Using SAT Solvers

in Combinatorics and Geometry

Manfred Scheucher

Boolean satisfiability problem

- Given Boolean formula, is there an assignment such that the formula is true?

Boolean satisfiability problem

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- NP-complete, but quite good heuristics

Why to use SAT Solvers?

- Prove/disprove existence of structure

usually faster

Why to use SAT Solvers?

- Prove/disprove existence of structure
- Find (counter)examples
- Counting occurences
- Induction base
- Induction step

Today's Topics

- L-shaped Point Set Embeddings of Trees
  (with Torsten Mütze)
- Orthogonal Symmetric Chain Decompositions
  (with Karl Däubel, Sven Jäger, and Torsten Mütze)
- (Disjoint) Holes in Point Sets
- Universal Point Sets for Planar Graphs
  (with Hendrik Schrezenmaier and Raphael Steiner)

L-Shaped Point Set Embeddings

joint work with Torsten Mütze

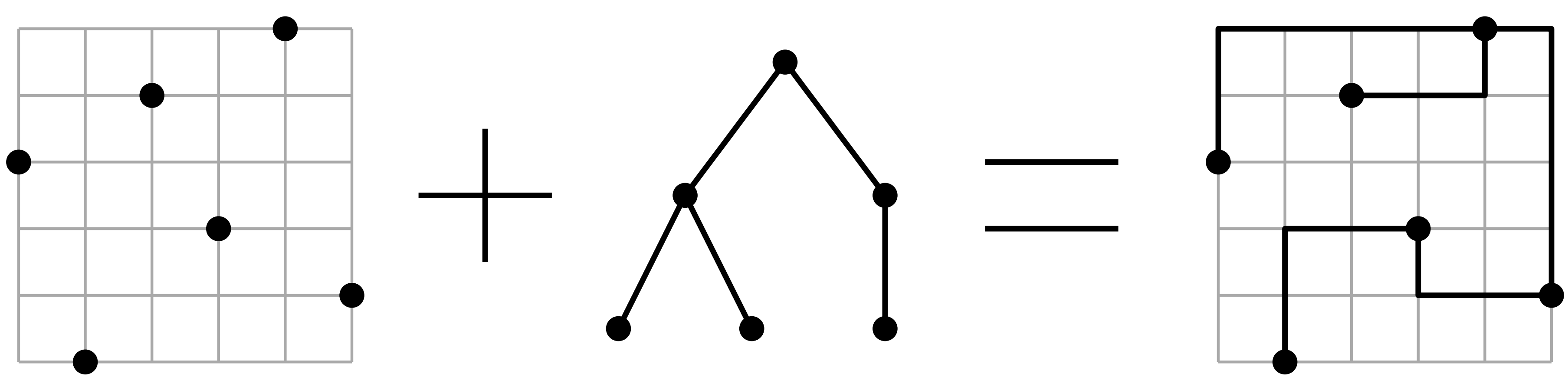
arXiv:1807.11043

L-Shaped Point Set Embeddings

T . . . tree on n vertices

P . . . set of m points

- vertices of T drawn as points of P
- edges drawn as unions of two axis parallel line segments



L-Shaped Point Set Embeddings

$f(T)$ ... minimum number m s.t. tree T admits a planar L-shaped embedding in any set of m points

$$f_d(n) := \max_{\substack{T\,:\,\text{tree on } n \text{ vertices} \\ \text{max. deg. } \Delta(T) \leq d}} f(T)$$

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- $f_4(n) \leq n^2$ [Di Giacomo, Frati, Fulek, Grilli, Krug '13]
- $f_4(n) \leq O(n^{1.58})$ [S.'15, Aichholzer-Hackl-S.'16]
- $f_3(n) \leq O(n^{1.22})$, $f_4(n) \leq O(n^{1.55})$ [Biedl et al.'17]
- no non-trivial lower bound ($\geq n$ is trivial)

SAT Model

- T ... tree on vertices $\{v_1, \ldots, v_n\}$
- P ... point set $\{P_1, \ldots, P_n\}$
- formulate Boolean satisfiability instance:
  $\exists$ solution iff. T admits an L-shaped embedding in P

SAT Model: Variables

- $M_{i,j}$ ... vertex v_i is mapped to point P_j
- $H_{a,b}$ ... edge ab is connected horizontally to a

SAT Model: Clauses

- Injective mapping V to P

every vertex v_i has to be mapped:

$$\bigvee_j M_{i,j}$$

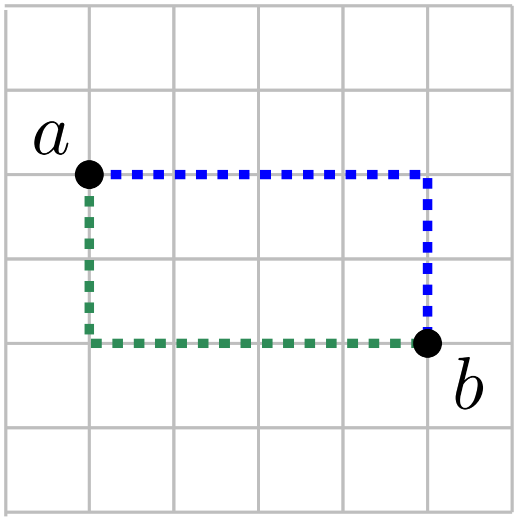
no two vertices v_{i_1}, v_{i_2} are mapped to the same point:

$$\neg M_{i_1,j} \vee \neg M_{i_2,j}$$

SAT Model: Clauses

- Injective mapping V to P
- L-shaped edges:
  ab connects either vertically or horizontally to a (and b)

$$H_{a,b} \lor H_{b,a}, \quad \neg H_{a,b} \lor \neg H_{b,a}$$



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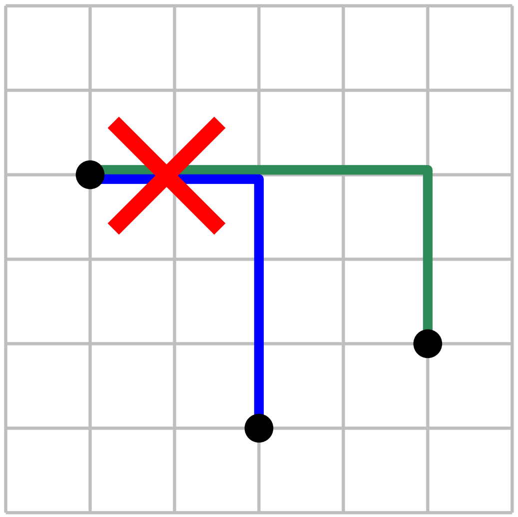

