $\mathbf{K}_\forall$ we denote the class of all ordered $L_\propto$-structures $\mathcal{A}_<$ where $\mathcal{A} \in \mathbf{K}$ and $<$ is a linear order on A (that is, we take L-structures from $\mathbf{K}$ and expand them with all possible linear orders). It is easy to show the following: if $\mathcal{C}$ is a countable structure, $\mathbf{K} = \mathrm{age}(\mathcal{C})$ and if $\mathbf{K}_\forall$ has the ordering property over $\mathbf{K}$, then $\mathcal{C}$ has ELOP.

Particular examples of structures with ELOP include the following: the random graph, the random digraph (the Fraïssé limit of the class of all finite digraphs), the random poset, the rational Urysohn space (the Fraïssé limit of the class of all finite metric spaces with rational distances), the rational Urysohn sphere (the Fraïssé limit of the class of all finite metric spaces with rational distances $\leqslant 1$), and the random rational ultrametric space.

References

[1] M. Bodirsky, M. Pinsker, T. Tsankov. Decidability of Definability. Proc. LICS 2011, 321–328

[2] J. Nešetřil. Ramsey classes and homogeneous structures. Combinatorics, probability and computing, 14 (2005), 171–189

[3] J. Nešetřil. Metric spaces are Ramsey. European Journal of Combinatorics 28 (2007), 457–468

[4] J. Nešetřil, V. Rödl. Partitions of finite relational and set systems. J. Comb. Th. A, 22 (1977), 289–312

[5] J. Nešetřil, V. Rödl. Combinatorial partitions of finite posets and lattices – Ramsey lattices. Alg. Univers. 19 (1984), 106–119

[6] M. Paoli, W. Trotter, J. Walker. Graphs and orders in Ramsey theory and in dimension theory. In: I. Rival, ed, Graphs and Order, Reidel Dordrecht 1984

[7] M. Sokić. Ramsey property of posets and related structures. PhD thesis, University of Toronto, 2010

Quasirandom hypergraphs

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Let $p(k)$ denote the partition function of k. For each $k \geq 2$, we describe a list of $p(k) - 1$ quasirandom properties that a k-uniform hypergraph can have. Our work connects previous notions on hypergraph quasirandomness, beginning with the early work of Chung and Graham and Frankl-Rödl related to strong hypergraph regularity, the spectral approach of Friedman-Wigderson, and more recent results of Kohayakawa-Rödl-Skokan and Conlon-Hàn-Person-Schacht on weak hypergraph regularity and its relation to counting linear hypergraphs.

For each of the quasirandom properties that are described, we define a hypergraph eigenvalue analogous to the graph case and a hypergraph extension of a graph cycle of even length whose count determines if a hypergraph satisfies the property. Our work can therefore be viewed as an extension to hypergraphs of the seminal results of Chung-Graham-Wilson for graphs.

Empty pentagons in point sets with collinearities

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Joint work with János Barát[2], Vida Dujmović[3], Gwenaël Joret[4], Ludmila Scharf[5], Daria Schymura[5], Pavel Valtr[6] and David R. Wood[7].

The Erdős-Szekeres Theorem [4], a classical result in discrete geometry, states that for every integer k there is a minimum integer $\mathrm{ES}(k)$ such that every set of at least $\mathrm{ES}(k)$ points in general position in the plane contains k points in convex position. Erdős [3] asked whether a similar result held for empty k-gons (k points in convex position with no other points inside their convex hull). Horton [8] answered this question in the negative by showing that there are arbitrarily large point sets in general position that contain no empty heptagon. On the other hand, Harborth [7] showed that every set of at least 10 points in general position contains an empty pentagon. More recently, Nicolás [11] and Gerken [6] independently settled the question for $k = 6$ by showing that sufficiently large point sets in general position always contain empty hexagons; see also [10, 14].

These questions are not interesting if the general position condition is abandoned completely, since a collinear point set contains no three points in convex position. However, considering point sets with a bounded number of collinear points does lead to interesting generalisations of these problems. First some definitions are needed. A point set X in the plane is in *weakly convex position* if every point in X lies on the boundary of $\mathrm{conv}(X)$, the convex hull of X. A point $x \in X$ is a *corner* of X if $\mathrm{conv}(X \setminus \{x\}) \neq \mathrm{conv}(X)$. The set X is in *strictly convex position* if every point in X is

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a corner of X. A *weakly* (respectively *strictly*) *convex k-gon* is a set of k points in weakly (respectively strictly) convex position. It is well known that the Erdős-Szekeres theorem generalises for point sets with bounded collinearities; see [1] for proofs. One generalisation states that every set of at least $\mathrm{ES}(k)$ points contains a weakly convex k-gon. For strictly convex position, the generalisation states that for all integers k and ℓ there exists a minimum integer $\mathrm{ES}(k, \ell)$ such that every set of at least $\mathrm{ES}(k, \ell)$ points in the plane contains ℓ collinear points or a strictly convex k-gon.

