The $\mathbb{Z}_2$-genus of Kuratowski minors

Radoslav Fulek and Jan Kynčl

Charles University, Prague

Drawings and embeddings

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vertices $\rightarrow$ points

edges $\rightarrow$ simple curves

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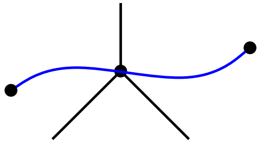

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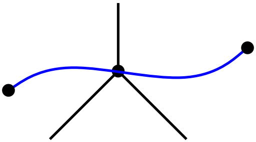

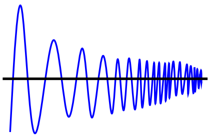

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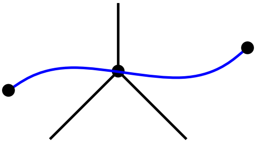

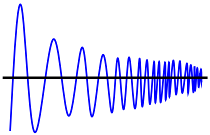

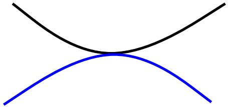

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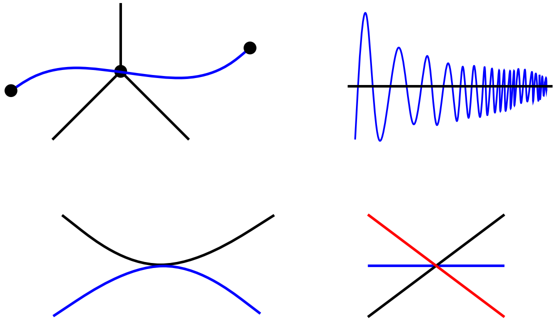

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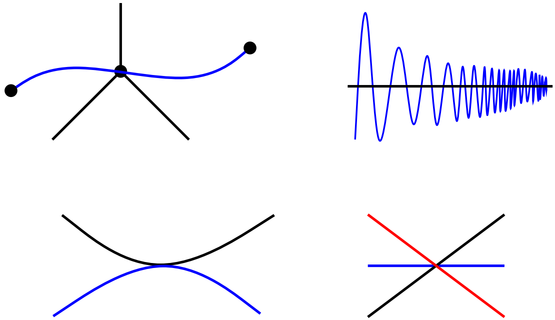

embedding = drawing with no crossings

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(Strong) Hanani–Tutte theorem: (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

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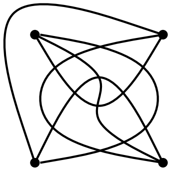

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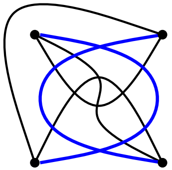

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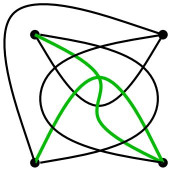

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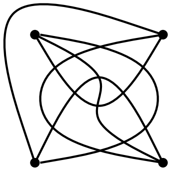

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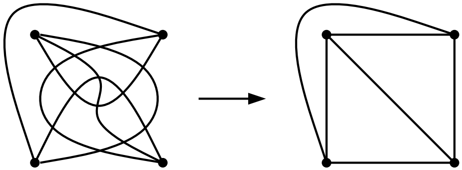

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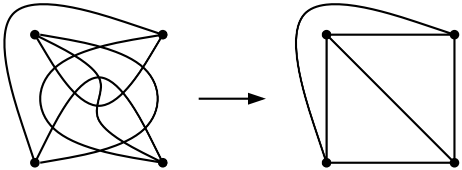

application: polynomial-time algorithm for testing planarity

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If a graph G has an **even** drawing D in the plane (every pair of edges crosses an even number of times), then G is planar.
(Moreover, G has a plane embedding with the same rotation system as D.)

Hanani–Tutte theorems on surfaces

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(Cairns–Nikolayevsky, 2000; Pelsmajer–Schaefer–Štefankovič, 2009)

If a graph G has an even drawing $\mathcal{D}$ on a surface S, then G has an embedding on S (that preserves the embedding scheme of $\mathcal{D}$).

