Fractional version of the domination game *
(Extended abstract)

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Abstract

We introduce and study the fractional version of the graph domination game, where the moves are ruled by the condition of fractional domination. Fundamental properties of this new game are proved, including the existence of an optimal strategy for each player.

1 Preliminaries

In this paper we state results on a competitive optimization game concerning graph domination. We should note at the beginning, however, that the main theorems can be formulated and proved on a higher level of generality, namely for vertex cover in hypergraphs (also called transversal, hitting set, blocking set in different areas of discrete mathematics and computer science) which is also equivalent to set cover by hypegraph duality. The vertex cover formulation has implications not only for the classical version of graph domination, but also for a variant called total domination. Proofs — in the more general setting — will be published in [9].

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We deal with finite undirected graphs $G = (V, E)$. A vertex $v \in V$ *dominates* itself and its neighbors; that is, exactly the vertices contained in the *closed neighborhood* $N[v]$ of v. A subset $S \subset V$ is called *dominating set* if every vertex of G is dominated by at least one vertex from S, i.e. $\bigcup_{v \in S} N[v] = V$. The smallest size of a dominating set in G is called the *domination number* and is denoted by $\gamma(G)$.

The *domination game*, introduced in [2], is a competitive optimization version of graph domination. It is played on a graph G by two players, namely Dominator and Staller, who take turns choosing a vertex such that at least one previously undominated vertex becomes dominated.[1] The game is over when all vertices are dominated, and the length of the game is the number of vertices chosen by the players. Dominator wants to finish the game as soon as possible, while Staller wants to delay the end. Assuming that both players play optimally and Dominator starts, the length of the game on G is uniquely determined; it is called the *game domination number* of G and is denoted by $\gamma_g(G)$. Analogously, the *Staller-start game domination number* of G, denoted by $\gamma'_g(G)$, is the length of the game under the same rules when Staller makes the first move.

Below we shall refer to these games as *integer games*, as opposed to their fractional versions which will be introduced.

Following [2], the domination game has been studied further in many papers, see e.g. [1, 3, 4, 11, 14, 18, 19, 20, 21, 22]. The notion also inspired the introduction of the total domination game on graphs [5, 10, 15, 16, 17], transversal game [6, 7] and domination game on hypergraphs [8].

2 Fractional domination game

A *fractional dominating function* of $G = (V, E)$ is a function $f : V \mapsto [0, 1]$ if $\sum_{u \in N[v]} f(u) \geq 1$ holds for every vertex v in G. The minimum of the sum $\sum_{v \in V} f(v)$ over all such f is called the *fractional domination number* $\gamma^*(G)$ of G, introduced in [12, 13]. Observe that a dominating set corresponds to a fractional dominating function f where every vertex is assigned to either 0 or 1.

A function $d : V \mapsto [0, 1]$ (omitting the local condition above on the vertices) is called a *partially dominating function*; we denote by $|d|$ the sum

[1]The condition turns out to be restrictive only for Staller.

$\sum_{v\in V} d(v)$. Given a partially dominating function d, the associated *(domination) load function* is the function ℓ defined on V as

$$\ell(v) = \ell(v,d) = \min\{1, \sum_{u\in N[v]} d(u)\}$$

for every vertex v.

The *fractional domination game* starts with the all-0 load function $\ell(v) \equiv 0$, and is finished when the all-1 function $\ell(v) \equiv 1$ is reached. Dominator and Staller take turns making moves of weight 1 each, except possibly in the last move which may be smaller. A *move* is a sequence $(v_{i_1}, w_1), (v_{i_2}, w_2), \dots$ of arbitrary length, with its *submoves* (v_{i_k}, w_k) $(k = 1, 2, \dots)$ where $v_{i_1}, v_{i_2}, \dots$ are vertices of G; any number of repetitions are allowed. Here $w_1, w_2, \dots$ are real numbers from $(0, 1]$; it is required that

$$\sum_{k\geq 1} w_k = 1$$

in each move except the last one, whereas $\sum_{k\geq 1} w_k \leq 1$ must hold in the last move.

At the beninning the partially dominating function is $d_0 \equiv 0$ and also the load function is $\ell_0 \equiv 0$. After the i^{th} move $(i = 1, 2, \dots)$ the values $d_i(v_j)$ are calculated to obtain the new partially dominating function d_i by the rule

$$d_i(v_j) = d_{i-1}(v_j) + \sum_{i_k=j} w_k$$

from which the corresponding load function ℓ_i is also derived. A move $(v_{i_1}, w_1), (v_{i_2}, w_2), \dots$ is legal if

($*$) For all $k = 1, 2, \dots,$ there exists a vertex $u \in N[v_{i_k}]$ with

$$\ell_{i-1}(u) + \sum_{u\in N[v_{i_s}],\ \ 1\leq s\leq k-1} w_s \leq 1 - w_k.$$

That is, in each submove there must exist a vertex whose load increases by exactly the weight in the submove.

The *value of the game* $\mathcal{G}$ is $|\mathcal{G}| = |d_q|$, provided that the all-1 load function is reached after a sequence of legal moves in the q^{th} move; i.e., $\ell_q \equiv 1$.

