A rainbow version of Mantel’s Theorem

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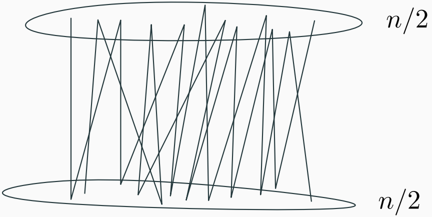
Motivation

Theorem (Mantel, 1907)

Suppose $V(G) = \{1, \ldots, n\}$ and G has $e(G) > n^2/4$ edges. Then G contains a triangle.

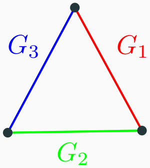
- and this is tight



Rainbow version

Theorem (ADGMS, 2018+; Culver, Lidický, Pfender, Volec 2018+)

*Suppose for $i = 1, 2, 3$ we have $V(G_i) = \{1, \dots, n\}$ and G_i has $e(G_i) > \alpha n^2$ edges for $\alpha = 0.2557\ldots = \frac{26-2\sqrt{7}}{81}$.
Then there is a rainbow triangle.*



- The result is asymptotically tight: we cannot decrease α.

Rainbow version

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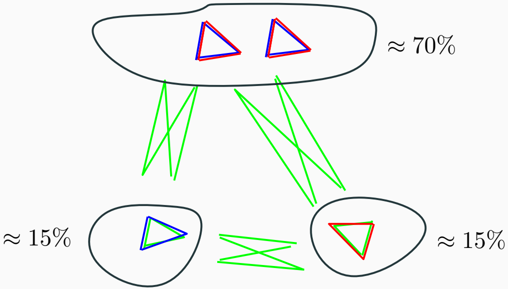


The approach

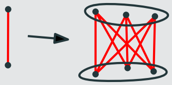
Lemma

Suppose $e(G_i) + e(G_j) \geq 2\alpha n^2 + \frac{3}{2}n$ for $1 \leq i < j \leq 3$. Then there is a rainbow triangle.

Lemma $\Rightarrow$ Theorem.

For contradiction, suppose that for some n, G_1, G_2, G_3 and ε we have $e(G_i) \geq (\alpha + \varepsilon)n^2$ and no rainbow triangle.

Then the "k-blowup" of G_i has



- kn vertices
- $\geq \alpha(kn)^2 + (\varepsilon kn)kn$ edges. We only need $\varepsilon kn \geq 3/4$.

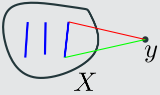
□

Property of minimal counterexample

Lemma

A counterexample to The Lemma with n minimum does not have $\emptyset \neq X \subsetneq V$ for which every $G_i[X]$ has a perfect matching.

Proof.



No rainbow triangle: $e_1(y, X) + e_2(y, X) \leq |X|$

Summing over all $y \in \bar{X}$:

$$e_1(\bar{X}, X) + e_2(\bar{X}, X) \leq |X| \cdot |\bar{X}|$$

Similarly, $e_1(X) + e_2(X) \leq |X|^2/2$

This gives a lower bound on $e(G_1 - X) + e(G_2 - X)$.
After some calculation: $G_1 - X$, $G_2 - X$, $G_3 - X$ are a smaller counterexample. □

Property of minimal counterexample

Lemma

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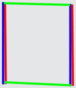
Corollary

Minimal counterexample does not have a triple edge 

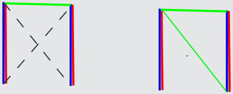
or a bad 4-cycle. 

Counting edges

Lemma

In a minimal counterexample, we have between two red-blue edges one of the following:

- *no green edge*
- *1 green edge, ≤ 2 red edges, ≤ 2 blue edges*
- *2 green edges, no red & no blue edges*



Proof.

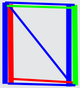
Easy case analysis. □

Counting edges II

Lemma

In a minimal counterexample, we have between a red-blue edge and a blue-green edge:

- *at most 5 edges*
- *the only way to get 5 edges is to have 3 blue ones, 1 red and one green.*



Proof.