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If a graph G has an independently even drawing on the projective plane, then G has an embedding on the projective plane.

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Partial answer: No to the orientable surface of genus 4 or larger
(Fulek–K., 2017)

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Theorem: (Fulek–K., 2017)
There is a graph G with $\mathbf{g}(G) = 5$ and $\mathbf{g}_0(G) \leq 4$.
Consequently, for every positive integer k there is a graph G with $\mathbf{g}(G) = 5k$ and $\mathbf{g}_0(G) \leq 4k$.

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Main result: YES, if a certain “folklore result” is true.

Bounding genus by $\mathbb{Z}_2$-genus—the plan

1) Ramsey-type statement:

If G has large genus $g = g(t)$, then G contains, as a minor, $G_1(t)$ or $G_2(t)$ or $\ldots$ or $G_r(t)$ of genus t.

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2) Easier subproblem:

Show that the $\mathbb{Z}_2$-genus of each of $G_i(t)$ is unbounded in t.

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Unpublished "folklore" result: (Robertson–Seymour)
There is a function g such that for every $t \geq 3$, every graph of Euler genus $g(t)$ contains a t-**Kuratowski graph** as a minor.

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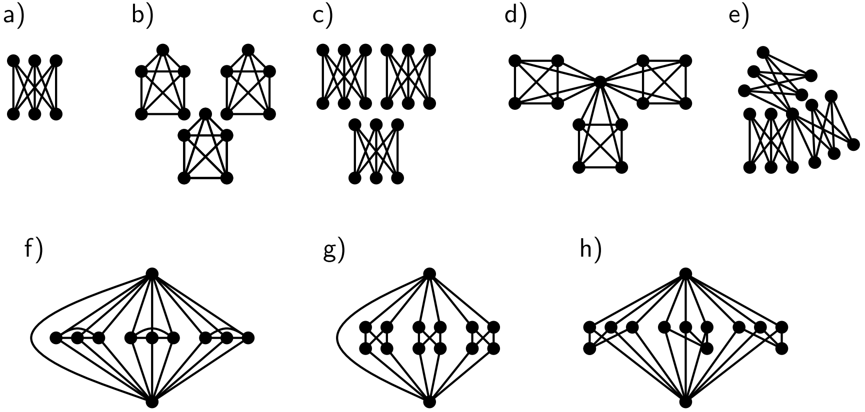
***t*-Kuratowski graph**:

- $K_{3,t}$, or
- t copies of K_5 or $K_{3,3}$ sharing at most 2 common vertices

3-Kuratowski graphs



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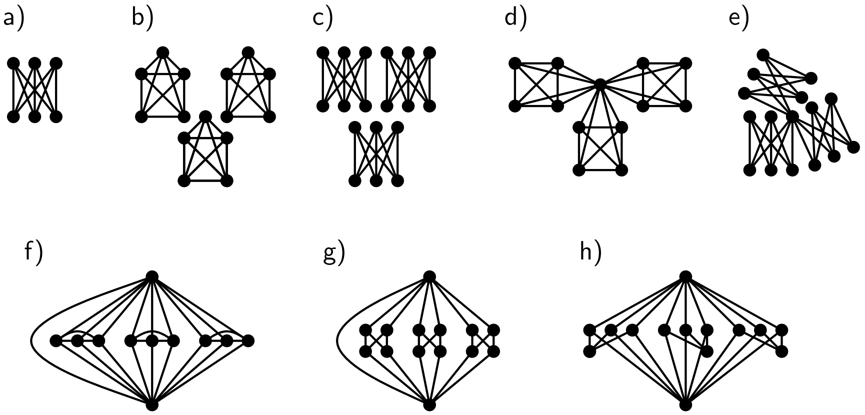
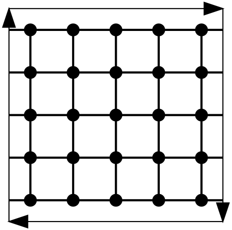


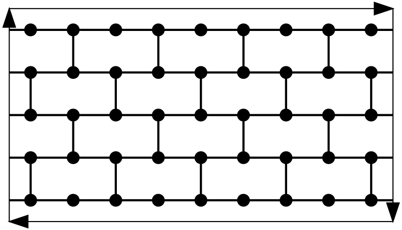
- what about graphs with large (orientable) genus and constant Euler genus?