Our work addresses the case of empty pentagons in point sets with collinearities. A subset X of a point set P is an *empty k-gon* if X is a strictly convex k-gon and $P \cap \mathrm{conv}(X) = X$. Abel et al. [1] showed that every finite set of at least $\mathrm{ES}(\frac{(2\ell-1)^{\ell}-1}{2\ell-2})$ points in the plane contains an empty pentagon or ℓ collinear points. The function $\mathrm{ES}(k)$ is known to grow exponentially [4, 5], so this bound is doubly exponential in ℓ. See [13, 2] for more on point sets with no empty pentagon. We prove the following theorem without applying the Erdős-Szekeres Theorem.

Theorem 0.1. *Let P be a finite set of points in the plane. If P contains at least $328\ell^2$ points, then P contains an empty pentagon or ℓ collinear points.*

This quadratic bound is optimal up to a constant factor since the $(\ell - 1) \times (\ell - 1)$ square grid has $(\ell - 1)^2$ points and contains neither an empty pentagon nor ℓ collinear points.

The proof of Theorem 0.1 is broken into two main steps. For contradiction, we assume that P has no empty pentagon and no ℓ collinear points. The first step is to show that P does not contain more than 8ℓ points in weakly convex position. The second step is to analyse the structure of the convex layers of P. By the first step, if P has many points there must be many layers, all with less than 8ℓ points. The main idea is to study the inner layers and show that if there are enough layers, the structure of the inner layers puts a limit on the number of points in the outer layers. This leads to an overall bound on the number of points.

One open problem is to characterise the finite point sets with no empty pentagon. The known examples of such sets are described in [2]. In particular, we may ask:

Problem 0.2. *Does every finite set of n points in the plane with no empty pentagon contain a subset of $\Omega(n)$ points that has the order type of a set that is the intersection of the integer lattice with a convex region?*

All the known point sets with no empty pentagon satisfy this property.

Abel et al. [1] asked if a result similar to Theorem 0.1 holds for empty hexagons in point sets with collinearities.

Problem 0.3. *Let P be a finite set of points in the plane. For what values of ℓ does there exist an integer $f(\ell)$ such that if P contains at least $f(\ell)$ points, then P contains an empty hexagon or ℓ collinear points?*

It is not clear how to adapt the proofs of Nicolás [11] and Gerken [6] to deal with collinearities, so this seems to be quite a difficult problem. Of course, the result of Horton [8] completely answers the analogous question for empty k-gons with $k \geq 7$.

We take this opportunity to mention an important related conjecture of Kára, Pór and Wood [9]. Two points in a point set P are said to be *visible* if there is no other point of P on the line segment between them.

Conjecture 0.4 (Big-Line-Big-Clique [9]). *There exists a function $f(k, \ell)$ such that every finite set of at least $f(k, \ell)$ points in the plane contains k pairwise visibile points or ℓ collinear points.*

Theorem 0.1 and its predecessor due to Abel et al. [1] show that this conjecture is true for $k = 5$ and all ℓ. It is open for $k \geq 6$ and $\ell \geq 4$. For more on the Big-Line-Big-Clique conjecture see [12].

References

[1] Z. Abel, B. Ballinger, P. Bose, S. Collette, V. Dujmović, F. Hurtado, S. D. Kominers, S. Langerman, A. Pór, and D. R. Wood, Every large point set contains many collinear points or an empty pentagon, Graphs Combin., 27 (2011), 47–60.

[2] D. Eppstein, Happy endings for flip graphs, J. Comput. Geom., 1 (2010), 3–28.

[3] P. Erdős, On some problems of elementary and combinatorial geometry, Ann. Mat. Pura Appl. (4), 103 (1975), 99–108.

[4] P. Erdős and G. Szekeres, A combinatorial problem in geometry, Compositio Math., 2 (1935), 463–470.

[5] P. Erdős and G. Szekeres, On some extremum problems in elementary geometry, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., 3–4 (1960/1961), 53–62.

[6] T. Gerken, Empty convex hexagons in planar point sets, Discrete Comput. Geom., 39 (2008), 239–272.

[7] H. Harborth, Konvexe Fünfecke in ebenen Punktmengen, Elem. Math., 33 (1978), 116–118.

[8] J. D. Horton, Sets with no empty convex 7-gons, Canad. Math. Bull., 26 (1983), 482–484.

[9] J. Kára, A. Pór, and D. R. Wood, On the chromatic number of the visibility graph of a set of points in the plane, Discrete Comput. Geom., 34 (2005), 497–506.

[10] V. A. Koshelev, The Erdős-Szekeres problem, Dokl. Akad. Nauk, 415 (2007), 734–736.

