Topological Drawings meet Classical Theorems of Convex Geometry

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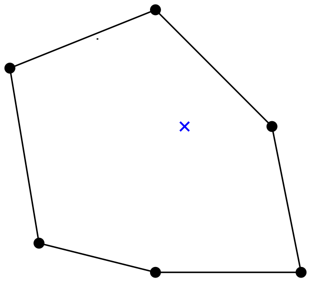
Theorems of Convex Geometry

- Carathéodory's Theorem
- Colorful Carathéodory Theorem
- Helly's Theorem
- Radon's Theorem
- Tverberg's Theorem

Theorem (Carathéodory)

For $P \subseteq \mathbb{R}^2$ and a point $x \in \operatorname{conv} P$ there are points $p_0, p_1, p_2 \in P$ such that x is inside the triangle p_0, p_1, p_2.



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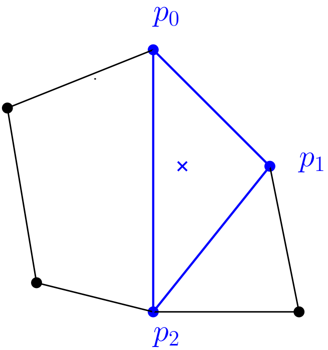

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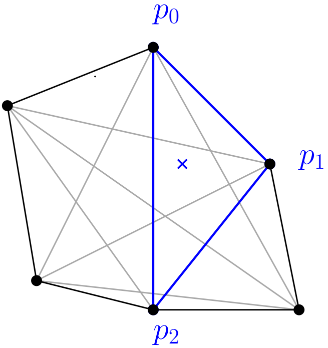

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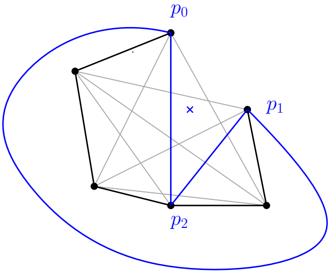

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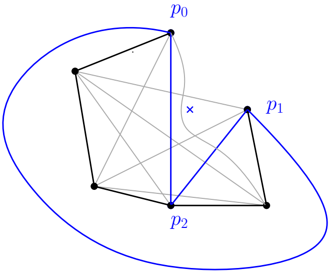
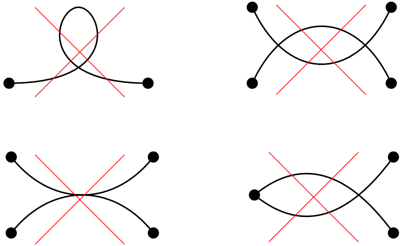


Topological Drawing

A topological drawing is a drawing of a complete graph such that:



Hierarchy [Arroyo, McQuillan, Richter, Salazar '17]

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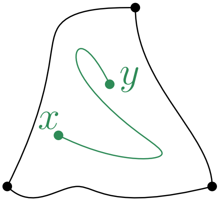
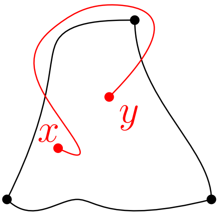
2. Convex drawing: every triangle has a *convex side*.





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5. straightline drawing

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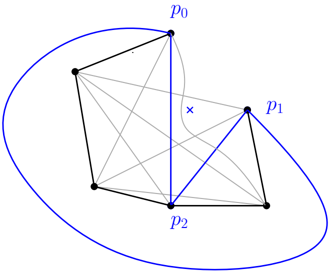

Carathéodory's Theorem in Topological Drawings

Theorem (Balko, Fulek, Kynčl '15)

Let D be a topological drawing of K_n, $x \in \mathbb{R}^2$ a point in a bounded connected component of $\mathbb{R}^2 - D$.
Then there is a triangle which contains x in the interior.