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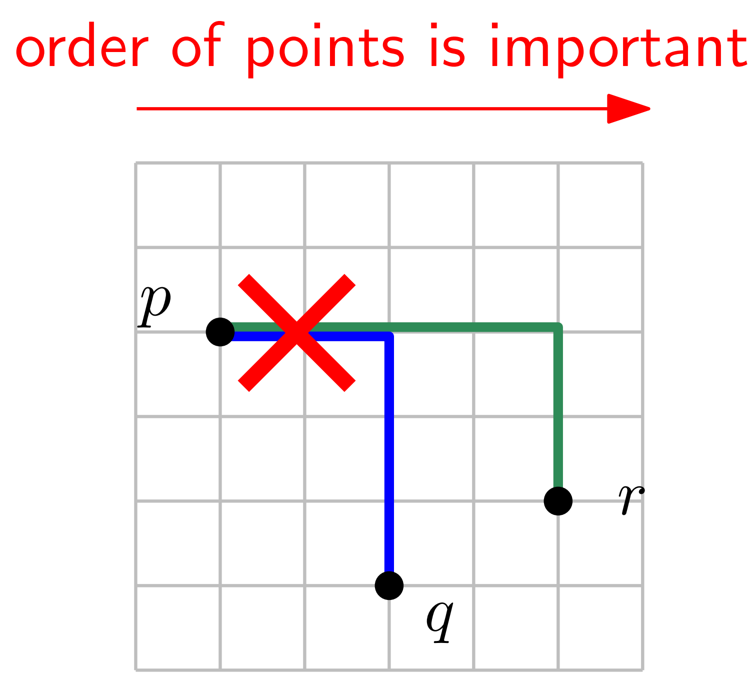

SAT Model: Clauses

- Injective mapping V to P
- L-shaped edges:
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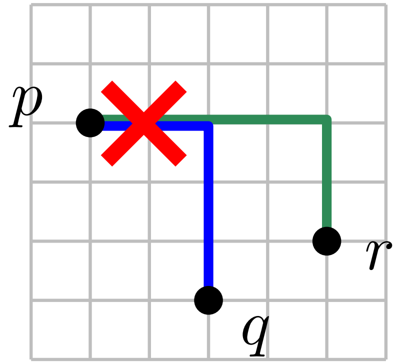
- No overlapping edges

If p left of q and r, ← only depends on point set
and $v_a \to p$, $v_b \to q$, $v_c \to r$, ← only depends on variables
then not both edges ab and ac horizonally connected



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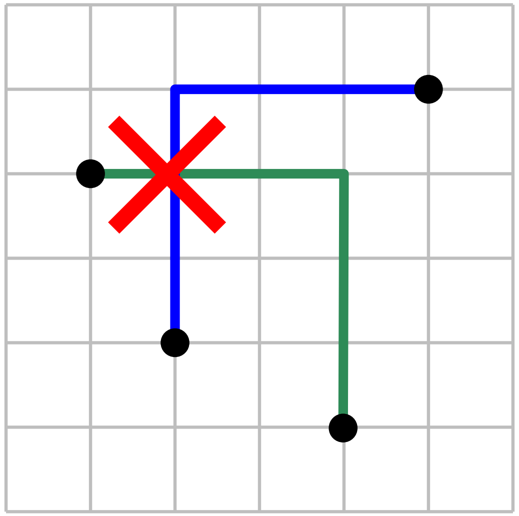
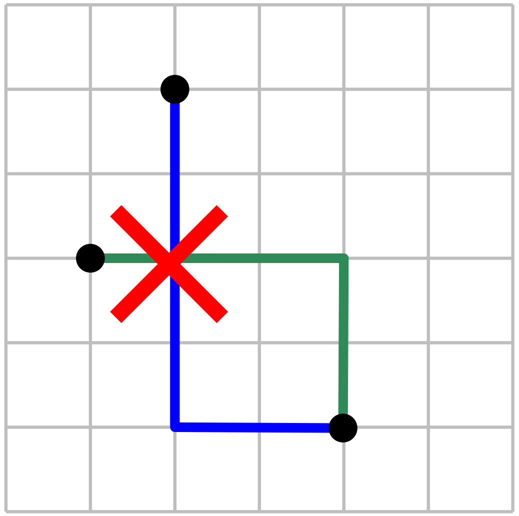
For each three points p, q, r with p left of q and r:

$$\neg M_{a,p} \vee \neg M_{b,q} \vee \neg M_{c,r} \vee \neg H_{a,b} \vee \neg H_{a,c}$$

SAT Model: Clauses

- Injective mapping V to P
- L-shaped edges:
  ab connects either vertically or horizontally to a (and b)
- No overlapping edges
- No crossing edges