[11] C. M. Nicolás, The empty hexagon theorem, Discrete Comput. Geom., 38 (2007), 389–397.

[12] A. Pór and D. R. Wood, On visibility and blockers, J. Computational Geometry, 1 (2010), 29–40.

[13] S. Rabinowitz, Consequences of the pentagon property, Geombinatorics, 14 (2005), 208–220.

[14] P. Valtr, On empty hexagons, in Surveys on discrete and computational geometry, vol. 453 of Contemp. Math., Amer. Math. Soc., 2008, 433–441.

Uniform Hypergraphs Containing no Grids

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1 Introduction

A hypergraph is called an $r \times r$ *grid*, $\mathbb{G}_{r\times r}$, if it is isomorphic to a pattern of r horizontal and r vertical lines, i.e., a family of sets $\{A_1, \dots, A_r, B_1, \dots, B_r\}$ such that $A_i \cap A_j = B_i \cap B_j = \emptyset$ for $1 \le i < j \le r$ and $|A_i \cap B_j| = 1$ for $1 \le i, j \le r$. Three sets C_1, C_2, C_3 form a *triangle*, $\mathbb{T}_3$ if they pairwise intersect in three distinct singletons, $|C_1 \cap C_2| = |C_2 \cap C_3| = |C_3 \cap C_1| = 1$, $C_1 \cap C_2 \neq C_1 \cap C_3$. A hypergraph is *linear*, if $|E \cap F| \le 1$ holds for every pair of edges.

In this paper we construct large linear r-hypergraphs which contain no grids. Moreover, a similar construction gives large linear r-hypergraphs which contain neither grids nor triangles. For $r \ge 4$ our constructions are almost optimal. For the triangle-free case we modify Behrend's construction to get the result. These investigations are motivated by Brown-Erdős-Sós conjecture and coding theory.

2 Avoiding grids in linear hypergraphs

Theorem 2.1. *For $r \ge 4$ there exists a real $c_r > 0$ such that there are linear r-uniform hypergraphs $\mathcal{F}$ on n vertices containing no grids and*

$$|\mathcal{F}| > \frac{n(n-1)}{r(r-1)} - c_r n^{8/5}.$$

The *Turán number* of the r-uniform hypergraph $\mathcal{H}$, denoted by $\mathrm{ex}(n, \mathcal{H})$, is the size of the largest $\mathcal{H}$-free r-graph on n vertices. If we want to emphasize r, then we write $\mathrm{ex}_r(n, \mathcal{H})$. Let $\mathbb{I}_{\ge 2}$ be (more precisely $\mathbb{I}^r_{\ge 2}$) the class

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of hypergraphs of two edges and intersection sizes at least two. This class consists of $r-2$ non-isomorphic hypergraphs, $\mathcal{I}_j$, $2 \le j < r$, $\mathcal{I}_j := \{A_j, B_j\}$ such that $|A_j| = |B_j| = r$, $|A_j \cap B_j| = j$. Using these notations the above Theorem can be restated as follows.

$$\frac{n(n-1)}{r(r-1)} - c_r n^{8/5} < \mathrm{ex}_r(n, \{\mathbb{I}_{\ge 2}, \mathbb{G}_{r\times r}\}) \le \frac{n(n-1)}{r(r-1)} \tag{2.1}$$

holds for every $n, r \ge 4$. In the case of $r = 3$ we only have

$$\Omega(n^{1.8}) = \mathrm{ex}_3(n, \{\mathbb{I}_{\ge 2}, \mathbb{G}_{3\times 3}\}) \le \frac{1}{6} n(n-1). \tag{2.2}$$

Conjecture 2.2. *The asymptotic* (2.1) *holds for $r = 3$, too.*
— Even more, for any given $r \ge 3$ there are infinitely many Steiner systems avoiding $\mathbb{G}_{r\times r}$.
— Probably there exists an $n(r)$ such that, for every admissible $n > n(r)$ (this means that $(n-1)/(r-1)$ and $\binom{n}{2}/\binom{r}{2}$ are both integers) there exists a grid-free $S(n, r, 2)$.

3 Neither grids nor triangles

A *perfect matching* is a subfamily $\mathcal{M}$ of the set system $\mathcal{F}$ such that the members of $\mathcal{M}$ cover every element of $V(\mathcal{F})$ exactly once. Our main result is a construction.

Theorem 3.1. *For $r \ge 4$ there exist an $n_0(r)$ and $\beta_r > 0$ such that*

$$\mathrm{ex}(n, \{\mathbb{I}_{\ge 2}, \mathbb{T}_3, \mathbb{G}_{r\times r}\}) > n^2 e^{-\beta_r \sqrt{\log n}} \tag{3.1}$$

holds for $n \ge n_0(r)$. In other words, there exists a linear r-uniform hypergraph $\mathcal{F}$ which contains neither grids nor triangles and $|\mathcal{F}| \ge n^2 \exp[-\beta_r \sqrt{\log n}]$. In addition, if r divides n, then $\mathcal{F}$ can be decomposed into perfect matchings, especially it is regular.

