The binary paint shop problem

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MCW, Prague Jul 30, 2019

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Outline

Introduction

Our results

The problem

- **double occurrence word** – every letter occurs twice

$$w = ADEBAFCBCDEF$$

- **want:** color all letters red&blue, every letter once red and once blue

$ADEBAFCBCDEF$ 4 changes

- **goal:** minimize the number of **color changes**

$ADEBAFCBCDEF$ 4 changes

$ADEBAFCBCDEF$ 2 changes

$$\gamma(w) = 2$$

Trivial observations

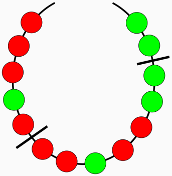
- $w_1 = A_1 A_1 A_2 A_2 \dots A_n A_n \qquad \gamma(w_1) = n$
- $w_2 = A_1 A_2 \dots A_n A_1 A_2 \dots A_n \qquad \gamma(w_2) = 1$
- W_n – set of words with letters $A_1, \dots, A_n$, each of them twice.

Natural questions

- value for nontrivial cases?
- algorithms?
- random $w \in W_n$?
- connection to some other parameters?
- motivation?

Motivation and previous results

- **paint shop**: a factory where a sequence of cars needs to be painted, for each sub-type we want one of each color, it is practical not to change the color too often.
- **necklace splitting**: [Image by Wikipedia user Kilom691, CC BY-SA 4.0]



Two (possibly more) thieves want to split a necklace with various types of gem-stones, using minimum number of cuts. N.Alon's theorem is more general, here it gives just $\gamma(w) \leq n$ for $w \in W_n$.

Hard problem

- APX-hard [Bonsma, Epping, Hochstättler (06); Meunier, Sebő (09)]
- Thus, the decision problem is NP-complete.
- some polynomial instances identified by Meunier and Sebő (09)

Heuristics

Results by Andres&Hochstättler, 2010.

- **greedy** – $g(w)$ – going from left to right, change color only if you must.

$$\mathbb{E}_{w \in W_n} g(w) = \mathbb{E}_n g(w) = 0.5n + o(n)$$

- **recursive greedy** – $rg(w)$ – remove the last letter, color recursively, choose the better way for the extra letter

$$\mathbb{E}_n rg(w) = 0.4n + o(n)$$

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Lower bounds

Observation

$$\gamma(w) \geq \alpha(G(w))$$

where $G(w)$ is the interval graph corresponding to the word w.

Scheinerman (1988) proved that for a random interval graph on n vertices, $\alpha \geq C\sqrt{n}$. Thus:

Corollary

$$\mathbb{E}_n\gamma \geq C\sqrt{n}$$

Linear lower bound

Theorem

$$\mathbb{E}_n\gamma \geq 0.214n - o(n)$$

This disproves a conjecture by Meunier, Neveu (2012). The conjecture was also mentioned at MCW 2012 (Andres) and MCW 2017 (Hochstättler).

Lower bound proof

- $w \in W_n$ – a random element
- will show $\Pr[\gamma(w) \le k] \le p$.
- This will prove that $\mathbb{E}_n \gamma \ge (1-p)k$.
- $C_n^{\le k}$ – colorings of $1, \ldots, 2n$ using n red and n blue, with at most k color changes.

$$\begin{aligned}
\Pr[\gamma(w) \le k] &= \Pr[w \text{ has a legal coloring in } C_n^{\le k}] \\
&\le \sum_{C \in C_n^{\le k}} \Pr[C \text{ is legal for } w] \\
&= \sum_{C \in C_n^{\le k}} \frac{n!^2}{(2n)!/2^n} \\
&= \cdots \le \frac{\sqrt{4n}}{2^n} \left(\frac{e \cdot 2n}{k}\right)^k
\end{aligned}$$

$p :=$ the latter, $k := 0.214n$ ... done.

Concentration

Theorem

Let w be a random element of W_n. Let $\gamma_n = \mathbb{E}_n \gamma$.

$$\Pr\left[|\gamma(w) - \gamma_n| \geq \sqrt{n \log n}\right] \leq 2n^{-1/8}$$

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Proof.

- Standard application of Azuma inequality.
- We let X_k be the expectation of $\gamma(w)$ after the positions of the letters $A_1, \ldots, A_k$ have been fixed.
- $X_0, X_1, \ldots, X_n$ is a martingale.
- $|X_k - X_{k+1}| \leq 2$.
- Azuma inequality gives the rest.

□

Improved upper bounds – theorem

Theorem

$$\gamma_n \leq (\frac{2}{5} - \varepsilon)n$$

for $\varepsilon \approx 1.64 \times 10^{-6}$.

Proof.

We run the recursive greedy algorithm, then observe that there is a linear number of local changes. □

Improved upper bounds – star heuristic

We propose a new heuristics – star heuristics. According to numerical evidence and rather convincing arguments, we believe that

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1. Similarly as in the recursive greedy, we take away the last letter and its second copy, we repeat.
2. We let the resulting words be $w_n = w, w_{n-1}, \ldots, w_1 = AA$.
3. Then we go forward, producing the coloring using red, blue, and * with the following condition:
4. The two copies of a letter must either be red/blue, blue/red or */*. We use the latter, if both red/blue and blue/red yield the same number of color changes.

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5. To get the coloring of w_{k+1} from that of w_k
   - do the greedy consideration of the new letter (possibly deciding about some *-colored letters).
   - possibly recolor the penultimate letter (and its copy) by a *.

Better bounds – open problem

Based on experiments (using a heuristics impossible to analyze), we believe the true value of γ_n is around $0.3n$. However, we have only the following bounds proved rigorously

$$0.214 \leq \lim \frac{\gamma_n}{n} \leq 0.4 - \varepsilon$$

We can imagine the upper bound can be decreased to around 0.361 with more work.

Question

What is $\lim \frac{\gamma_n}{n}$*? Does the limit even exist?*