Ramsey-type statement for genus

projective 5×5 **grid**



projective 5**-wall**



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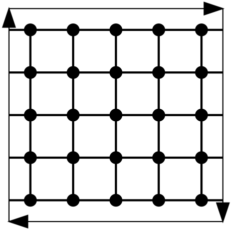

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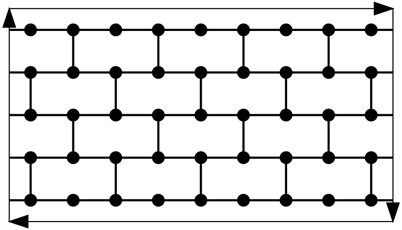

Theorem:
The “folklore result” implies that there is a function h such that for every $t \geq 3$, every graph of genus $h(t)$ contains, as a minor, a t-Kuratowski graph or the projective t-wall.

Lower bounds on the $\mathbb{Z}_2$-genus

Theorem: (Schaefer–Štefankovič, 2013)
If G consists of t copies of K_5 or $K_{3,3}$ sharing at most 1 vertex, then $\mathbf{g}_0(G) = \mathbf{g}(G) = t$.
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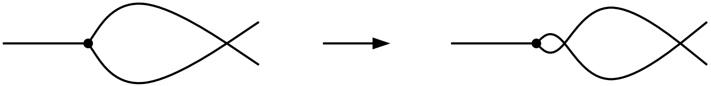
- "correct" the rotation of each vertex to obtain an even drawing



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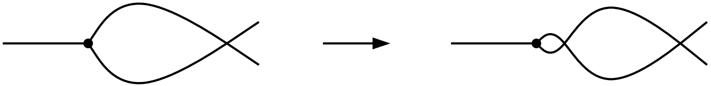

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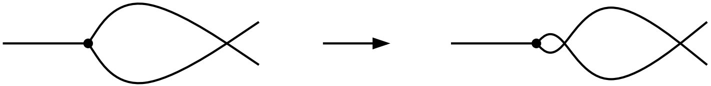

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Theorem:
We have $\mathbf{g}_0(G) = \mathbf{g}(G)$ also for each of the remaining t-Kuratowski graphs G: $K_{3,t}$ and 2-amalgamations of t copies of K_5 or $K_{3,3}$.

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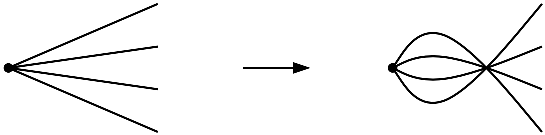

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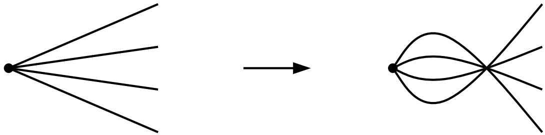

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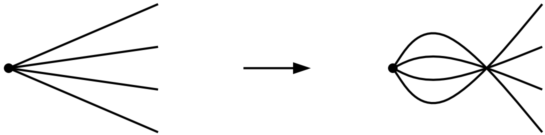

- use the weak Hanani–Tutte theorem for surfaces
- use the fact $\mathbf{g}(K_{3,n}) = \lceil (n-2)/4 \rceil$

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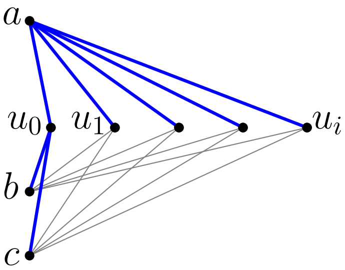