For the case $r = 3$ we have the same with a much weaker lower bound

$$\mathrm{ex}(n, \{\mathbb{I}_{\geq 2}, \mathbb{T}_3, \mathbb{G}_{3\times 3}\}) > n^{1.6} e^{-\beta_3 \sqrt{\log n}}. \tag{3.2}$$

Note that $|\mathcal{F}| = o(n^2)$ by [2] the lower bound (3.1) is almost optimal. This result slightly improves the Erdős-Frankl-Rödl [2] construction in two ways. We make the hypergraph regular, and avoid not only triangles but grids, too.

Problem 3.2. *Determine the order of magnitude of* $\mathrm{ex}(n, \mathbb{G}_{r\times r})$*.*

In the proofs we use tools from combinatorial number theory and discrete geometry. Let $r_k(q)$ be the maximum number of integers which can be selected from $\{1, \ldots, q\}$ containing no k-term arithmetic progression. Call a set $M \subset [q]$ *r-sum-free* if the equation

$$c_1 m_1 + c_2 m_2 = (c_1 + c_2) m_3$$

has no solutions with $m_1, m_2, m_3 \in M$ and c_1, c_2 are positive integers with $c_1 + c_2 \leq r$ except the one with $m_1 = m_2 = m_3$.

Lemma 3.3. (Generalized Behrend) *For arbitrary positive integer r there exists a $\gamma_r > 0$ such that for any integer q one can find an r-sum-free subset $M \subseteq \{0, 1, ..., q\}$ such that $|M| > q e^{-\gamma_r \sqrt{\log q}}$.*

The case $r = 2$ (and $c_1 = c_2 = 1$) is the original statement of Behrend [1]. Ruzsa also notes that an upper bound $O(q/(\log q)^{\alpha_r})$ for the general case can be proved.

Call a set of numbers *A_6-free* if it does not contain a subset of the form

$$\{x - a - b, x - b, x - a, x + a, x + b, x + a + b\}$$

for some $a, b > 0$, $a \neq b$. Call it *A_4-free* if it does not contain a fourtuple of the form $\{x - 2a, x - a, x + a, x + 2a\}$ for some $a > 0$, and call it *AP_k-free* if it contains no k-term arithmetic progression. Let $r(n, P_1, P_2, \ldots)$ denote the maximum number of integers which can be selected from $\{1, \ldots, n\}$ avoiding the patterns $P_1, P_2, \ldots$. With this notation $r_3(n) := r(n, AP_3)$.

Since an A_4-free set has no 5-term arithmetic progression we get $r(n, A_4) \leq r_5(n) = o(n)$ by Szemerédi's theorem. A 4-sum-free sequence is A_4-free as well (one has, e.g., $1 \times (x-2a) + 3 \times (x+2a) = 4 \times (x+a)$), thus Lemma 3.3 gives a lower bound showing

$$r(n, A_4) = n^{1-o(1)}.$$

Lemma 3.4.

$$\frac{2}{5} r_3(n)^{3/5} < r(n, A_6, A_4, AP_3).$$

Conjecture 3.5. *There is an* $\varepsilon > 0$ *such that*

$$n^{3/5+\varepsilon} < r(n, A_6)$$

holds for large enough n. *Possibly the order of magnitude of this function is* $n^{1-o(1)}$.

Unfortunately, the recent general construction of Shapira [3] does not seem to be applicable here. (We are thanksfull for I. Ruzsa (Budapest) and J. Wolf (Paris) for this observation.)

References

[1] F. A. Behrend, On sets of integers which contain no three terms in ariht-metical progression, Proc. Nat. Acad. Sci. USA 32 (1946), 331–332.

[2] P. Erdős, P. Frankl, and V. Rödl, The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent, Graphs and Combin. 2 (1986), 113–121.

[3] A. Shapira, Behrend-type constructions for sets of linear equations, Acta Arith. 122 (2006), 17–33.

Hunting for rainbow bulls and diamonds

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1 Introduction

We use [1] for terminology and notation not defined here and consider finite and simple graphs only. If K_n is edge-coloured in a given way and a subgraph H contains no two edges of the same colour, then H will be called a totally multicoloured (TMC) or rainbow subgraph of K_n and we shall say that K_n contains a TMC or rainbow H. For a graph H and an integer n, let $f(n,H)$ denote the maximum number of colours in an edge-colouring of K_n with no TMC H. The numbers $f(n,H)$ are called *anti-ramsey numbers* and have been introduced by Erdős, Simonovits and Sós [2].