Analogously to the integer game, also here Dominator wants a small $|\mathcal{G}|$, while Staller wants a large $|\mathcal{G}|$. To define the *fractional game domination number* $\gamma_g^*(G)$, assume that Dominator starts the game on G, and consider the set

$$D_G = \{a : \text{Dominator has a strategy which ensures } |\mathcal{G}| \leq a\}.$$

Now, the fractional game domination number is defined as $\gamma_g^*(G) = \inf D_G$. Assuming that Staller starts the game, the Staller-start fractional game domination number $\gamma_{Sg}^*(G)$ is defined similarly.

3 General results

As mentioned in the preliminaries, the main results can be stated in the more general framework of vertex cover; here we formulate them for the domination game only.

Optimal strategies. Consider the following set which correspont to D_G from Staller's viewpoint:

$$S_G = \{b : \text{Staller has a strategy which ensures } |\mathcal{G}| \geq b\}.$$

Theorem 1. *For every graph G,*

$$\inf(D_G) = \min(D_G) = \sup(S_G) = \max(S_G) = \gamma_g^*(G)$$

and the corresponding equalities also hold for the Staller-start fractional game domination number $\gamma_{Sg}^*(G)$.

Finite moves. The definition of legal move admits the option that a player splits the value 1 into an infinite number of pieces; e.g., $w_k = 2^{-k}$ may also be feasible (with suitable v_{i_k} respecting $(*)$). It turns out, however, that each player can design an optimal strategy without moves consisting of infinitely many submoves.

Theorem 2. *For every graph G, each player has an optimal strategy such that, in every move, each vertex occurs in at most one submove.*

The Rational Game. One may consider a restricted version of the fractional domination game, in which the players are required to use only *rational* weights w_k in each submove. We define $\gamma_g^{\mathbb{Q}}(G)$ to be $\sup S_G = \inf D_G$ under this extra condition. Also for the Staller-start game, the notation $\gamma_{Sg}^{\mathbb{Q}}(G)$ can be introduced.

Theorem 3. *For every graph G, the equalities $\gamma_g^{\mathbb{Q}}(G) = \gamma_g^*(G)$ and $\gamma_{Sg}^{\mathbb{Q}}(G) = \gamma_{Sg}^*(G)$ are valid.*

Continuation Principle. A monotone property of the fractional domination number is expressed in the following idea, which provides a useful tool in simplifying several arguments. Given G and a load function ℓ, let the fractional game ℓ-domination number be denoted by $\gamma_g^*(G|\ell) = \inf(D|\ell) = \sup(S|\ell)$, where $\inf(D|\ell)$ and $\sup(S|\ell)$ are defined analogously to D_G and S_G, for the game starting with a load function ℓ on G with the move of Dominator.

Theorem 4. *If ℓ_1 and ℓ_2 are load functions on G such that $\ell_1(v) \leq \ell_2(v)$ holds for every $v \in V(G)$, then $\gamma_g^*(G|\ell_1) \geq \gamma_g^*(G|\ell_2)$.*

An immediate consequence is

Theorem 5. *The fractional game domination numbers for the Staller-start and for the Dominator-start games on G may differ by at most 1.*

4 Paths and cycles

The determination of exact values for $\gamma_g^*(G)$ seems hard, even if G has a very simple structure. We have estimates for paths and cycles in which the difference of the upper and lower bounds does not exceed a small constant; yet the numbers do not coincide. The currently known best bounds for cycles are summarized in Table 1.

5 Comparison of domination parametes

Proposition 6. *For any graph G, the following inequalities hold:*

- $\gamma^*(G) \leq \gamma_g^*(G) < 2\gamma^*(G)$

	$\leq \gamma_g^*(C_n)$	$\gamma_g^*(C_n) \leq$
$n \equiv 0 \pmod 4$	$\frac{n}{2} - \frac{4}{3} + \frac{4}{3n}$	$\frac{n}{2}$
$n \equiv 1 \pmod 4$	$\frac{n}{2} - \frac{3}{2} + \frac{4}{3n}$	$\frac{n}{2} - \frac{1}{6}$
$n \equiv 2 \pmod 4$	$\frac{n}{2} - 1 + \frac{4}{3n}$	$\frac{n}{2} - \frac{1}{3}$
$n \equiv 3 \pmod 4$	$\frac{n}{2} - \frac{7}{6} + \frac{4}{3n}$	$\frac{n}{2} - \frac{1}{2}$

Table 1: Lower and upper bounds for cycles.

- $\gamma^*(G) \leq \gamma_{Sg}^*(G) < 2\gamma^*(G) + 1$
- *There does not exist any universal constant $c > 0$ with*

$$c\gamma_g(G) \leq \gamma_g^*(G).$$

- *If every block of G is a complete graph (and in particular if G is a tree), then*

$$\frac{\gamma_g(G) + 1}{2} \leq \gamma(G) = \gamma^*(G) \leq \gamma_g^*(G).$$

6 Some open problems

We close these notes with three conjectures and a further problem.

Conjecture 1. *If each of the first $2k - 1$ ($k \geq 1$) moves was integer move, i.e. of the form $(v_{i_1}, 1)$, then Staller has an integer move in the $(2k)^{\text{th}}$ turn, which is optimal in the fractional game.*

This means that fractional moves would be advantageous for Dominator only. If true then this conjecture implies the following weaker one.

Conjecture 2. *For every graph G,* $\gamma_g^*(G) \leq \gamma_g(G)$.

Conjecture 3. *For every isolate-free graph G,* $\gamma_g^*(G) \leq 3n/5$.

Problem 1. *Is there a constant $c < 3/5$ such that $\gamma_g^*(G) \leq c \cdot n$ holds for every isolate-free G?*

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