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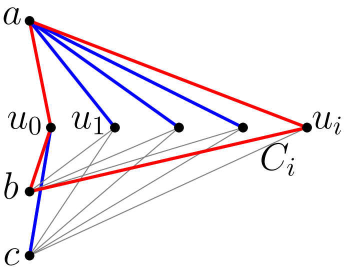

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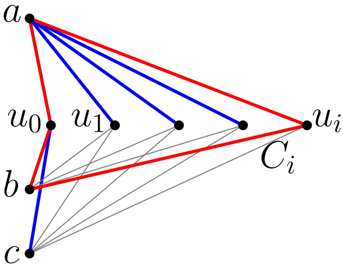

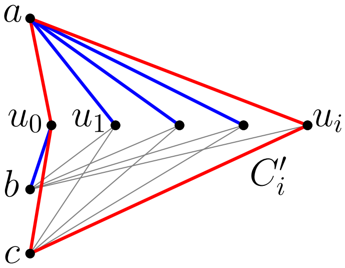

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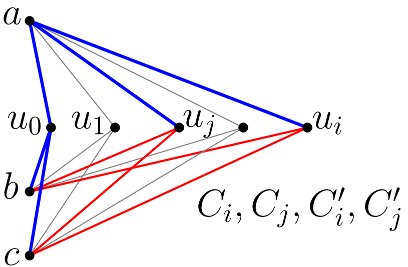
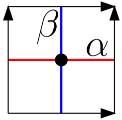


$\mathbb{Z}_2$-homology of closed curves on M_g

Fact: $H_1(M_g; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2^{2g}$.



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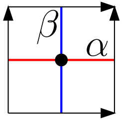

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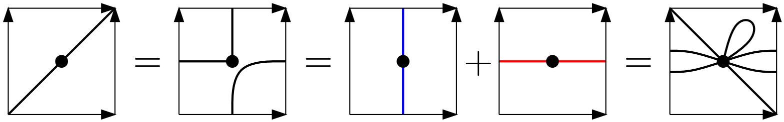

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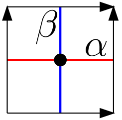

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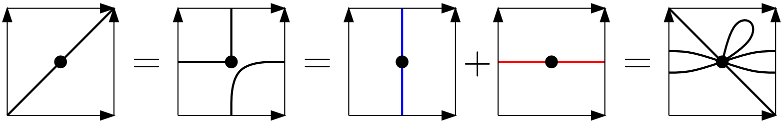

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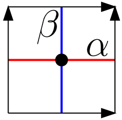

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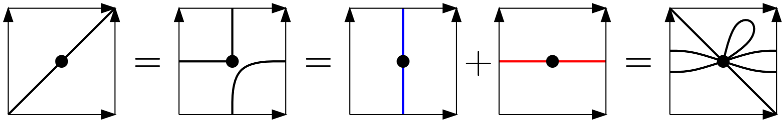

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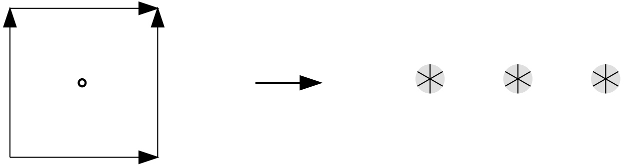

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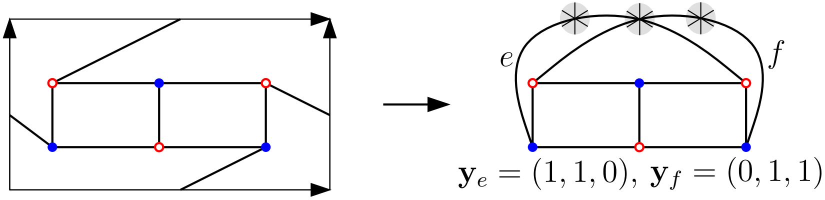

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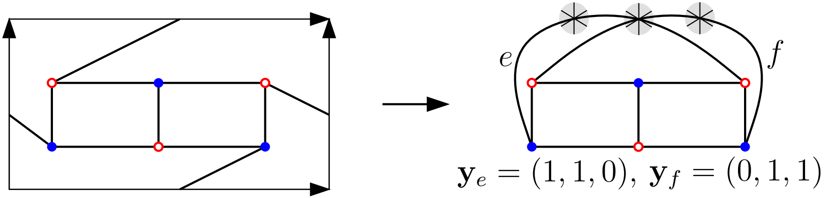