We now define $rb(n,H)$ as the minimum number of colours such that any edge-colouring of K_n with at least $rb(n,H) = f(n,H)+1$ colours contains a TMC or rainbow subgraph isomorphic to H. The numbers $rb(n,H)$ will be called *rainbow numbers.*

For a given family $\mathcal{H}$ of finite graphs $ext(n,\mathcal{H}) =: max\{|E(G)| \mid H \not\subset G \text{ if } H \in \mathcal{H}\}$, that is, let $ext(n,\mathcal{H})$ be the maximum number of edges a graph G of order n can have if it has no subgraph from $\mathcal{H}$. The graphs attaining the maximum for a given n are called extremal graphs. The numbers $ext(n,\mathcal{H})$ are called *Turán numbers* [9].

For a given graph H, let $\mathcal{H}$ be the family of all graphs which are obtained by deleting one edge from H. If G is a graph of order n having no subgraph isomorphic to H, then a TMC copy of G and one extra colour for all remaining edges (of K_n) has no TMC subgraph H. Hence, $f(n,H) \geq ext(n,\mathcal{H})+1$. Moreover,

$$ext(n,\mathcal{H}) + 2 \leq f(n,H) + 1 = rb(n,H) \leq ext(n,H).$$

The lower bound is sharp for some graph classes. This has been shown if H is a complete graph on $k \geq 3$ vertices in [6, 8] and if H is a matching with k edges and $n \geq 2k+1$ in [3, 8].

Erdős, Simonovits and Sós [2] showed that $f(n,H)/\binom{n}{2} \to 1 - \frac{1}{d}$ as $n \to \infty$, where $d+1 = min\{\chi(H-e) \mid e \in E(H)\}$, and that $f(n,H) -$

$ext(n, \mathcal{H}) = o(n^2)$. Hence the rainbow numbers are asymptotically known if $min\{\chi(H - e) \mid e \in E(H)\} \geq 3$. If $min\{\chi(H - e) \mid e \in E(H)\} \leq 2$, then the situation is quite different.

2 Rainbow numbers for bulls and diamonds

For cycles the following result (which has been conjectured by Erdős, Simonovits and Sós [2]) has been shown by Montellano-Ballesteros and Neumann-Lara [7].

Theorem 2.1. *[7] Let $n \geq k \geq 3$. Then $rb(n, C_k) = \lfloor \frac{n}{k-1} \rfloor \binom{k-1}{2} + \binom{r}{2} + \lceil \frac{n}{k-1} \rceil$, where r is the residue of n modulo $k - 1$.*

Gorgol [4] has considered a cycle C_k with a pendant edge, denoted C_k^+, and computed all rainbow numbers.

Theorem 2.2. *[4]*
$rb(n, C_k^+) = rb(n, C_k)$, for $n \geq k + 1$.

However, if we add two (or more) edges to a cycle C_k, the situation becomes surprisingly interesting.

Theorem 2.3. *Let F be a graph of order $n \geq k \geq 3$ containing a cycle C_k. If F has cyclomatic number $v(F) \geq 2$, then $rb(n, F)$ has no upper bound which is linear in n.*

We first consider the graph $D = K_4 - e$, which is called the *diamond*. This graph contains a C_3 and has cyclomatic number $v(D) = 2$. Montellano-Ballesteros [5] has shown an upper bound for the rainbow number of the diamond.

Theorem 2.4. *[5] For every $n \geq 4$,*

$$ext(n, \{C_3, C_4\}) + 2 \leq rb(n, D) \leq ext(n, \{C_3, C_4\}) + (n + 1).$$

Using this we can show the following theorem.

Theorem 2.5. *$rb(n, D) = \Theta(n^{\frac{3}{2}})$.*

If $v(F) = 1$, then the situation is quite different. Let B be the unique graph with 5 vertices and degree sequence $(1, 1, 2, 3, 3)$, which is called the *bull*. Here we have been able to compute all rainbow numbers for the bull.

Theorem 2.6. *$rb(5, B) = 6$ and $rb(n, B) = n + 2$ for $n \geq 6$.*

References

[1] J. A. Bondy and U. S. R. Murty. Graph Theory with Applications. Macmillan, London and Elsevier, New York, 1976.

[2] P. Erdős, M. Simonovits and V. T. Sós. Anti-Ramsey theorems. Infinite and finite sets, Vol. II, edited by A. Hajnal, R. Rado, and V. T. Sós, Colloq. Math. Soc. János Bolyai 10 (1975) 633-643.

[3] S. Fujita, A. Kaneko, I. Schiermeyer, and K. Suzuki. A Rainbow k-Matching in the Complete Graph with r Colors. The Electronic Journal of Combinatorics 16, (2009), # R51.

[4] I. Gorgol. Rainbow Numbers for Cycles with Pendant Edges. Graphs and Combinatorics 24 (2008) 327-331.

[5] J. J. Montellano-Ballesteros. An Anti-Ramsey Theorem on Diamonds. Graphs and Combinatorics 26 (2010) 283-291.