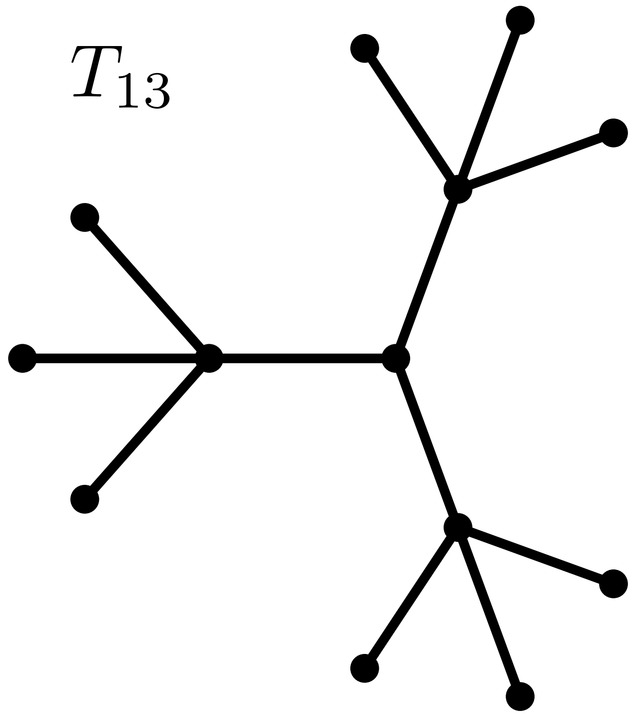


Results

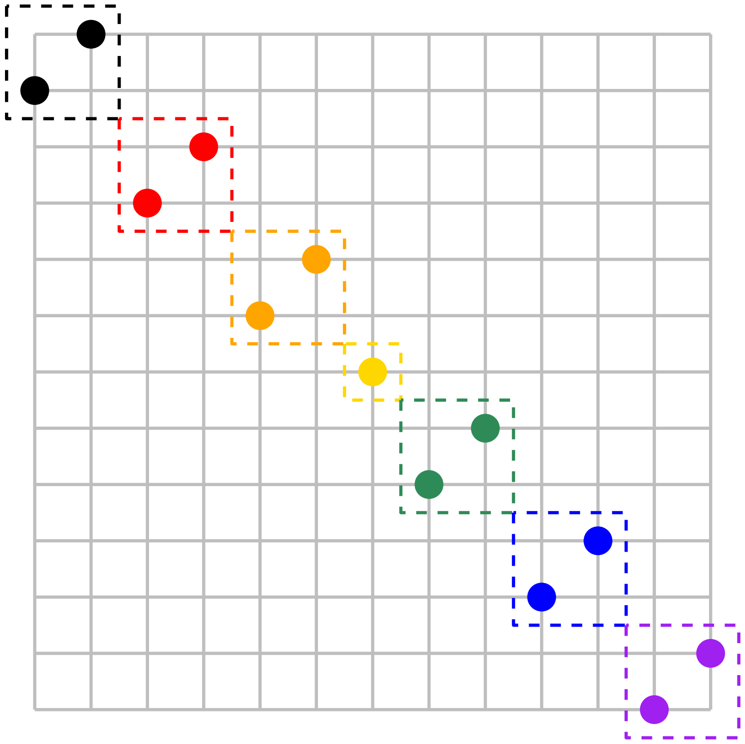
Theorem: Every tree on $n \leq 12$ vertices admits an L-shaped embedding in every set of n points.

Theorem: T_{13} has no L-shaped embedding in P_{13}.

T_{13}



P_{13}

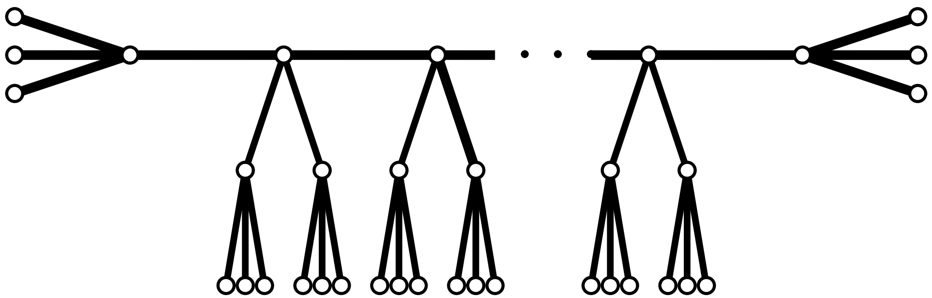


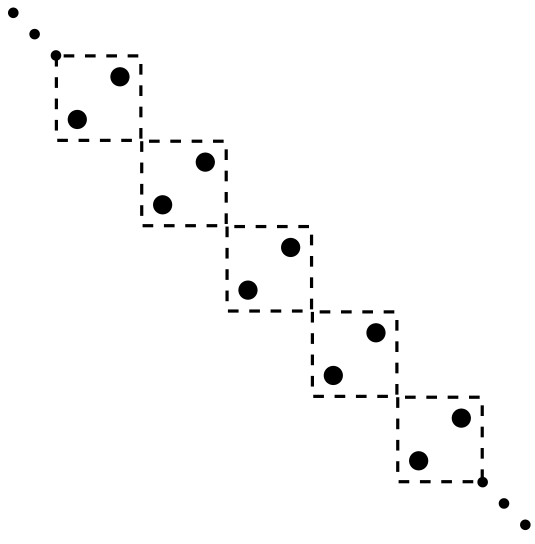
Results

Theorem: Every tree on $n \leq 12$ vertices admits an L-shaped embedding in every set of n points.

Theorem: T_{13} has no L-shaped embedding in P_{13}.

- Further examples for $n \in \{13, 14, 16, 17, 18, 19, 20\}$
- If cyclic order fixed, infinite family