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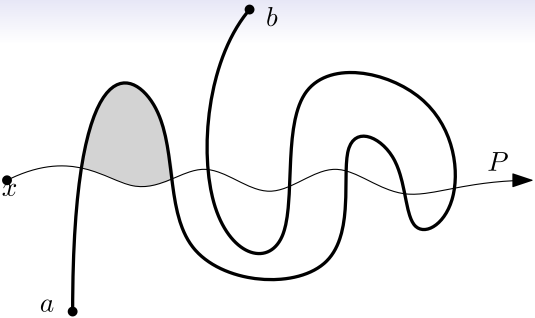
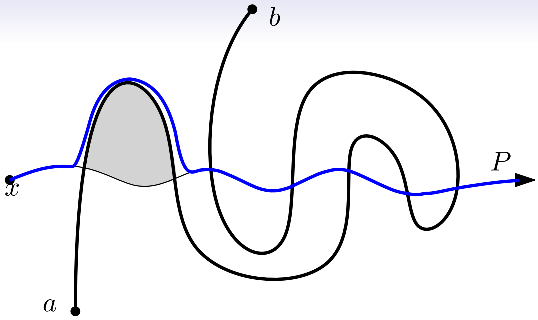
- Then there is a path P in $D' - ab$ which connects x to infinity. Choose P with no crossings in $D' - ab$ and minimal crossings with ab.
- **Claim:** P has exactly <u>one</u> crossing with ab.

□



- a is the vertex, we started to delete edges, i.e. b is still connected to all vertices in D'
- P does not cross an edge in $D' - ab$ and minimal number of crossings in D' with the edge ab
- Consider another path P' with fewer crossings with the edge ab. By Minimality an edge crossing P' exists
- c is connected to b. Contradiction.



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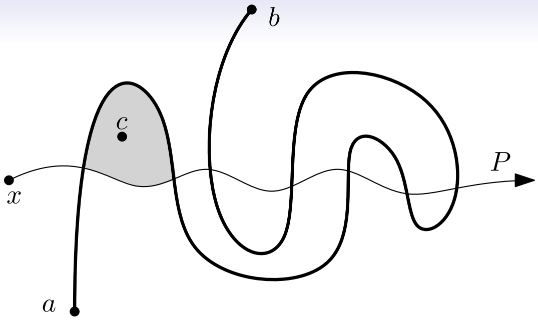

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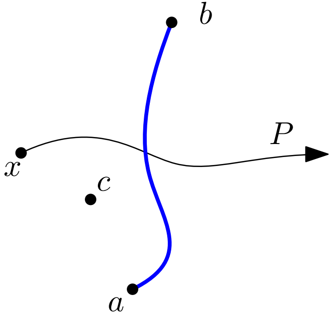

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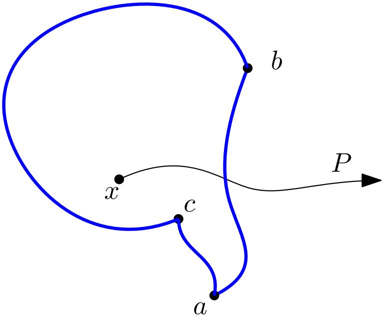

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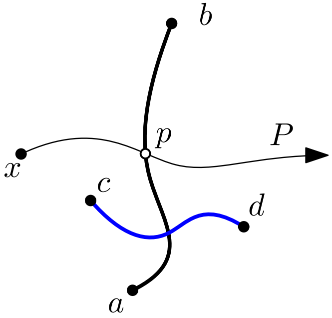

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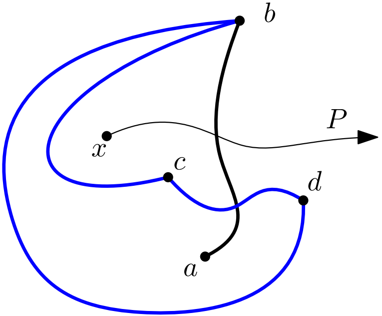