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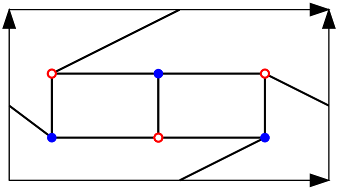

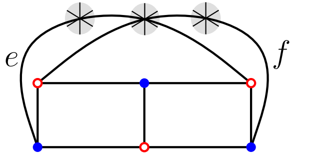

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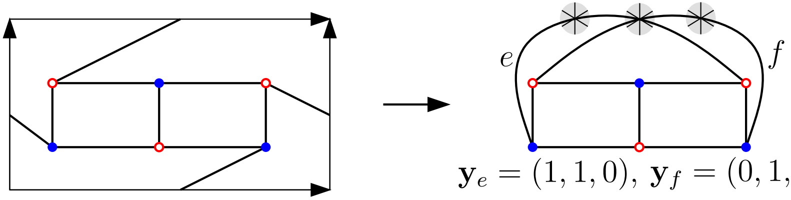

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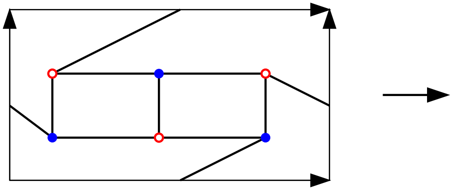

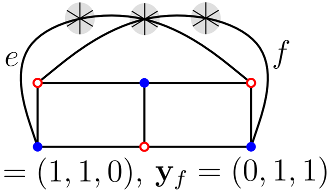

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Lemma: In every independently even drawing of $K_{3,3}$ (induced by $\{a, b, c, u_0, u_1, u_2\}$ from $K_{3,t}$) on M_g, we have

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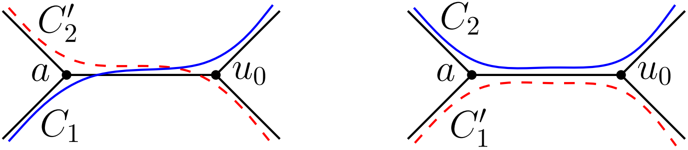

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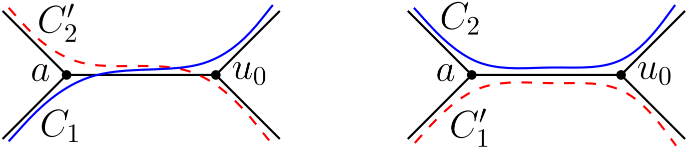

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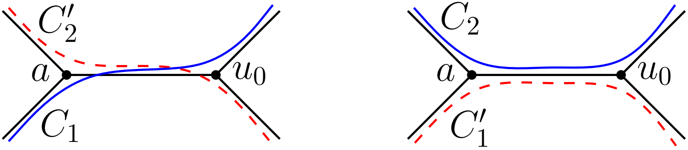

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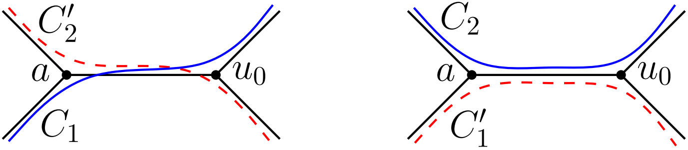

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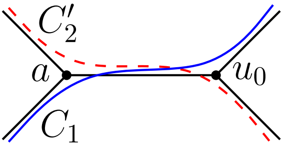

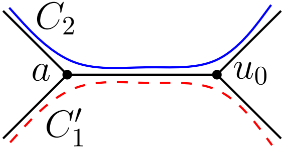

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- the rank of the intersection form is at least $(t-2)/2$ and so $2g \geq (t-2)/2$.