[6] J. J. Montellano-Ballesteros and V. Neumann-Lara. An anti-Ramsey theorem. Combinatorica 22 (2002), no. 3, 445-449.

[7] J. J. Montellano-Ballesteros and V. Neumann-Lara. An Anti-Ramsey Theorem on Cycles. Graphs and Combinatorics 21 (3) (2005) 343-354.

[8] I. Schiermeyer. Rainbow Numbers for Matchings and Complete Graphs. Discrete Mathematics 286 (2004) 157-162.

[9] P. Turán. Eine Extremalaufgabe aus der Graphentheorie. Mat. Fiz. Lapok 48 (1941) 436-452.

Coloring problems on interval hypergraphs

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1 Introduction

Twenty years ago, Voloshin [9, 10] opened a new dimension for the coloring theory of hypergraphs, by introducing *mixed hypergraphs*. The point was that, beside the classical condition of excluding monochromatic edges, a further type of edges was introduced, which are not allowed to be totally multicolored. The two kinds of restrictions make the situation quite complex; some hypergraphs are not colorable [8], while among the colorable ones the possible numbers of colors are practically unrestricted [6]. More precisely, for every finite set $S \subset \mathbb{N} \setminus \{1\}$ there exists a mixed hypergraph $\mathcal{H}$ such that $\mathcal{H}$ admits a coloring with exactly k colors if and only if $k \in S$. Moreover, as proved in [7], also the number of different proper color partitions can be prescribed for every $k \in S$. Similar results were obtained for r-uniform hypergraphs in [2] and [11].

The structure class of mixed hypergraphs has been extended to several levels, the first of them being the *color-bounded hypergraphs* [1], in which the number of colors is bounded from below and from above in each edge. The most general model of locally restricted hypergraph coloring is called *pattern hypergraph* [5]; then every edge has a given collection of partitions, and a coloring is feasible if and only if its color classes split each edge according to one of the partitions allowed for that edge.

Disregarding the arbitrarily definable condition sets of pattern hypergraphs, the most general class studied so far is that of stably bounded hypergraphs [3]. Then, beside the number of colors occurring in an edge, also the number of occurrences of the most frequent color in an edge is bounded from above and from below. In the current note we collect some problems on algorithmic complexity, which remain open after the recent paper [4].

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2 Stably bounded hypergraphs

Formally, we consider hypergraphs $\mathcal{H} = (X, \mathcal{E})$ with vertex set $X = \{x_1, \ldots, x_n\}$ and edge set $\mathcal{E} = \{E_1, \ldots, E_m\}$, assuming $E_i \neq \emptyset$ for all $1 \leq i \leq m$. *Stably bounded hypergraph* means that four functions $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b} : \mathcal{E} \to \mathbb{N}$ are also given, such that $1 \leq \boldsymbol{s}(E_i) \leq \boldsymbol{t}(E_i) \leq |E_i|$ and $1 \leq \boldsymbol{a}(E_i) \leq \boldsymbol{b}(E_i) \leq |E_i|$ hold for all $E_i \in \mathcal{E}$. A *proper vertex coloring* of $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ is a mapping $\varphi : X \to \mathbb{N}$ which satisfies the following conditions for every edge $E_i \in \mathcal{E}$.

(i) The number of distinct colors in E_i is at least $\boldsymbol{s}(E_i)$ and at most $\boldsymbol{t}(E_i)$.

(ii) The number of vertices in E_i on which the most frequent color of E_i occurs is at least $\boldsymbol{a}(E_i)$ and at most $\boldsymbol{b}(E_i)$.

The hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ is *colorable* if it admits at least one proper coloring. If $\mathcal{H}$ is colorable, then its *upper chromatic number* is defined as the maximum possible number of colors in a proper vertex coloring of $\mathcal{H}$, denoted by $\overline{\chi}(\mathcal{H})$, and its *lower chromatic number* is the minimum number of colors in a proper vertex coloring.

If, for all $E_i \in \mathcal{E}$, one or more of the four conditions

$$\boldsymbol{s}(E_i) = 1 \ ; \quad \boldsymbol{t}(E_i) = |E_i| \ ; \quad \boldsymbol{a}(E_i) = 1 \ ; \quad \boldsymbol{b}(E_i) = |E_i|$$

hold, then the corresponding functions are non-restrictive and can be disregarded. In this way we obtain functional subclasses; for instance, an (S,T)-hypergraph is one where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-restrictive, i.e. only $\boldsymbol{s}$ and $\boldsymbol{t}$ can be restrictive. Analogously, in an A-hypergraph only $\boldsymbol{a}$ can be restrictive. The corresponding subclass can be defined for any subset of $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$ in a similar way.

3 Open problems on interval hypergraphs

Interval hypergraph means that there is an order $v_1, \ldots, v_n$ of the vertices such that each $E_i \in \mathcal{E}$ consists of consecutive vertices in this order. The time complexity of many algorithmic questions has been settled in [4]. At the time of writing this summary, the following cases remain unsolved.

