Towards the Classification of Polymorphism-homogeneous Metric Spaces

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Polymorphism-homogeneity

Let **A** be a relational structure.

HH

A is **homomorphism-homogeneous** if every homomorphism between finite substructures of **A** extends to an endomorphism of **A**.

PH

A is **polymorphism-homogeneous** if every partial polymorphism of **A** with finite domain extends to a global polymorphism of **A**.

Is PH a sheer generalisation? Or, maybe not:

Proposition (Pech,Pech 2015)

A structure **A** is PH iff $\mathbf{A}^k$ is HH, for every $k \in \mathbb{N}_+$.

Homomorphisms of metric spaces

non-expansive mappings

A mapping $f : M_1 \to M_2$, between two metric spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ is **1-Lipschitz** if

$$d_{\mathcal{M}_2}(f(x), f(y)) \leqslant d_{\mathcal{M}_1}(x, y), \text{ for all } x, y \in M_1.$$

Example

Any graph **G** with all loops can be seen as a metric space:

- As $(V(\mathbf{G}), d)$ with

$$d(x,y) := \begin{cases} 0, & \text{when } x = y; \\ 1, & \text{when } x \neq y \text{ but } (x,y) \in E(\mathbf{G}); \\ 2, & \text{when } (x,y) \notin E(\mathbf{G}), \end{cases}$$

  for any $x, y \in V(\mathbf{G})$.
- As $(V(\mathbf{G}), d_{\mathbf{G}})$, where $d_{\mathbf{G}}$ is the **usual graph metric.**

Classification of HH metric spaces

One nice result:

Theorem (Dolinka 2012)

The rational Urysohn space is HH.

The daunting truth:

Deciding HH of finite metric spaces

It is a **coNP**-complete problem!

Evidentally, there is NO reasonable classification of HH metric spaces.

But, what about PH metric spaces?! Do we stand a chance?

The rise of the $\star$-property

$\star$-metric space

A metric space $\mathcal{M}$ has the **$\star$-property** when

$$\text{there exist no such } a, b > 0 \text{ in } \mathrm{im}(d_{\mathcal{M}}) \text{ that } a < b \leqslant 2 \cdot a.$$

In any $\star$-metric space for any $a, b \in \mathrm{im}(d_{\mathcal{M}}) \backslash \{0\}$:

$$\text{if } b \leqslant 2a \text{ then } b \leqslant a.$$

Theorem

All finite and countably infinite $\star$-metric spaces are PH.

The class of metric spaces that do **not** have the $\star$-property “includes” graphs. Hence,

the problem of deciding whether a finite metric space
without the $\star$-property is HH is **coNP**-complete.

Skeletal metric spaces

A metric space is **normalised** if the smallest nonzero distance in it is 1, in case it exists.

A **skeletal metric space** := a finite normalised metric space without the $\star$-property.

skeleton

We associate each skeletal metric space $\mathcal{M}$ with a unique graph $\mathbf{G}_{\mathcal{M}}$ such that $V(\mathbf{G}_{\mathcal{M}}) = M$, whereas:

$$(x, y) \in E(\mathbf{G}_{\mathcal{M}}) \text{ iff } d_{\mathcal{M}}(x, y) \in \{0, 1\}, \quad \text{for any } x, y \in M.$$

The usual graph metric on it, will be denoted by $\delta_{\mathcal{M}}$.

Useful new terms

Let $\mathcal{M}$ be a skeletal metric space.

An $a \in \mathrm{im}(d_{\mathcal{M}})$ is a k**-distance** in $\mathcal{M}$ if

$$\exists x, y \in M : \quad d_{\mathcal{M}}(x, y) = a \ \text{ and } \ \delta_{\mathcal{M}}(x, y) = k.$$

A set of points of $\mathcal{M}$ is a (k)-**set** in $\mathcal{M}$ if

$$\forall x, y \in \mathbf{G}_{\mathcal{M}} : \quad \delta_{\mathcal{M}}(x, y) \leqslant k.$$

