Completing edge-labelled graphs

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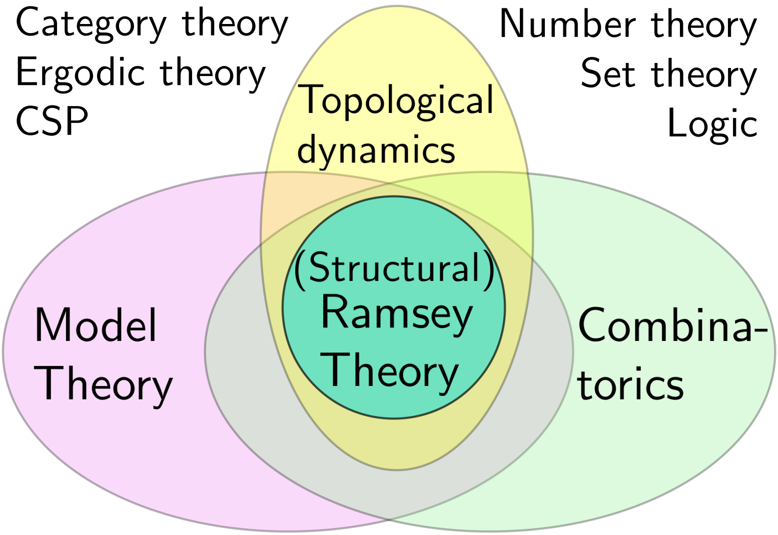
Joint work with Jan Hubička and Jaroslav Nešetřil

The structural Ramsey theory



Completion (extension) problems

Given a class $\mathcal{C}$ of some objects and a partial object X, is it possible to complete it to an object $\overline{X} \in \mathcal{C}$?

- Algebraic topology
- Partial Steiner systems
- Partial geometric representation of graphs
- A zillion NP-complete problems
- ...

The completion problem for edge-labelled graphs

Let L be a set and $\mathcal{C}$ be a class of (not necessarily all) finite **complete** L-edge-labelled graphs. Given an L-edge-labelled graph $\mathbf{G}$, is it possible to add the remaining edges and their labels to get a graph from $\mathcal{C}$? We call such a graph from $\mathcal{C}$ a *completion* of $\mathbf{G}$.

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$L = \{N, E\}$, $\mathcal{G}$ is the class of all finite complete L-edge-labelled graphs. Then every $\mathbf{G}$ has a completion in $\mathcal{G}$.

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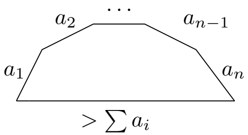
Example (Graphs without a $\overline{K_3}$)

$L = \{N, E\}$, $\mathcal{G}_{\overline{K_3}}$ is the class of all finite complete L-edge-labelled graphs without a triangle with all labels N. Then $\mathbf{G}$ has a completion in $\mathcal{G}_{\overline{K_3}}$ if and only if it contains no triangle with all labels N.

Integer-valued metric spaces

$L = \{1, 2, \dots\}$, **M** is in $\mathcal{M}_{\mathbb{Z}}$ if and only if every triangle of **G** satisfies the triangle inequality. Then **G** has a completion if and only if it contains no *non-metric cycle*:



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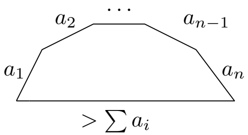

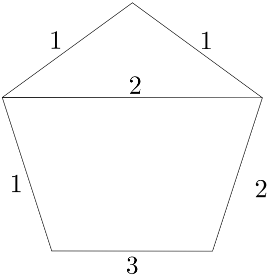

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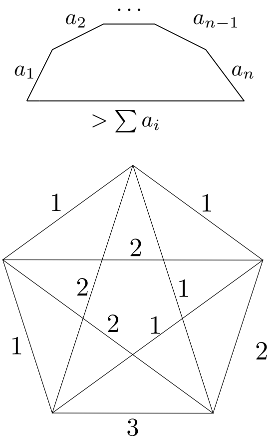

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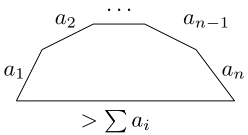

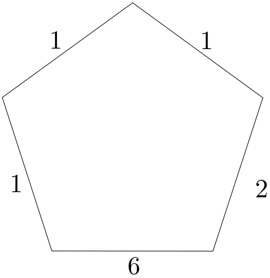

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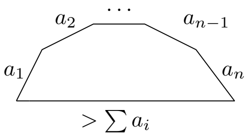

Shortest path completion

Let $\mathbf{G} = (V, E, \ell)$, $\ell \colon E \to L$ be an L-edge-labelled graph. Define $d \colon \binom{V}{2} \to L$ by ($\|\mathbf{P}\|$ is the sum of labels of $\mathbf{P}$)

$$d(x, y) := \min\{\|\mathbf{P}\| : \mathbf{P} \text{ is a path from } x \text{ to } y \text{ in } \mathbf{G}\}.$$

We call $\overline{\mathbf{G}} = (V, \binom{V}{2}, d)$ the *shortest path completion of* **G**.