And now for something completely different

Orthogonal Chain Decompositions

joint work with
Karl Däubel, Sven Jäger, and Torsten Mütze

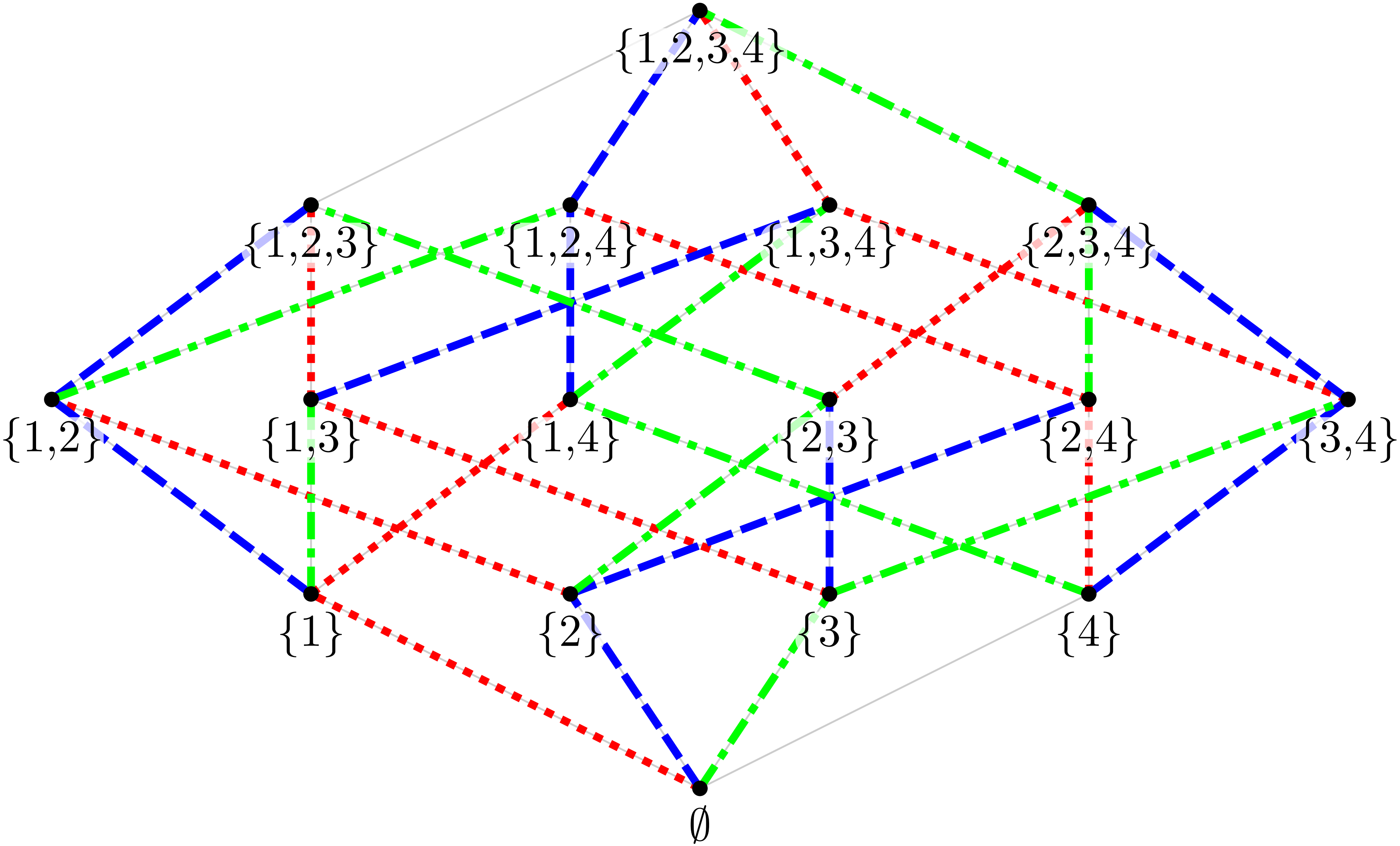
arXiv:1810.09847

Orthogonal Chain Decompositions

- Boolean lattice Q_n

can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains [Dilworth '50]

Orthogonal Chain Decompositions



Orthogonal Chain Decompositions

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- *orthogonal* chain-decompositions:
  any two chains from two decompositions
  have at most one element in common

Orthogonal Chain Decompositions

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Proof Idea:

1. (specific) orthogonal CDs for Q_5 and Q_7
2. Lemma "Q_a, $Q_b \Rightarrow Q_{a+b}$"

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n	1	2	3	4	5	6	7	8	9	10	11
almost-orth.	1	2	2	2	3	3*	4*	3*	3*	3	4*
$\lfloor n/2 \rfloor + 1$	1	2	2	3	3	4	4	5	5	6	6

found via SAT solvers!

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Theorem: $\exists$ four orthogonal CDs in $Q_{n\geq 60}$.
[Däubel-Jäger-Mütze-S. '18]

SAT Instance

- Problem: naive CNF formulation much too big

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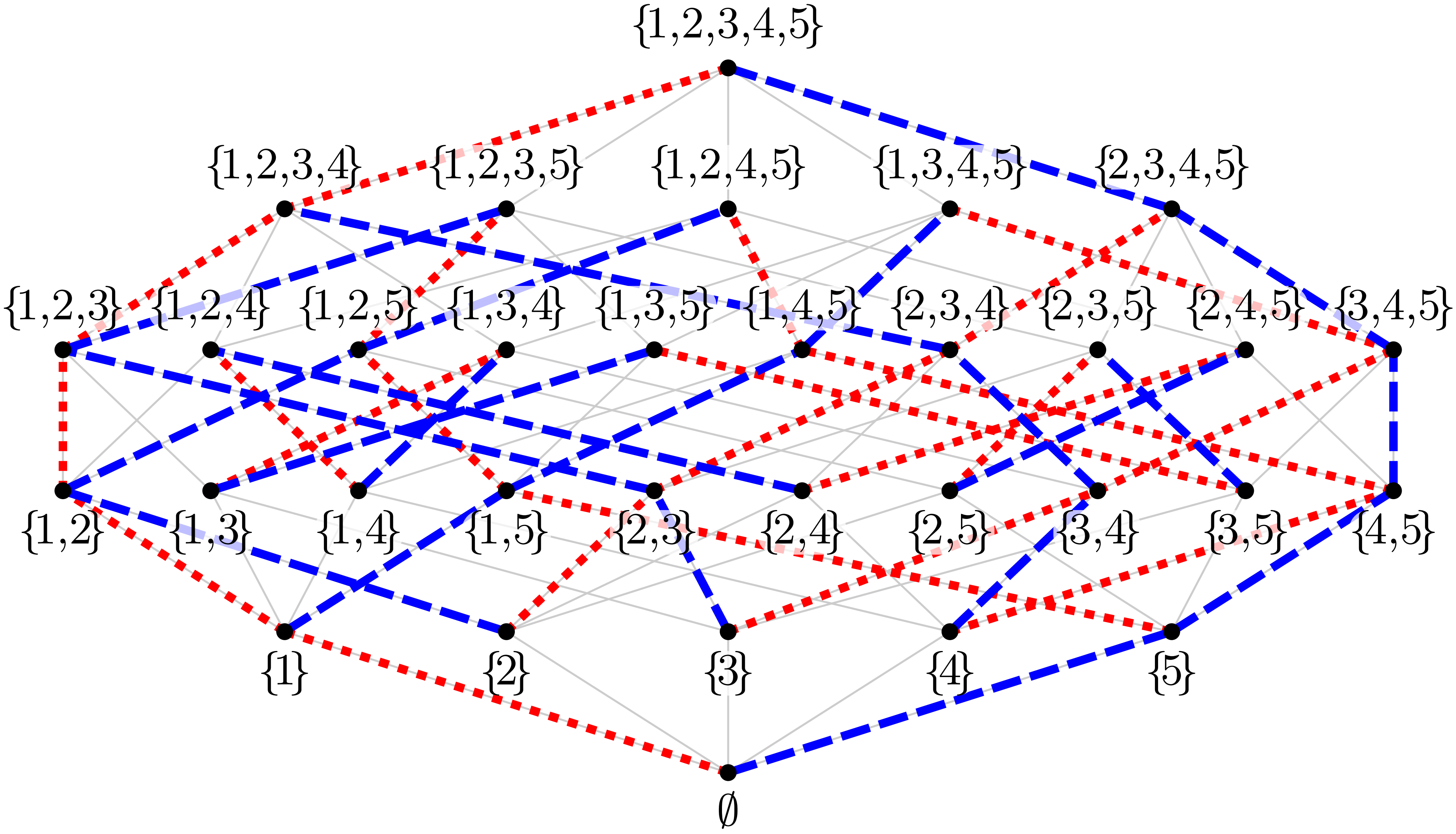