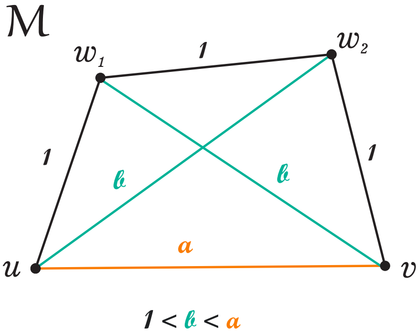
Proposition

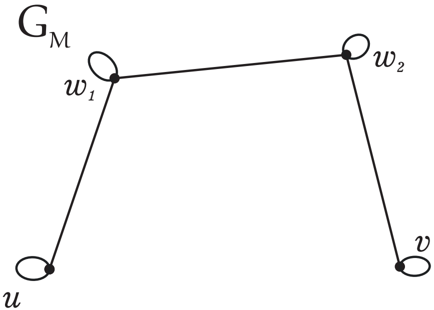
If $\mathcal{M}$ is HH then

$$\forall u, v, x, y \in M \ \ (\delta_{\mathcal{M}}(u, v) < \delta_{\mathcal{M}}(x, y) < \infty \Longrightarrow d_{\mathcal{M}}(u, v) < d_{\mathcal{M}}(x, y)). \quad (\maltese)$$

$\looparrowright$ *Distances make a partition order in the natural order.*

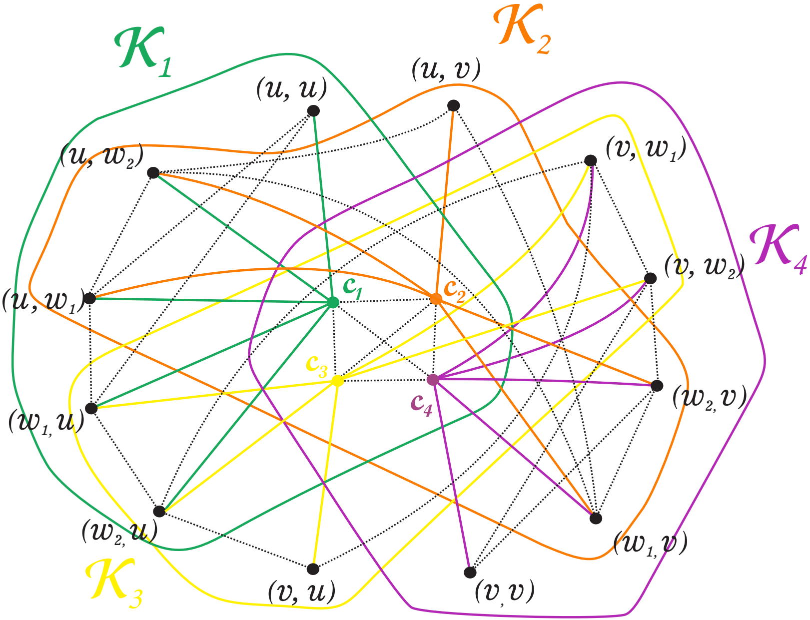
Lets visualise it!





A point of $\mathcal{M}$ is a **middle point** of a (k)-set in $\mathcal{M}$ in case it is at most at distance $\left\lceil \frac{k}{2} \right\rceil$ in $\mathbf{G}_{\mathcal{M}}$ from all the points of that (k)-set.

Middle points of $\mathcal{M}^2$



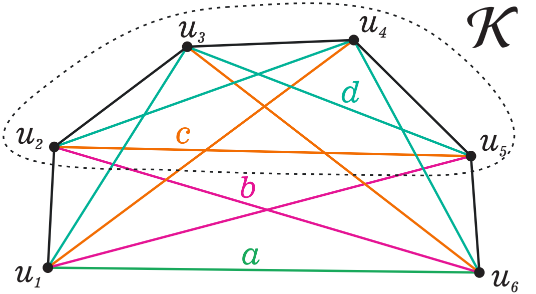
The intuition behind middle points

Let $\mathcal{M}$ be a skeletal metric space with connected skeleton.

Let $\mathcal{L}$ be a maximal (k)-set in $\mathcal{M}$, where k is even. If $\mathcal{L}$ has a middle point x then

$$\mathcal{L} = \left\{ y \in M \mid \delta_{\mathcal{M}}(x, y) \leqslant \frac{k}{2} \right\}.$$

Note: $\mathcal{L}$ is a metric ball.



$1 < d < c < b < a$

The mighty middle points

Let $\mathcal{M}$ be a PH skeletal metric space with connected skeleton.

Proposition

There exists a middle point of any (k)-set in $\mathcal{M}$, where $k \leqslant \mathrm{diam}(\mathbf{G}_{\mathcal{M}})$.

Important observation

Every maximal (2)-set in $\mathcal{M}$ possesses a middle point, and is associated with a different one.

Proposition

Let $\mathcal{M}$ be of diameter 3, and suppose that each maximal (2)-set in it has a middle point. Then:

- there exists a cover of $\mathcal{M}$ which consists of maximal (2)-sets, and
- middle points of those maximal (2)-sets induce a complete graph in $\mathbf{G}_{\mathcal{M}}$.