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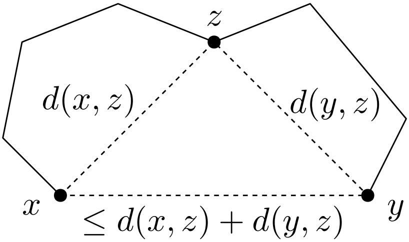
Lemma

$\bar{\mathbf{G}} = (V, \binom{V}{2}, d) \in \mathcal{M}_{\mathbb{Z}}$.

Proof.



□

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Lemma

$d|_E = \ell$ if and only if $\mathbf{G}$ contains no non-metric cycle.

Proof.

Straightforward. □

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Corollary

Let $\mathcal{F}$ be the family of all non-metric L-edge-labelled cycles. Then $\mathbf{G}$ has a completion in $\mathcal{M}_{\mathbb{Z}}$ if and only if $\mathbf{G} \in \mathrm{Forb}(\mathcal{F})$, that is, there is no $\mathbf{F} \in \mathcal{F}$ with a homomorphism $\mathbf{F} \to \mathbf{G}$.

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Observation

There are only finitely many integer non-metric cycles with largest edge n.

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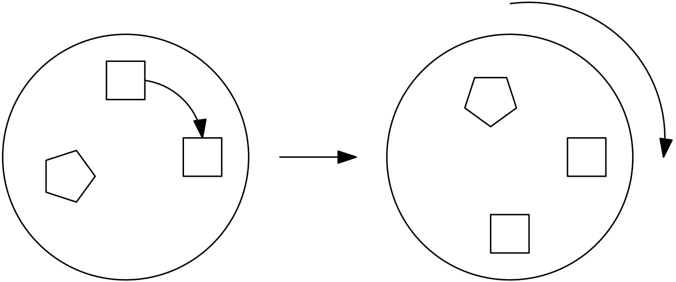
Definition

Let $\mathcal{C}$ be a class of finite complete L-edge-labelled graphs. We say that *the completion problem is easy for $\mathcal{C}$* if there is a family $\mathcal{F}$ of L-edge-labelled cycles such that

1. $\mathbf{G}$ has a completion in $\mathcal{C}$ if and only if $\mathbf{G} \in \mathrm{Forb}(\mathcal{F})$; and
2. for every finite $S \subseteq L$ there are only finitely many S-edge-labelled cycles in $\mathcal{F}$.

Homogeneous structures

A (not necessarily relational) structure **A** is *homogeneous* if every isomorphism of its finite substructures can be extended to an automorphism of **A**.



Definition

A class of finite structures $\mathcal{C}$ is an *amalgamation class* if it is the class of all finite substructures of some homogeneous structure.

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Example (Countably infinite homogeneous graphs, Lachlan–Woodrow 1980)

If $\mathbf{G}$ is a countably infinite homogenous graph, then $\mathbf{G}$ or its complement $\overline{\mathbf{G}}$ is one of the following:

1. the random (Rado) graph,
2. the generic K_n-free graph for $3 \leq n < \infty$,
3. the disjoint union of infinitely many K_n's for $1 \leq n \leq \omega$ or the disjoint union of finitely many K_ω's.

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Example

The class $\mathcal{M}_\mathbb{Z}$ of all finite integer-valued metric spaces is an amalgamation class, the corresponding homogeneous structure is called the integer Urysohn's space.

Ramsey classes

Theorem (Nešetřil, 2005)

Every reasonable Ramsey class is an amalgamation class.

Theorem (Hubička–Nešetřil, 2015; imprecise statement)

Every reasonable amalgamation class for which the completion problem is easy is a Ramsey class.

Theorem (Herwig–Lascar, 2000; imprecise statement)

Every amalgamation class for which the completion problem is easy and which furthermore has an automorphism-preserving completion has EPPA.

Examples of binary symmetric Ramsey classes (imprecise)

- All graphs and K_n-free graphs (Nešetřil–Rödl 1977)
- Metric spaces with distances from $S \subseteq \mathbb{R}^{>0}$ whenever they form an amalgamation class (Sauer 2013; Hubička–Nešetřil 2016).
- Generalised metric spaces with distances from a linearly ordered monoid (Conant 2015; Hubička–K–Nešetřil 2017).
- Cherlin's metrically homogeneous graphs (Cherlin 2011; AB-WHHKKKP 2017).
- Lattice-like nested equivalences (Braunfeld 2017).

$\mathfrak{M}$-valued metric spaces

Definition (Partially ordered commutative semigroup)

The tuple $\mathfrak{M} = (M, \oplus, \preceq)$ is a *partially ordered commutative semigroup* if the following hold:

1. $(M, \oplus)$ is a commutative and associative operation,
2. $(M, \preceq)$ is a partial order,
3. $\oplus$ is monotone in $\preceq$ ($a \preceq b \Rightarrow a \oplus c \preceq b \oplus c$).