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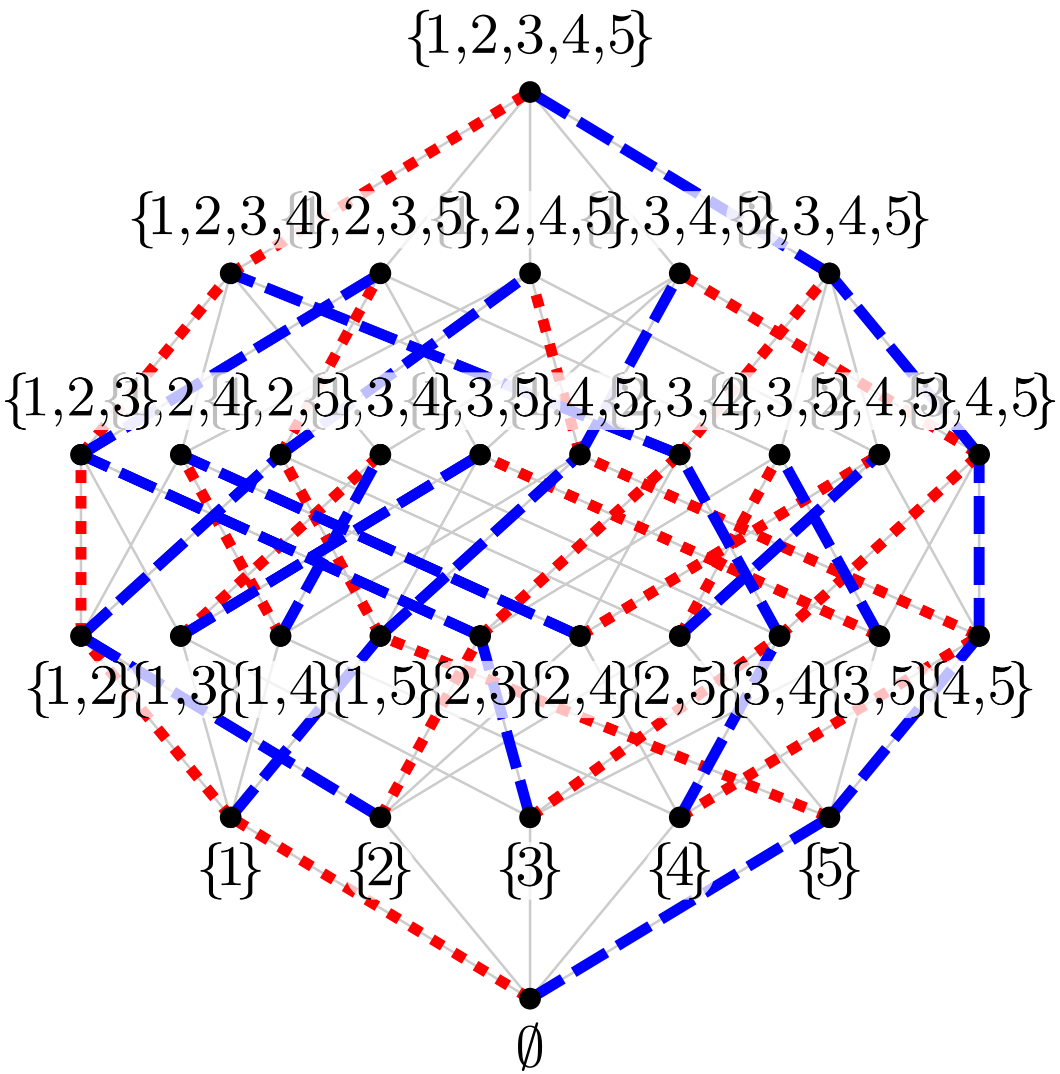
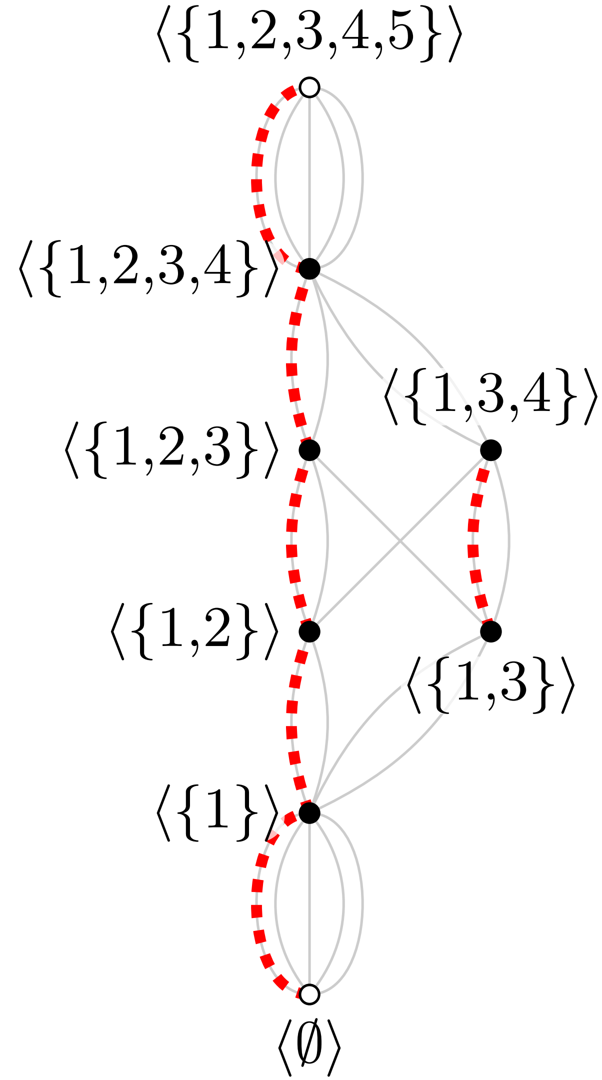