Main result

Theorem

Let $\mathcal{M}$ be a skeletal metric space with connected skeleton. Then:

- If $\mathrm{diam}(\mathbf{G}_{\mathcal{M}}) = 2$, then

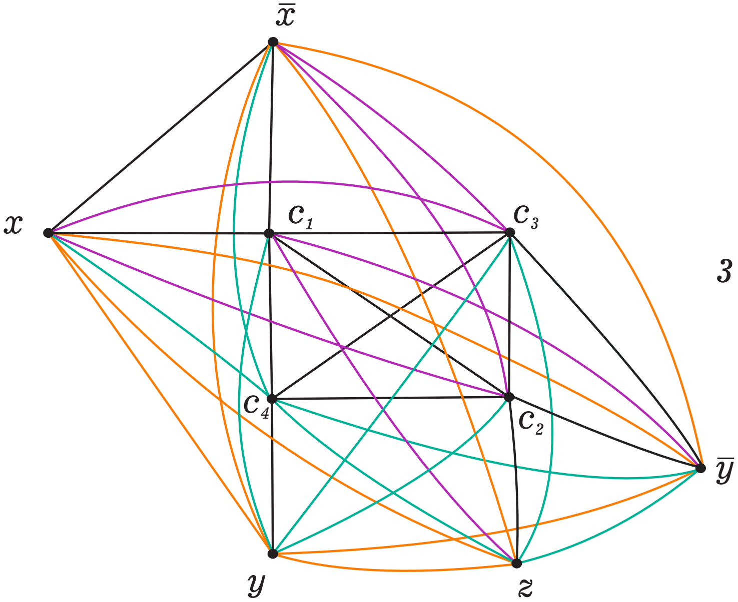
  $\mathcal{M}$ is PH iff its skeleton has a universal vertex.

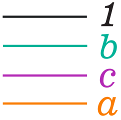
- If $\mathrm{diam}(\mathbf{G}_{\mathcal{M}}) = 3$, then

  $\mathcal{M}$ is PH whenever all of the following is satisfied:

  - (✠) holds,
  - the intersection of all of its maximal (2)-sets is non-empty,
  - every maximal (2)-set in $\mathcal{M}$ contains a middle point, and
  - there exists only one 2-distance.

A non-PH skeletal metric spaces with $\mathrm{diam}(\mathbf{G}_{\mathcal{M}}) = 3$





$3 \geqslant a > c > b > 1$

Skeletal metric spaces with disconnected skeletons

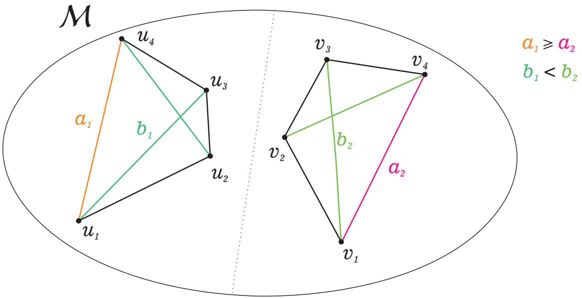
Let $\mathcal{M}$ be a skeletal metric space.

Let $P \subseteq M$ be such that $\mathbf{G}_{\mathcal{M}}[P]$ is a connected component of $\mathbf{G}_{\mathcal{M}}$. Then, $\mathcal{M}[P]$ is referred to as an $\mathcal{M}$-**connected component.**

Proposition

If $\mathcal{M}$ is PH then all the $\mathcal{M}$-connected components are PH, too.

The opposite implication, however, does **not** hold!



Metrically PH connected graphs

A connected graph is **metrically PH** if it is PH when considered as a metric space in the graph metric.

Theorem

Given:

- $\mathbf{H}$ - a finite connected graph, $\mathrm{diam}(H) \leqslant 3$
- $\overline{\mathbf{H}} := (V(\mathbf{H}), d_{\mathbf{H}})$

$\mathbf{H}$ is metrically PH iff

- either $\mathrm{diam}(\mathbf{H}) \leqslant 1$
- or the following conditions hold:
  1) $\overline{\mathbf{H}}$ is a skeletal metric space with connected skeleton,
  2) every maximal (2)-set in $\overline{\mathbf{H}}$ contains a middle point each, and
  3) the intersection of all the maximal (2)-sets in $\overline{\mathbf{H}}$ is non-empty.

The search continues!

Open problems:

1. Does there have to exist **only one** single 2-distance in a PH skeletal metric space with a connected skeleton of diameter 3?
2. What about skeletal metric spaces with connected skeletons of diameter $\geqslant 4$?
3. Why not consider the correlation between HH graphs and metrically HH graphs of diameter 2.

Thank you for your attention! :)