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Definition ($\mathfrak{M}$-valued metric space)

Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup. A complete $\mathfrak{M}$-edge-labelled graph $\mathbf{G} = (V, d)$ with $d\colon \binom{V}{2} \to \mathfrak{M}$ is an *$\mathfrak{M}$-valued metric space* if for every $x \neq y \neq z \in V$ it holds that

$$d(x, y) \oplus d(y, z) \succeq d(x, z).$$

We denote by $\mathcal{M}_{\mathfrak{M}}$ the class of all finite $\mathfrak{M}$-valued metric spaces.

Examples

- If $\mathfrak{M} = (\mathbb{R}^{>0}, +, \leq)$ then $\mathcal{M}_{\mathfrak{M}}$ is the class of all finite metric spaces.
- If $\mathfrak{M} = (\mathbb{N}, \cdot, |)$, then the $\mathfrak{M}$-valued metric spaces are the divisibility metric spaces.
- If $\Lambda = (\Lambda, \vee, \leq)$ is a distributive lattice, then Λ-valued metric spaces are Braunfeld's nested equivalences.
- If $S \subseteq \mathbb{R}^{>0}$ is one of Sauer's subset, $x \oplus_S y := \sup\{s \in S : s \leq x + y\}$ and $\mathfrak{M} = (S, \oplus_S, \leq)$, then $\mathcal{M}_{\mathfrak{M}}$ is one of Sauer's classes.

Our result

Theorem

Let $\mathfrak{M} = (M, \oplus, \preceq)$ *be a partially ordered commutative semigroup and let* $\mathcal{F}$ *be a sufficiently nice family of* $\mathfrak{M}$*-edge-labelled cycles. Then the class* $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ *is an amalgamation class, has EPPA and a Ramsey expansion.*

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Shortest path completion for $\mathfrak{M}$

$\mathbf{G} = (V, E, \ell)$ is a finite $\mathfrak{M}$-edge-labelled graph. Define

$$d(x, y) := \inf_{\preceq}\{\|\mathbf{P}\| : \mathbf{P} \text{ is a path from } x \text{ to } y \text{ in } \mathbf{G}\},$$

where $\|P\|$ is the $\oplus$-sum of the labels of $\mathbf{P}$. We call $\overline{\mathbf{G}} = (V, \binom{V}{2}, d)$ the $\mathfrak{M}$-*shortest path completion of* $\mathbf{G}$.

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Definition (Sufficiently nice family $\mathcal{F}$)

$\mathcal{F}$ is sufficiently nice if it ensures that the shortest path completion is defined, behaves well w.r.t. it and the completion problem for $\mathrm{Forb}(\mathcal{F})$ is easy.

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Theorem

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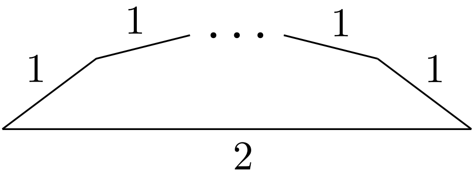
Obstacle

Let $\mathfrak{M} = (\{1,2\}, \max, \leq)$. Then the completion problem for $\mathcal{M}_{\mathfrak{M}}$ is not easy.

Proof.



Proof overview

1. Prove strong amalgamation and completion — not easy
2. Study semigroup structure
   2.1 "Blocks" (subsets defining equivalences)
   2.2 Approximation of block maxima
3. Elminate imaginaries
   3.1 Define an expansion
   3.2 Prove strong amalgamation and completion — easy
   3.3 EPPA
4. Introduce an order
   4.1 Define an expansion
   4.2 Prove strong amalgamation and completion — "easy"
   4.3 Prove Ramsey and the expansion property
5. Applications

Corollaries

Class	**Ramsey**	**EPPA**
S-metric spaces	HN16	new (part Conant15)
Conant's generalized metric spaces	HKN18	new (part Conant15)
Braunfeld's nested equivalences	Braunfeld17	new
Metrically homogeneous graphs	AB-WHHKKKP 2017	AB-WHHKKKP 2017
Cherlin's 4-edge-labelled graphs	new using Li 2018+	new using Li 2018+
Divisibility metric spaces, ...	new	new

Conjecture

Let $\mathcal{C}$ be an amalgamation class of M-edge-labelled graphs and assume that the completion problem for $\mathcal{C}$ is easy. Then there is a partially ordered commutative semigroup $\mathfrak{M} = (M, \oplus, \preceq)$ and a sufficiently nice family $\mathcal{F}$ of $\mathfrak{M}$-edge-labelled cycles such that $\mathcal{C} = \mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$.

The end!

Thank you for your attention

Questions?