Problem 3.1. *Determine the complexity of the problems of*

1. *deciding the colorability of (S,T)-hypergraphs;*
2. *determining the lower chromatic number of (S,A)-hypergraphs;*

3. *determining the lower chromatic number of (S,T,A)-hypergraphs;*
4. *determining the upper chromatic number of A-hypergraphs;*
5. *determining the upper chromatic number of (T,A)-hypergraphs;*
6. *determining the upper chromatic number of (S,T)-hypergraphs;*
7. *determining the upper chromatic number of (T,B)-hypergraphs;*
8. *determining the upper chromatic number of (S,T,B)-hypergraphs;*

under the assumption that the input is restricted to <u>*interval*</u> *hypergraphs.*

References

[1] Cs. Bujtás and Zs. Tuza, Mixed colorings of hypergraphs. *Electronic Notes in Discrete Mathematics* 24 (2006), 273–275.

[2] Cs. Bujtás and Zs. Tuza, Uniform mixed hypergraphs: The possible numbers of colors. *Graphs and Combinatorics* 24 (2008), 1–12.

[3] Cs. Bujtás and Zs. Tuza, Color-bounded hypergraphs, III: Model comparison. *Applicable Analysis and Discrete Mathematics* 1 (2007), 36–55.

[4] Cs. Bujtás and Zs. Tuza, Color-bounded hypergraphs, VI: Structural and functional jumps in complexity. *Discrete Mathematics* to appear.

[5] Dvořák, Z., J. Kára, D. Král' and O. Pangrác, Pattern hypergraphs. *Electronic Journal of Combinatorics* 17 (2010), #R15, 20 pp.

[6] T. Jiang, D. Mubayi, V. Voloshin, Zs. Tuza and D. West, The chromatic spectrum of mixed hypergraphs. *Graphs and Combinatorics* 18 (2002), 309–318.

[7] D. Král', On feasible sets of mixed hypergraphs. *Electronic Journal of Combinatorics* 11 (2004), #R19, 14 pp.

[8] Zs. Tuza and V. Voloshin, Uncolorable mixed hypergraphs. *Discrete Applied Mathematics* 99 (2000), 209–227.

[9] V. Voloshin, The mixed hypergraphs. *Computer Science Journal of Moldova* 1 (1993), 45–52.

[10] V. Voloshin, On the upper chromatic number of a hypergraph. *Australasian Journal of Combinatorics* 11 (1995), 25–45.

[11] P. Zhao, K. Diao and K. Wang, The chromatic spectrum of 3-uniform bi-hypergraphs. *Discrete Mathematics* 311 (2011), 2650–2656.

The removal lemma for products of systems

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In [6] the authors showed a removal lemma for linear systems on abelian groups provided a coprimality condition between the order of the group and the determinantal of maximal order of the matrix. The result extended the removal lemma for groups introduced by Green in [4] to linear systems of equations for abelian groups, following the extensions to the non-abelian case [5], for finite fields [7, 8] or to compact abelian groups [2].

Following the lines of [6], we present a notion of representability, compatible with product structures, that allows us to show a removal lemma. Moreover, the representability result for product structures, allows us to eliminate the condition regarding the determinantal for [6, Theorem 1].

Definition 0.1 (Representable system)**.** Let G be a finite set. Let A be a property on G^m, $A : G^m \to \{0,1\}$. Let $S(A,G) = \{\mathbf{x} \in G^m \ : \ A(\mathbf{x}) = 1\}$. A pair of colored hypergraphs (K,H) is said to represent (A,G) if the following holds:

- K and H are s-uniform m-colored hypergraphs. H has m edges with different colors and $h > s$ vertices. Each edge in K bears a label in G. s and h are bounded by functions depending on m.

- The labels of the edges of each copy of H in K, ordered by colors, form an element of $S(A,G)$.

- For each solution $\mathbf{x} = (x_1,\dots,x_m) \in S(A,G)$ there exist a set $Q(\mathbf{x})$ that equipartitions the copies of H in K related to $\mathbf{x}$ and, for each $q \in Q(\mathbf{x})$ and each $i \in [1,m]$, there exist an edge set $E(\mathbf{x},i,q) \subset E(K)$ of edges coloured i, labeled x_i, with $|E(\mathbf{x},i,q)| = c|K|^s/|G|$, for some c lower bounded by a function of m, with the following property. For each edge $e \in E(\mathbf{x},i,q)$, there are p copies of H in K related to the solution $\mathbf{x}$ through q containing e. p is independent on $\mathbf{x}$, i, q or e. $|Q(\mathbf{x})|$ is independent of $\mathbf{x}$.

- Any copy of H in K related to $\mathbf{x}$ through q, intersects $E(\mathbf{x},i,q)$ for all $i \in [1,m]$.

