Johanna Wiehe:

The NL-coflow polynomial

(joint work with W. Hochstättler)

motivation

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- generalization: Tutte Polynomial

$$T_G(x, y) = \sum_{S \subseteq E} (x-1)^{rk(E)-rk(S)} (y-1)^{|S|-rk(S)}$$

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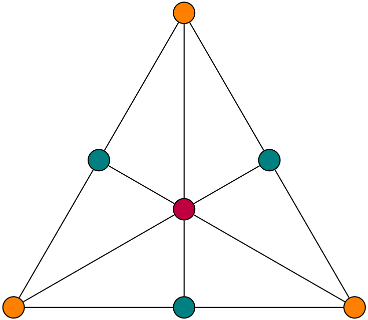

a proper coloring with 3 colors

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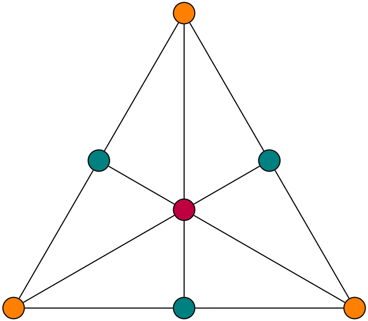

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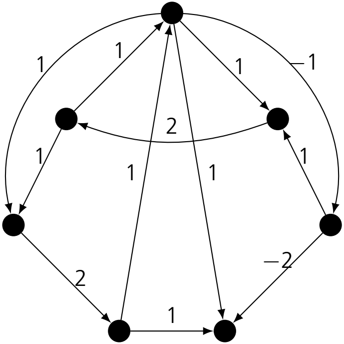

an NZ-3-flow

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- conjecture (Neumann-Lara)

  every orientation of a simple planar graph can be acyclically colored with two colors

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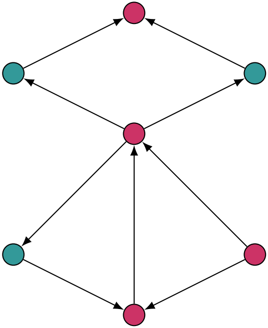

an acyclic coloring with 2 colors

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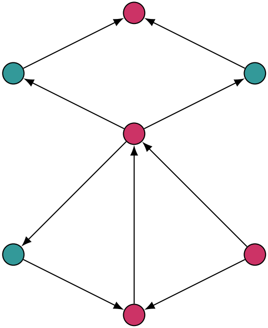

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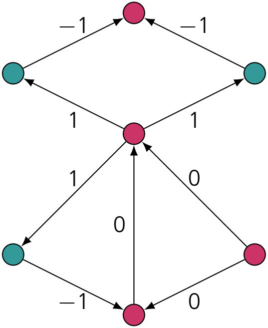

an NL-2-coflow

NL-Coflows

Definition (Hochstättler)

- Let $D = (V, A)$ be a digraph. A *coflow* is a map f, that satisfies Kirchhoff's law of flow conservation for (weak) cycles

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- Let G be a finite Abelian group. An *NL-G-coflow* in D is a coflow $f : A \longrightarrow G$, such that $\text{supp}(f)$ contains a feedback arc set (i.e. $S \subseteq A$ s.t. $D - S$ is acyclic).

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- For $k \geq 2$, a coflow $f : A \longrightarrow \{0, \pm 1, ..., \pm(k-1)\}$ is an *NL-k-coflow*, if $\text{supp}(f)$ is contains a feedback arc set.

Möbius inversion

Definition
Let $(P, \leq)$ be a finite poset, then the *Möbius function* is defined as follows

$$\mu : P \times P \to \mathbb{Z},\ \ \mu(x,y) := \begin{cases} 0 & \text{, if } x \nleq y \\ 1 & \text{, if } x = y \\ -\sum_{x \leq z < y} \mu(x,z) & \text{, if } x < y. \end{cases}$$

Theorem (*inversion from above*)
Let $(P, \leq)$ be a finite poset, $f, g : P \to \mathbb{K}$ functions and μ the Möbius function. Then the following equivalence holds

$$f(x) = \sum_{y \geq x} g(y),\ \text{ for all } x \in P \iff g(x) = \sum_{y \geq x} \mu(x,y) f(y),\ \text{ for all } x \in P.$$

Let $D = (V, A)$ be a digraph and let

$$f_k : 2^A \to \mathbb{Z} \text{ count all } G\text{-coflows and let}$$
$$g_k : 2^A \to \mathbb{Z} \text{ count all NL-}G\text{-coflows.}$$

Using

$$\mathcal{C} := \{A/C \mid \exists\, C_1, ..., C_r \text{ directed cycles, such that } C = \bigcup_{i=1}^{r} C_i\} \text{ with } “\supseteq“$$

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that is

$$\boxed{\psi^D_{NL}(k) = g_k(A) = \sum_{B \in \mathcal{C}} \mu(A, B) \cdot k^{rk(B)}.}$$

another representation

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we obtain

$$\Rightarrow \psi_{NL}^{D}(k) = \sum_{B \in \mathcal{P}} (-1)^{rk_{\mathcal{P}}(B)} k^{rk(A/B)}.$$

Symmetric digraphs

Let $D = (V, A)$ be a symmetric digraph and $G = (V, E)$ its underlying undirected graph. Then we have

$$\psi_{NL}^{D}(x) = P(G, x) \cdot x^{-c(G)}.$$

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Proof. The polyhedron described by

$$(M, -M)\begin{pmatrix}\overrightarrow{x}\\ \overleftarrow{x}\end{pmatrix} = 0$$

$$\overrightarrow{x_i} + \overleftarrow{x_i} \geq 1 \quad \forall i$$

$$\overrightarrow{x}, \overleftarrow{x} \geq 0$$

is unbounded, thus the face poset is contractible!

open problems

- How does the NL-coflow polynomial (= # acyclic colorings) of totally cyclic digraphs look like?

- In general, how does the NL-coflow polynomial of complete digraphs look like?

- Is there a meaningful two variable polynomial combining the dichromatic and the NL-flow polynomial as the Tutte polynomial does in the classical case?

- How many vertices suffice to create a 5-chromatic tournament?
  (a 3-chromatic tournament has at least 7 vertices, a 4-chromatic tournament at least 11)

- ...

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Thank you for your attention.