SAT Instance

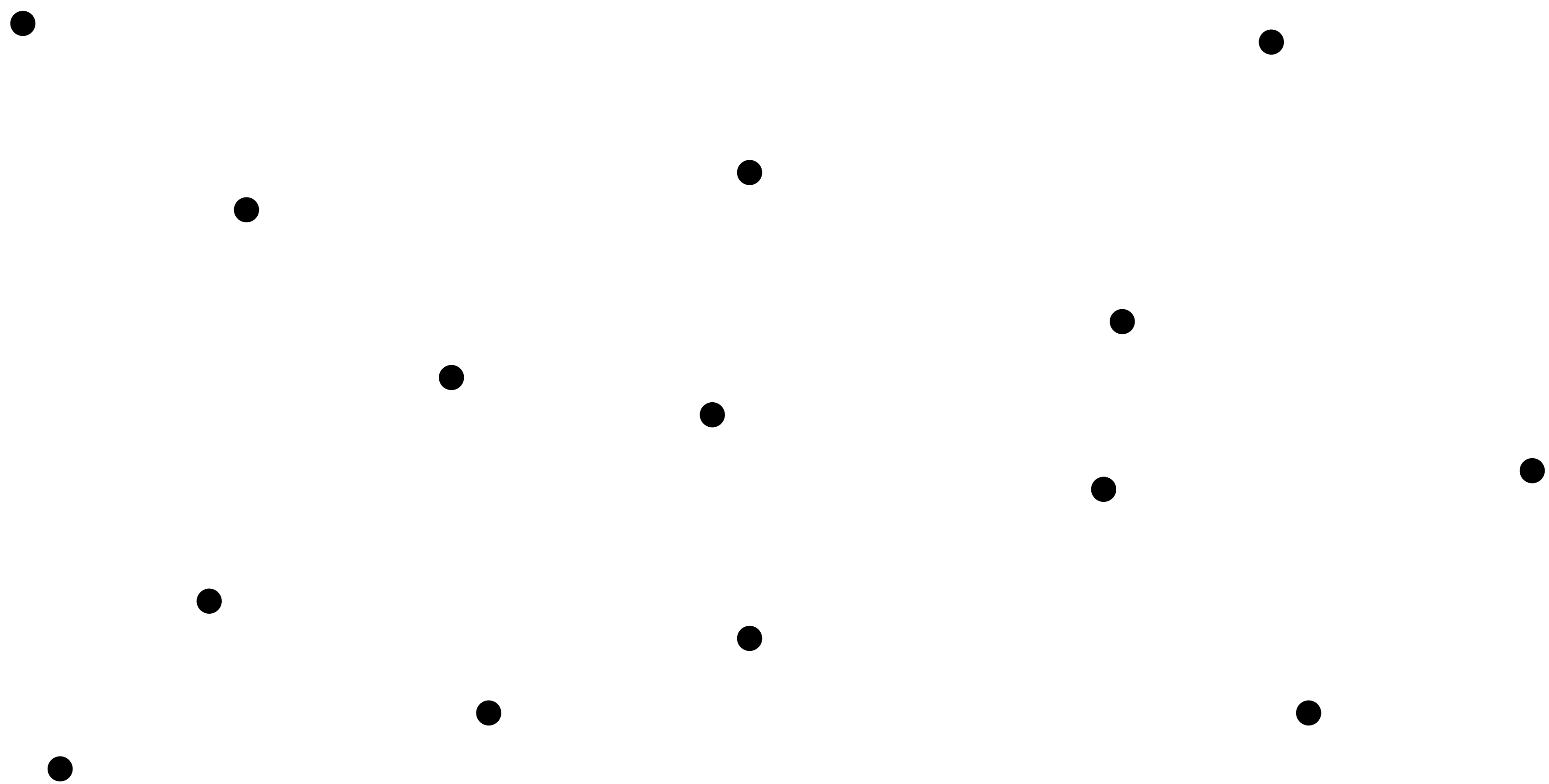
- Problem: naive CNF formulation much too big
- use symmetries
- iteratively add clauses to "correct errors"

  (incremental solving supported by minisat/glucose)

And now for something more completely different

Classical Erdős–Szekeres

- Given n points in the plane in general position, is there subset size k in convex position ("k-gon")?