This definition captures the conditions to allow the transference of the removal lemma for colored hypergraphs [1] to algebraic settings using the arguments in [5, 8, 7, 6]. In particular, the representation for [6, Theorem 1] uses $|Q(\mathbf{x})| = 1$ and $p = 1$. Although the prime examples are the linear systems, Definition 0.1 is open to other relational systems.

Let A_1 and A_2 be two systems on m variables on G_1 and G_2 respectively. Then $(A_1, A_2) : G_1^m \times G_2^m \to \{0, 1\}$ with $(A_1, A_2)(x, y) = 1 \iff A_1(x) = 1$ and $A_2(y) = 1$ is said to be the independent product of A_1 and A_2. The independent product of representable systems is representable.

Lemma 0.2 (Representability of the product of systems)**.** *Let A_1, A_2 be representable systems on m variables over G_1 and G_2 respectively. Then the independent product $((A_1, A_2), (G_1, G_2))$ is representable.*

The construction starts by blowing up K_1 and K_2 (each vertex of K_1 turns into $|K_2|$ vertices and viceversa), and the edges are the preimages of the edges by the projection from the blown up hypergraph to the original. Finally, we take the union of K_1 and K_2, as well as H_1 and H_2, respecting the colours of the edges, and check that the new hypergraph pair represents the independent product.

Let A be a $k \times m$, $m \geq k$, integer matrix and let $D_k(A)$ denote the k-th determinantal of A. There is a matrix A', with $D_k(A') = 1$, such that $S(A', G) \subset S(A, G)$ for any finite abelian group G. In particular, $S(A, G)$ is the union of systems $A'x = b$, for different independent vectors b (see [6, Section 5]). Lemma 0.2 can be used to remove the determinantal condition from [6, Theorem 1], by observing that the solution set (A, G) is the product of the solution sets of the maximal p-subgroups in G.

Indeed, the result for products allows us to combine the representations of the solution sets for each of the maximal p-subgroups of G with $p|D_k(A)$ and the solutions set of the remaining factor (which has order coprime with $D_k(A)$). The representation for each of the p-groups involve representing $(A, \mathbb{Z}_{p^i}^t)$ for $i \in [1, d]$, for some d depending on A, and combining it with the representation of $(A, L_p) = (A, \prod_{i>d}^r \mathbb{Z}_{p^i}^{t_i})$ using again Lemma 0.2. Furthermore, the product lemma is used to represent (A, L_p) by combining the representations for $(A, L_p') = (A, \mathbb{Z}_{p^d}^{\prod_{i>d}^r t_i} \subset \prod_{i>d}^r \mathbb{Z}_{p^i}^{t_i})$ with the one of (A', L_p) using that any solution to $A'x = b$, $x \in L_p$ for some b, is a sum of a solution to $A'x = b$, $x \in L_p'^m$, and a solution to $A'x = 0$, $x \in L_p^m$.

In all the cases, we use additional variables $\mathbf{y}$ to paramaterize the different systems $A'x = b$, for different independent vectors b. The condition on d is rather technical and strongly depends on the construction of the auxiliary matrix A' from [6, Lemma 10] when the additional variables $\mathbf{y}$ are used.

References

[1] T. Austin, T. Tao, Testability and repair of hereditary hypergraph properties. Random Structures & Algorithms (2010) vol. 36 (4) pp. 373-463.

[2] P. Candela, O. Sisask, A removal lemma for linear configurations in subsets of the circle, to appear in Proc. Edinburgh Math. Soc., arXiv:1109.2939 (2011).

[3] T. Gowers, Hypergraph regularity and the multidimensional Szemerédi theorem, Annals of Mathematics, Second Series (2007) vol. 166 (3) pp. 897-946.

[4] B. Green, A Szemerédi-type regularity lemma in abelian groups, with applications, Geometric and Functional Analysis (2005) vol. 15 (2) pp. 340-376.

[5] D. Král', O. Serra, L. Vena, A combinatorial proof of the removal lemma for groups, Journal of Combinatorial Theory, Series A (2009) vol. 116 (4) pp. 971-978.

[6] D. Král', O. Serra, L. Vena, On the removal lemma for linear systems over Abelian groups. European J. Combin. 34 (2013), no. 2, pp. 248-259.

[7] D. Král', O. Serra, L. Vena, A removal lemma for systems of linear equations over finite fields, Israel Journal of Mathematics (2012) vol. 187, pp. 193-207

[8] A. Shapira, A proof of Green's conjecture regarding the removal properties of sets of linear equations, J. Lond. Math. Soc. (2) (2010), vol. 81 (2), pp. 355-373.

[9] T. Tao, A variant of the hypergraph removal lemma, Journal of Combinatorial Theory, Series A (2006) vol. 113 (7) pp. 1257-1280.