Finite big Ramsey degrees in countable universal structures

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Structural Ramsey property

K — a class of finite structures

Definition. **K** has the Ramsey property if for all $k \geqslant 2$ and all $\mathcal{A}, \mathcal{B} \in$ **K** there is a $\mathcal{C} \in$ **K** such that $\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$.

Here, $\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$ means:

> for every coloring $\chi : \mathrm{Emb}(\mathcal{A}, \mathcal{C}) \to k$
> there is a $w \in \mathrm{Emb}(\mathcal{B}, \mathcal{C})$
> such that $|\chi(w \circ \mathrm{Emb}(\mathcal{A}, \mathcal{B}))| \leqslant 1$

NB. Coloring embeddings VS coloring substructures

Alas...

The class of finite graphs does not have the Ramsey property.

Ramsey property $\Rightarrow$ severe restrictions on Aut($\mathcal{A}$) for $\mathcal{A} \in$ **K**:

- either Aut($\mathcal{A}$) $=$ Sym(A) [e.g. sets, complete graphs],
- or Aut($\mathcal{A}$) $= \{\text{id}_A\}$ [e.g. ordered graphs].

Is there a "Ramsey property" for the class of finite graphs?

(Small) Ramsey degrees

K — a class of finite structures

$\mathcal{A}$ — a structure in **K**.

$t \in \mathbb{N}$ is a (small) Ramsey degree of $\mathcal{A}$ in **K** if
t is the least integer such that
for all $k \geqslant 2$ and all $\mathcal{B} \in$ **K** there is a $\mathcal{C} \in$ **K** with $\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_{k,t}$.

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$$t(\mathcal{A}, \mathbf{K}) = \begin{cases} t, & \text{if such an integer exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Ex. Finite graphs, posets, metric spaces etc have (small) Ramsey degrees in corresponding classes.

Going back to Ramsey's Theorem

Theorem. (Ramsey) For all $n \geqslant 1$ and $k \geqslant 2$ and every coloring $\chi : \binom{\omega}{n} \to k$ there is an infinite $S \subseteq \omega$ such that χ restricted to $\binom{S}{n}$ is constant.

In other words, $\omega \longrightarrow (\omega)^n_k$ for all $n \geqslant 1$ and $k \geqslant 2$.

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Is there a "corresponding Ramsey property" for $\mathbb{Q}$ and $\mathcal{R}$?

Big Ramsey degrees

$\mathcal{U}$ — a countably infinite structure

$\mathcal{A}$ — a finite structure

Definition. $T \in \mathbb{N}$ is a big Ramsey degree of $\mathcal{A}$ in $\mathcal{U}$ if
T is the least integer such that
for all $k \geqslant 2$ we have that $\mathcal{U} \longrightarrow (\mathcal{U})^{\mathcal{A}}_{k,T}$.

Here, $\mathcal{U} \longrightarrow (\mathcal{U})^{\mathcal{A}}_{k,T}$ means:

for every coloring $\chi : \mathrm{Emb}(\mathcal{A}, \mathcal{U}) \to k$
there is a $w \in \mathrm{Emb}(\mathcal{U}, \mathcal{U})$
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$$T(\mathcal{A},\mathcal{U}) = \begin{cases} T, & \text{if such an integer exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Big Ramsey degrees

Finite big Ramsey degrees in some Fraïssé limits:

- Finite chains … in $\mathbb{Q}$
  [Devlin 1979]
- Finite graphs … in the Rado graph $\mathcal{R}$
  Finite tournaments … in the random torunament $\mathcal{T}$
  (a few more graph-like structures)
  [Sauer 2006]
- Finite S-ultrametric spaces … in the “Urysohn space” $\mathcal{Y}_S$
  [Van Thé 2008]
- Finite local orders … in $\mathcal{S}(2)$ (and a few more)
  [Laflamme, Van Thé, Sauer 2010]
- Finite triangle-free graphs … in the Henson graph $\mathcal{H}_3$
  [Dobrinen 2018+]

Big Ramsey degrees

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Why?