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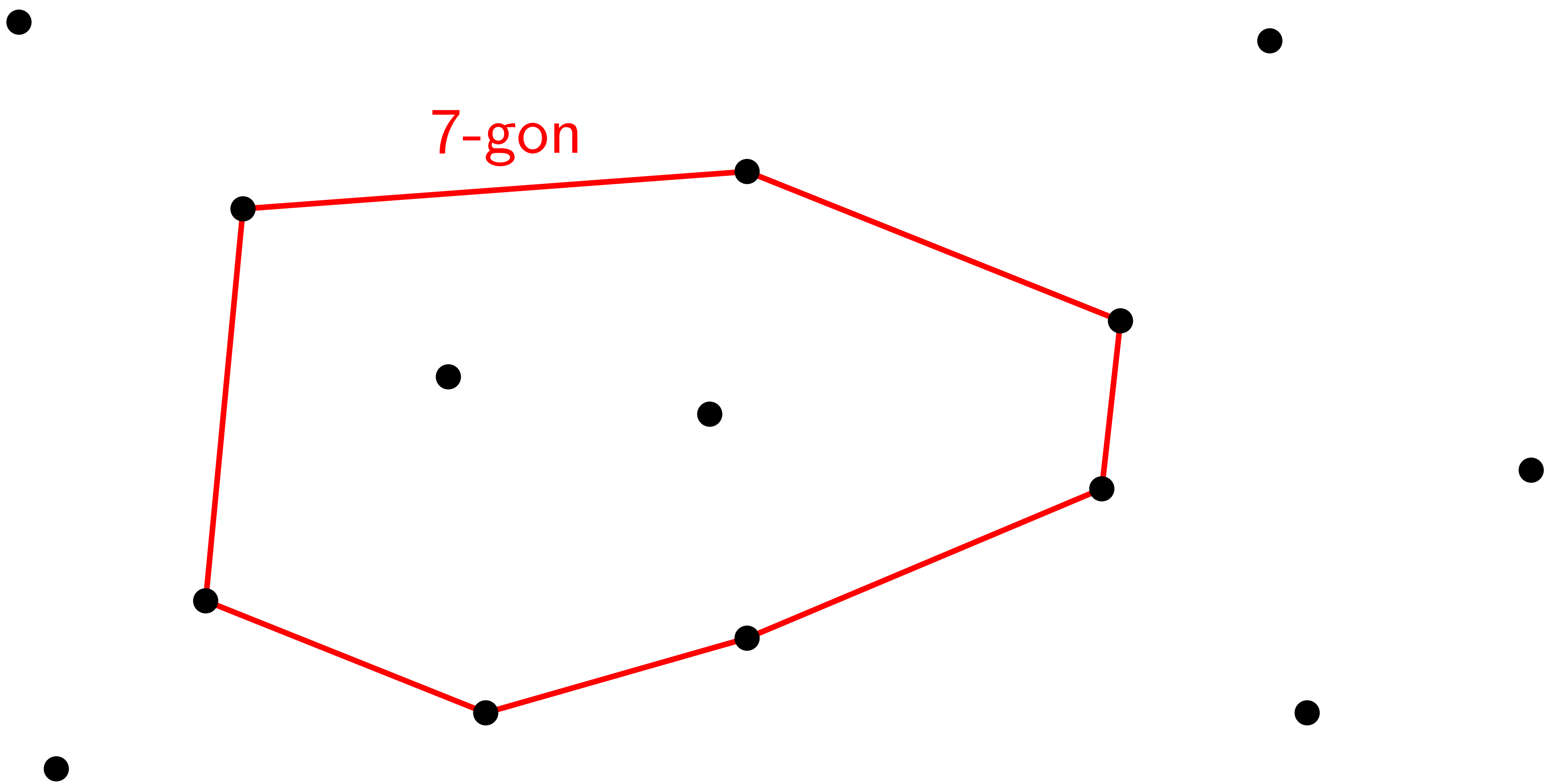

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Theorem. $2^{k-2} + 1 \leq g(k) \leq \binom{2k-4}{k-2}$. [Erdős–Szekeres '35]

equality conjectured by Szekeres, Erdős offered 500$ for a proof

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Known: $g(4) = 5$, $g(5) = 9$, $g(6) = 17$

↑

computer assisted proof, 1500 CPU hours [Szekeres–Peters '06]

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1 hour using SAT solvers!

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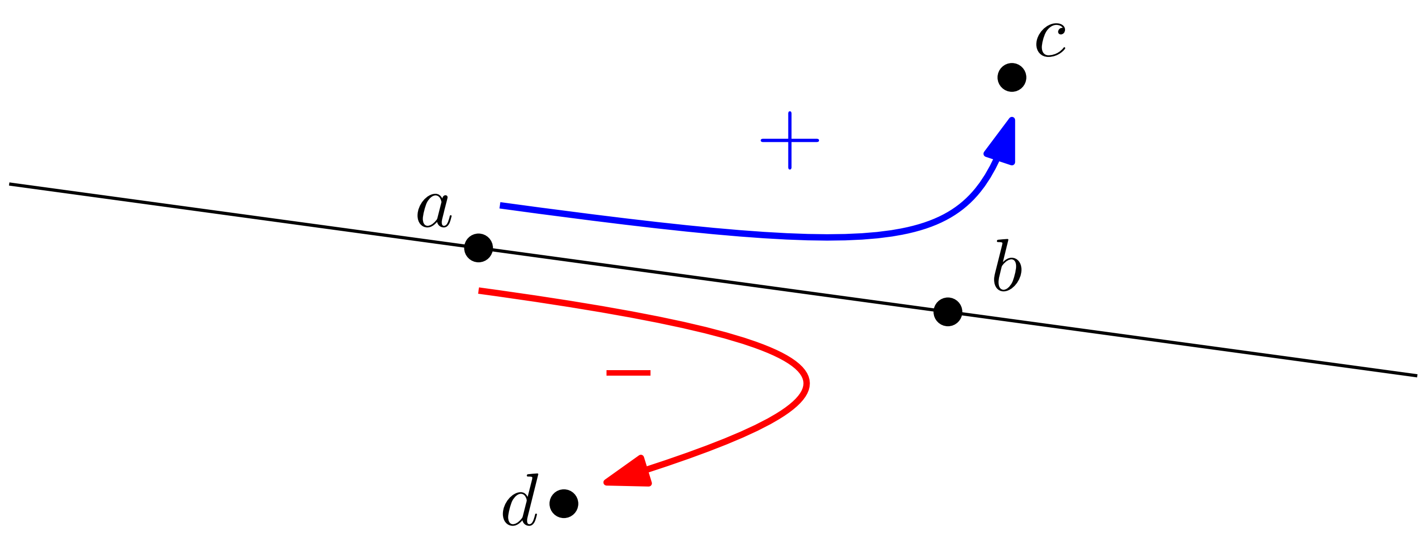
SAT Formulation