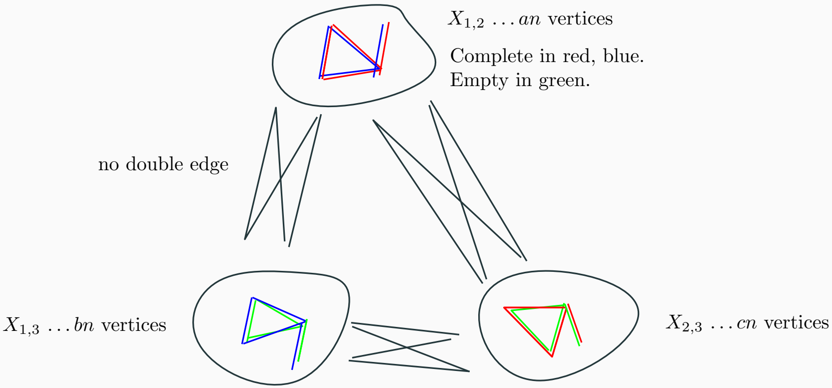
Easy case analysis. □

The proof sketch

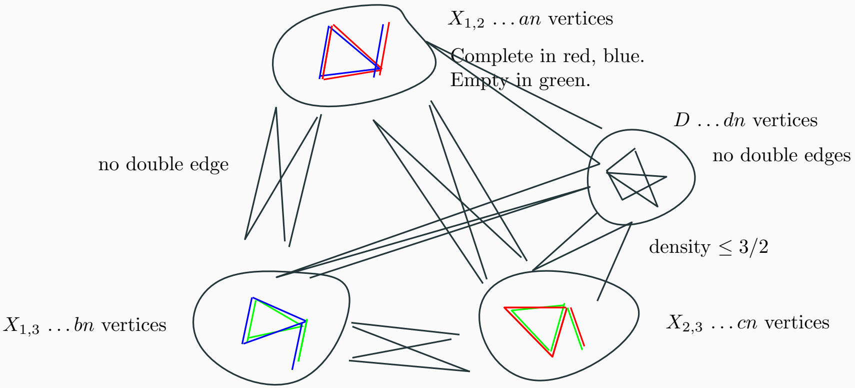
Consider a minimal counterexample.

- No triple edge.
- M – maximum matching of double edges.
- $M_{i,j}$ — edges of M that have color i and j
- $X_{i,j}$ vertices covered by $M_{i,j}$.
- $D = V \setminus (X_{1,2} \cup X_{2,3} \cup X_{1,3})$
- Rearrange the edges to preserve densities between $X_{1,2}$ and $X_{1,3}$ and also between ...
- Next, we get a system of inequalities.

The proof sketch

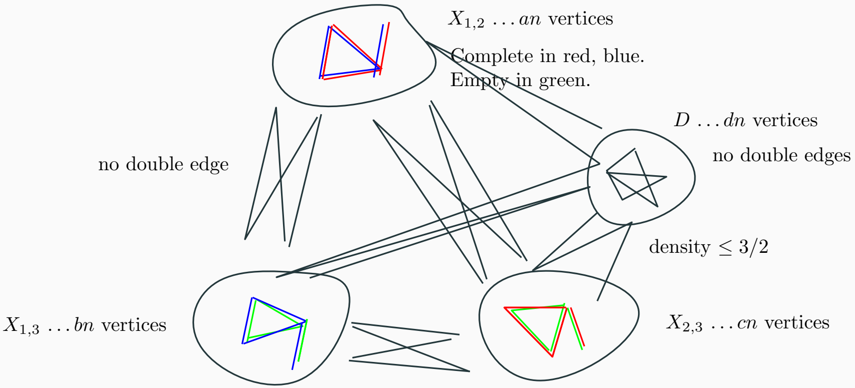


The proof sketch



$$a + b + c + d = 1$$

The proof sketch



$e(G_1'') + e(G_2'') + e(G_3'') \leq \binom{n}{2} + \binom{an}{2} + \binom{bn}{2} + \binom{cn}{2} + \frac{1}{2}(a+b+c)n \cdot dn$
which gives us the inequality

$$6\alpha \leq 1 + a^2 + b^2 + c^2 + d(a+b+c)$$

Future work

Problem (Nonsymmetric version)

For what real numbers $\alpha_1, \alpha_2, \alpha_3 > 0$ *is it true that every triple of graphs* G_1, G_2, G_3 *satisfying* $|E(G_i)| > \alpha_i n^2$ *must have a rainbow triangle?*

Problem (Rainbow Turán theorem)

For every positive integer r*, what is the smallest real number* δ_r *so that whenever* $G_1, \ldots, G_{\binom{r}{2}}$ *are graphs on a common set of* n *vertices with* $|E(G_i)| \geq \delta_r n^2$ *for every* $1 \leq i \leq \binom{r}{2}$ *there exists a "rainbow"* K_r *(i.e., a set of* r *vertices and one edge from each* G_i *that together form a clique on this set of vertices).*