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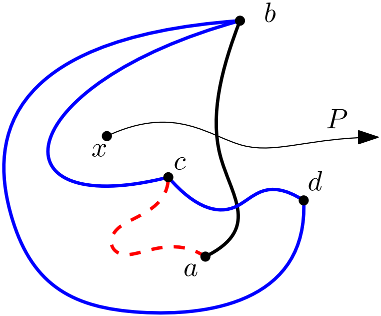
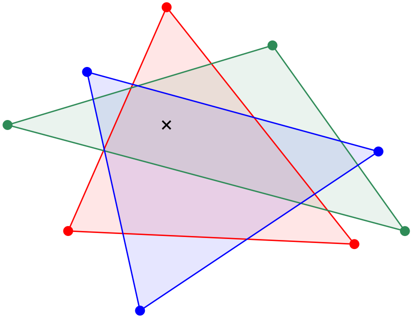


Colorful Carathéodory

Theorem (Bárány '82)

Consider the sets $P_0, P_1, P_2 \subset \mathbb{R}^2$, $x \in \mathbb{R}^2$.
If $x \in \operatorname{conv} P_i$ *for all* $i \in \{0,1,2\}$, *there are* $p_i \in P_i$ *for every* $i \in \{0,1,2\}$ *such that* $x \in \operatorname{conv}\{p_0, p_1, p_2\}$.



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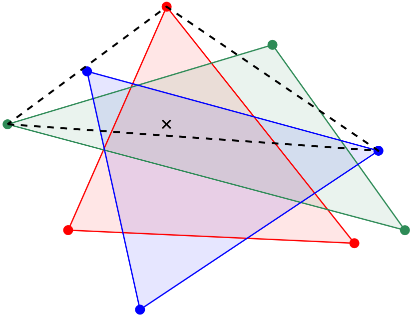

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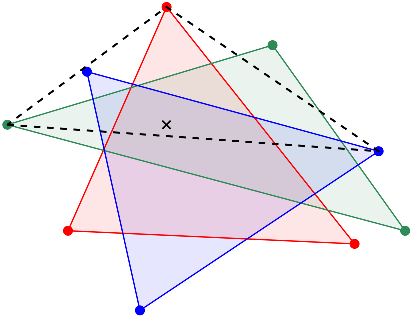
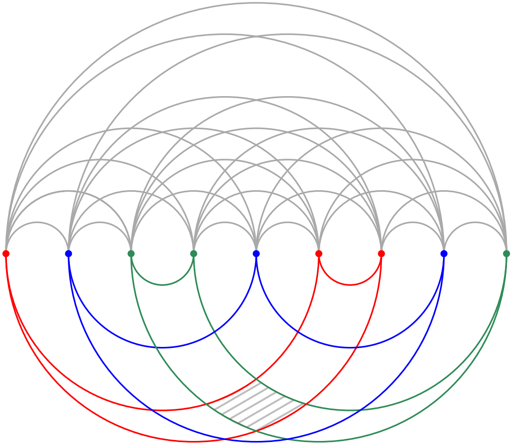


For pseudolinear drawings the Colorful Carathéodory Theorem holds. [Holmsen '16]

Colorful Carathéodory in Topological Drawings

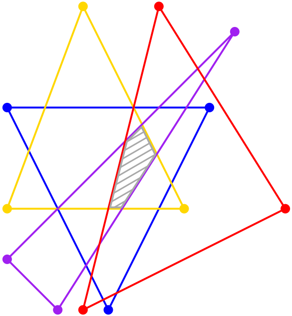


Counterexample of Colorful Carathéodory for *pseudocircular drawing*.

Helly's Theorem

Theorem

Let $V_1, \ldots, V_n$ convex subsets of $\mathbb{R}^2$. If every 3 subsets have a non-empty intersection, then the intersection of all n subsets is non-empty.