Theorem. (Ramsey) For all $n \geqslant 1$ and $k \geqslant 2$ and every coloring $\chi : \binom{\omega}{n} \to k$ there is an infinite $S \subseteq \omega$ such that χ restricted to $\binom{S}{n}$ is constant.

In other words, $T(m, \omega) = 1$ for all m.

A piggyback result

Theorem. Assume the following:

- $\mathcal{F}$ is a countably infinite relational structure
- Age($\mathcal{F}$) has the strong amalgamation property
- **K** $= \{(\mathcal{A}, \prec) : \mathcal{A} \in$ Age($\mathcal{F}$) and $\prec$ is a linear order on *A* such that $(A, \prec)$ is finite or has order type $\omega\}$
- $\sqsubset$ is a linear order on *F* of order type ω.

Then:

- Age($\mathcal{F}, \sqsubset$) $=$ **K**.
- If $\mathcal{A}$ has finite big Ramsey degree in $\mathcal{F}$ then $(\mathcal{A}, \prec)$ has finite big Ramsey degree in $(\mathcal{F}, \sqsubset)$.

A piggyback result

Corollary.

- Every finite linearly ordered graph has finite big Ramsey degree in $(\mathcal{R}, \sqsubset)$, where $\sqsubset$ is a linear order on *R* of order type ω.
- Every finite permutation has finite big Ramsey degree in the permutation $(\mathbb{Q}, <, \sqsubset)$, where $<$ is the usual ordering of the rationals and $\sqsubset$ is a linear order on $\mathbb{Q}$ of order type ω.

NB. A *linearly ordered graph* is a pair $(G, <)$ where G is a graph and $<$ is a linear order on the vertices of G.

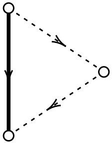
NB. A *permutation* is a structure $(A, <_1, <_2)$ where $<_1$ and $<_2$ are linear orders on A.

Acyclic digraphs

Theorem. There exists a countably infinite acyclic digraph $\mathcal{D}$ such that every finite acyclic digraph $\mathcal{A}$ has finite big Ramsey degree in $\mathcal{D}$.

A class of posets

Theorem. Let **K** be a class of all finite linearly ordered posets which omit



There exists a countably infinite linearly ordered poset $\mathcal{P}$ such that every $\mathcal{A} \in$ **K** has finite big Ramsey degree in $\mathcal{P}$.

NB. A linearly ordered poset $= (A, \leqslant, \prec)$ where $(A, \leqslant)$ is a poset and $\prec$ is a linear extension of $\leqslant$.

A class of metric spaces

Without going into technicalities:

Theorem. Let

- S be a "*special distance set*",
- $\mathcal{L}$ be a finite "S^+*-metric space*", and
- $\mathbf{K}_{S,\mathcal{L}}$ be the class of all fin. S-met.spcs "*spanned*" by $\mathcal{L}$.

Then

- $\mathbf{K}_{S,\mathcal{L}}$ is a "*sort of a Fraïssé class*",
- it has the Fraïssé limit $\mathcal{U}_{S,\mathcal{L}}$, and
- every $\mathcal{M} \in \mathbf{K}_{S,\mathcal{L}}$ has finite big Ramsey degree in $\mathcal{U}_{S,\mathcal{L}}$.

Tools

Let $\mathbb{C}$ be a category and $A, B, C \in \mathrm{Ob}(\mathbb{C})$.

$C \longrightarrow (B)^A_k$ if:

for every mapping $\chi : \hom(A, C) \to k$ there is a $\mathbb{C}$-morphism $w : B \to C$ such that $|\chi(w \cdot \hom(A, B))| = 1$.

A category $\mathbb{C}$ has the Ramsey property if:

for all $k \geqslant 2$ and all $A, B \in \mathrm{Ob}(\mathbb{C})$ such that $\hom(A, B) \neq \varnothing$ there is a $C \in \mathrm{Ob}(\mathbb{C})$ satisfying $C \longrightarrow (B)^A_k$

Tools

Let $\mathbb{C}$ be a category and $A, B, C \in \mathrm{Ob}(\mathbb{C})$.

$C \longrightarrow (B)^A_k$ if:

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$T \in \mathbb{N}$ is a big Ramsey degree of A in U if:

for all $k \geqslant 2$ we have that $U \longrightarrow (U)^A_{k,T}$.