- Variables for triple-orientations: $\chi_{abc} \in \{+, -\}$



SAT Formulation

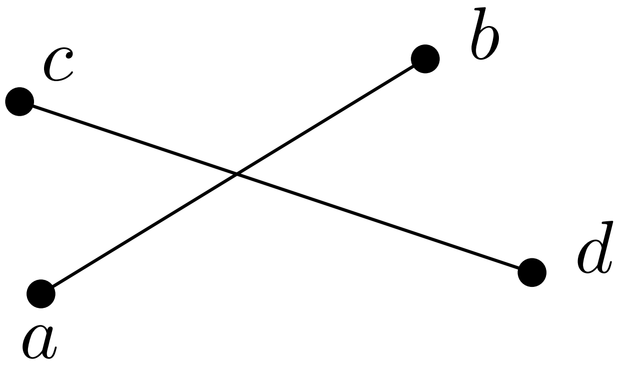
- Variables for triple-orientations: $\chi_{abc} \in \{+,-\}$
- Axiomatize "point set": chirotope/signotope axioms
  - Alternating axioms:
    $\chi_{i_{\pi(1)},i_{\pi(2)},i_{\pi(3)}} = \text{sgn}(\pi) \cdot \chi_{i_1,i_2,i_3}$
  - Exchange axioms:
    For any $x_1, \ldots, x_r, y_1, \ldots, y_r$:
    If $\chi_{y_i,x_2,\ldots,x_r} \cdot \chi_{y_1,\ldots,y_{i-1},x_1,y_{i+1},\ldots,y_r} \geq 0$ for every i,
    then $\chi_{x_1,\ldots,x_r} \cdot \chi_{y_1,\ldots,y_r} \geq 0$

$\Theta(n^6)$ many

SAT Formulation

- Variables for triple-orientations: $\chi_{abc} \in \{+, -\}$
- Axiomatize "point set": chirotope/signotope axioms
- Crossings via triple-orientations

$\chi_{abc} \neq \chi_{abd}$ and $\chi_{cda} \neq \chi_{cdb}$



SAT Formulation

- Variables for triple-orientations: $\chi_{abc} \in \{+, -\}$
- Axiomatize "point set": chirotope/signotope axioms
- Crossings via triple-orientations
- k-gons via crossings
  (every 4-tuple in k-gon is in convex positon)

Variant: k-Holes

- *k-hole* . . . k-gon with no interior points

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- $h(4) = 5$, $h(5) = 10$, $30 \leq h(6) \leq 463$, $h(7) = \infty$

Harborth '78

Horton'83

Overmars '02

Gerken '08, Nicolas '07, Koshelev '09

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- $h_3(n)$ and $h_4(n)$ quadratic

↑ ↑

number of 3- and 4-holes, resp.

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- $h_5(n) = \Omega(n \log^{4/5} n)$ and $h_6(n) = \Omega(n)$
- $h_k(n)$ determined for small values of n

n	9	10	11	12	13	14	15	16	17	18	19
$h_5(n)$	0	1	2	3	3	6	9	11	≤ 16	≤ 21	≤ 26

Harborth '78 & Dehnhardt '87 (n = 10–12) S.'13 (n = 13–15) via SAT (n = 16)

Couting via SAT

- containment $p \in \triangle(a, b, c)$ via "no crossings"
- k-holes via crossings and containments
- *count* with variables $X_{a,b,c,\ldots;\ell}$:
  $a, b, c, \ldots$ form the ℓ-th k-hole (in lexicographic order)

And now for something more completely different

Universal Point Sets for Planar Graphs

joint work with
Hendrik Schrezenmaier and Raphael Steiner

arXiv:1811.06482

Universal Point Sets for Planar Graphs

Definition: *n-universal* point set S:
$\forall$ planar n-vertex graph G can be drawn straight-line on S.

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- $n \times n$ grid is n-universal [De Fraysseix–Pach–Pollack '90]
- $\dots$
- $|S| \leq \frac{n^2}{4} - O(n)$ [Bannister–Cheng–Devanny–Eppstein '14]

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- $|S| \geq n + \Omega(\sqrt{n})$ [De Fraysseix–Pach–Pollack '90]
- ...
- $|S| \geq 1.235n(1 + o(1))$ [Kurowski '04]

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- $\exists$ n-universal sets for $n \leq 10$ [Cardinal–Hoffmann–Kusters '15]
- $\nexists$ n-universal sets for $n \geq 15$ [Cardinal–Hoffmann–Kusters '15]

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$|S| \geq 1.293n(1 + o(1))$

$\nexists$ 11-universal set on 11 points

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Universal Point Sets for Planar Graphs

SAT model for a fixed set S and fixed graph $G \in \mathcal{G}$:

- Injective mapping $V(G) \to S$
- No two edges cross (planarity)

(similar idea as in L-shaped model)

Universal Point Sets for Planar Graphs

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(similar idea as in L-shaped model)

All in one SAT instance:

- all graphs can be handled simultaneously
- all point sets simultaneously (chirotope/signotope axioms)

... but solvers do not terminate ...

Universal Point Sets for Planar Graphs

- Enumerate all triangulations on 11 vertices (plantri) (1,249)

Universal Point Sets for Planar Graphs

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- Enumerate all chirotopes on 11 points (2,343,203,071)

via chirotope axioms, 20 CPU hours, 100 GB storage

Universal Point Sets for Planar Graphs

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- Enumerate all chirotopes on 11 points (2,343,203,071)

- Start with $\mathcal{G}$ as 11-vertex triangulations with maximum degree 10
- Test each pair S and G (SAT)

priority queue ("hard" triangulations first)

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- 500 CPU days later:

conflict collection of 23 stacked triangulations !

previously: 7393

Thank you for your attention!