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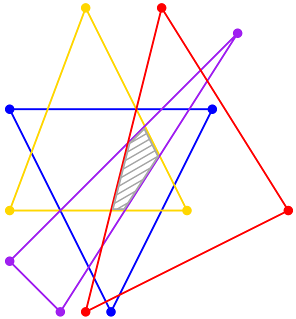
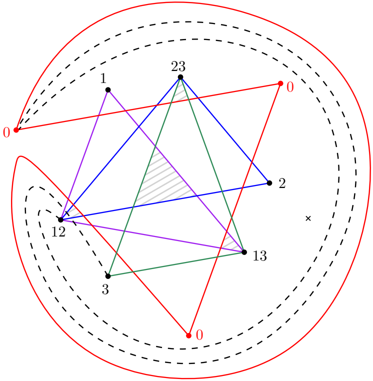


Is valid for pseudolinear drawings. [Bachem, Wanka '88] & [Goodman, Pollack '82]

Helly's Theorem in Topological Drawings



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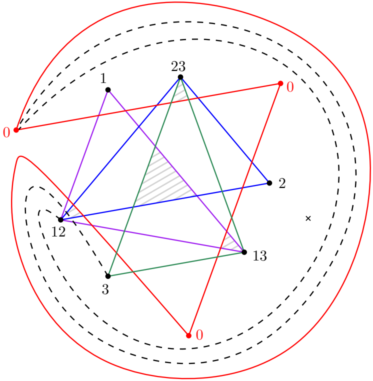

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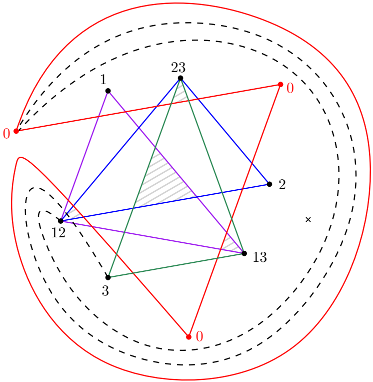
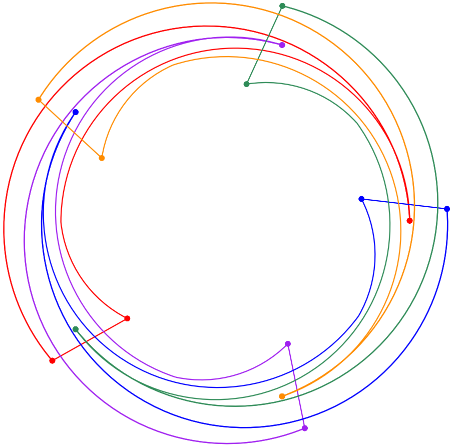


This drawing is pseudocircular.
Question: Is there a finite number $k \geq 3$ such that Helly's Theorem holds with k instead of $d + 1$ (Helly Number) ?

Helly number



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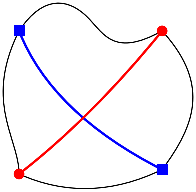

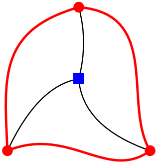

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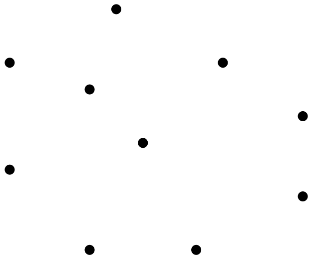
Theorem (Tverberg '66)

Each set of $(d+1)(r-1)+1$ *points in* $\mathbb{R}^d$ *can be partitioned into* r *subsets such that the convex hulls intersect.*



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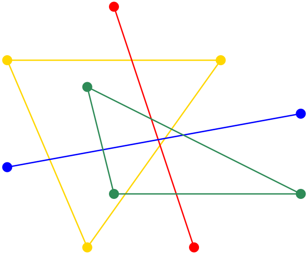

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- Roudneff showed Tverberg for pseudolinear drawings ['88].

Other Classical Theorems

- Birch's Theorem
- Kirchberger's Theorem
- Ramsey Theorem
- ...

Thank you for your attention!