Tools

Theorem. Let

- $\mathbb{B}$ and $\mathbb{C}$ be categories,
- $B \in \mathrm{Ob}(\mathbb{B})$ and $C \in \mathrm{Ob}(\mathbb{C})$,
- there is a forgetful functor $U : \overline{\mathrm{Age}}_{\mathbb{B}}(B) \to \overline{\mathrm{Age}}_{\mathbb{C}}(C)$,
- $U(B) = C$;
- if $U(B') = C$ then $\hom_{\mathbb{B}}(B, B') \neq \varnothing$; and
- for every $f \in \hom_{\mathbb{C}}(C, C)$ there is a $B' \in \mathrm{Ob}(\mathbb{B})$ such that $U(B') = C$ and $f \in \hom_{\mathbb{B}}(B', B)$.

Then $T_{\mathbb{B}}(A, B) \leqslant T_{\mathbb{C}}(U(A), C)$ for all $A \in \overline{\mathrm{Age}}_{\mathbb{B}}(B)$.

NB. $\overline{\mathrm{Age}}_{\mathbb{C}}(C) = \{A \in \mathrm{Ob}(\mathbb{C}) : \hom(A, C) \neq \varnothing\}$

Tools

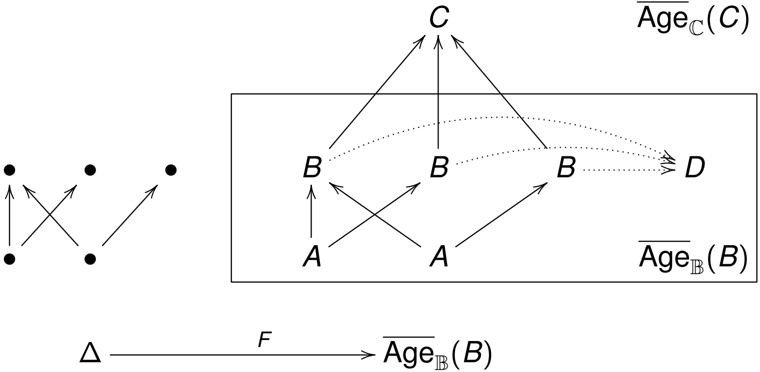
Theorem. Let

- $\mathbb{C}$ be a category whose every morphism is monic,
- $\mathbb{B}$ be a (not necessarily full) subcategory of $\mathbb{C}$,
- $B \in \mathrm{Ob}(\mathbb{B})$ and $C \in \mathrm{Ob}(\mathbb{C})$ be such that $\hom_{\mathbb{C}}(B, C) \neq \varnothing$,
- $A \in \overline{\mathrm{Age}}_{\mathbb{B}}(B)$
- for every (A, B)-diagram $F : \Delta \to \overline{\mathrm{Age}}_{\mathbb{B}}(B)$ the following holds: if F has a commuting cocone in $\overline{\mathrm{Age}}_{\mathbb{C}}(C)$ whose tip is C, then F has a commuting cocone in $\overline{\mathrm{Age}}_{\mathbb{B}}(B)$.

Then $T_{\mathbb{B}}(A, B) \leqslant T_{\mathbb{C}}(A, C)$.

Tools

Theorem. In other words,



Future work

Finite chains have finite big Ramsey deg's both in ω and in $\mathbb{Q}$.

Question: What other countable chains $\mathcal{C}$ have the property that $T(m, \mathcal{C}) < \infty$ for all m?

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Non-scattered countable chains ✓

Observation. If $\mathcal{C}$ is a non-scattered countable chain, then $T(m, \mathcal{C}) = T(m, \mathbb{Q})$.

Future work

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Question: What other countable chains $\mathcal{C}$ have the property that $T(m, \mathcal{C}) < \infty$ for all m?

Non-scattered countable chains ✓

Observation. If $\mathcal{C}$ is a non-scattered countable chain, then $T(m, \mathcal{C}) = T(m, \mathbb{Q})$.

Goal: Prove that scattered countable chains also have the property.

Easier Goal: Prove that countable ordinals have the property.