Universal Orderings for Generalised Colouring Numbers

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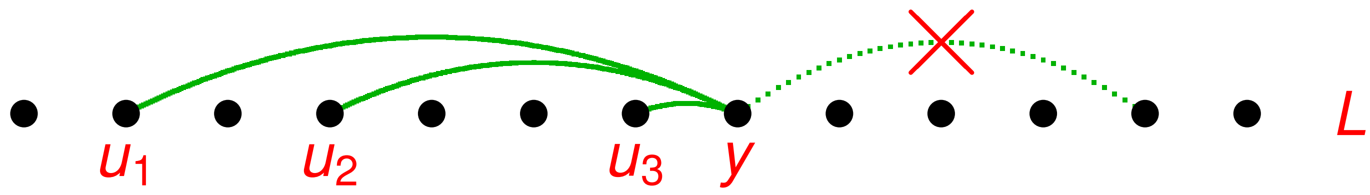
The normal colouring number

- - let $\boldsymbol{L}$ be a linear ordering of the vertices of a graph $\boldsymbol{G}$
  - for a vertex $y \in V(G)$,
    let $\boldsymbol{S}(\boldsymbol{G}, \boldsymbol{L}, \boldsymbol{y})$ be the neighbours u of y with $u <_L y$



  - and set $\boldsymbol{S}[\boldsymbol{G}, \boldsymbol{L}, \boldsymbol{y}] = S(G, L, y) \cup \{y\}$

- then the **colouring number** $\mathbf{col}(\boldsymbol{G})$ is defined as

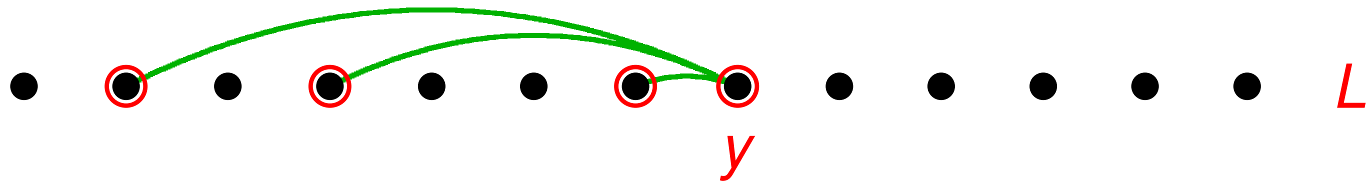
$$\mathrm{col}(G) = \min_{L} \max_{y \in V(G)} \left| S[G, L, y] \right|$$

Generalising the colouring number

- the set $S[G, L, y]$ can also be defined as

  "the set of vertices $u \leq_L y$

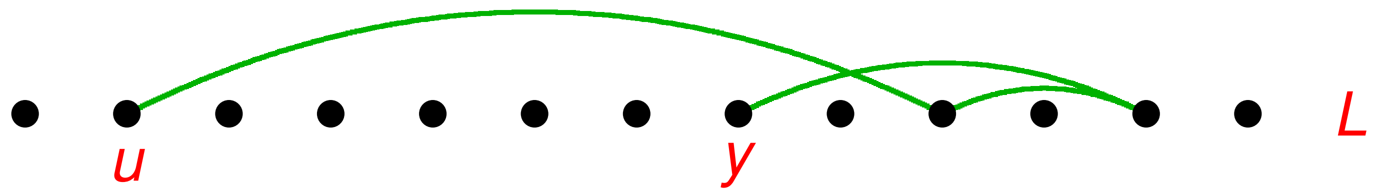
  for which there is a yu-path of length at most 1"



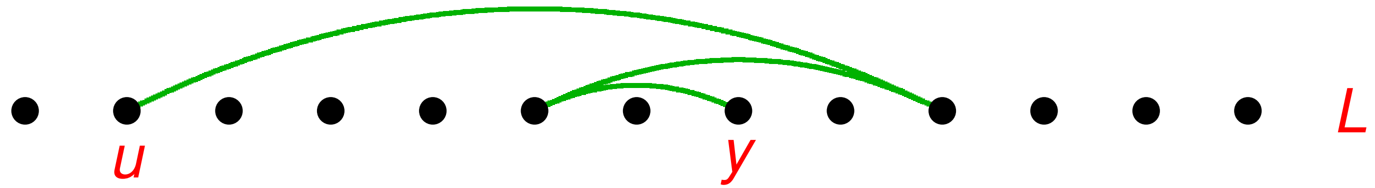
- what would happen if we allow longer paths ?

Generalising the colouring number

- what would happen if we allow longer paths ?
- for $u \leq_L y$:
  - a **strong *yu*-path** has all internal vertices larger than y

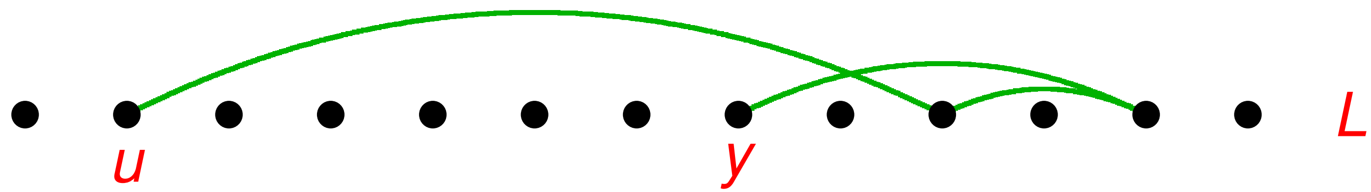


  - a **weak *yu*-path** has all internal vertices larger than u



Strong generalised colouring numbers

- a **strong yu-path** has all internal vertices larger than y

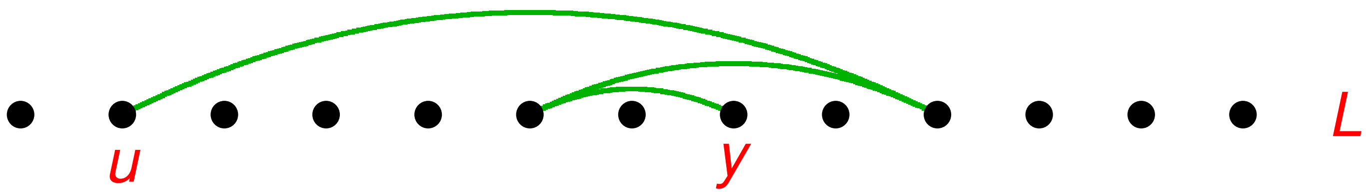


- let $S_r[G, L, y]$ be the set of vertices $u \leq_L y$ for which there exists a strong uy-path with length at most r

- then define the **strong r-colouring number $\mathrm{scol}_r(G)$** by
  - $\mathrm{scol}_r(G, L) = \max_{y \in V(G)} |S_r[G, L, y]|$
  - $\mathrm{scol}_r(G) = \min_{L} \mathrm{scol}_r(G, L)$

Weak generalised colouring numbers

- - a **weak *yu*-path** has all internal vertices larger than u



  - let $\boldsymbol{W_r[G, L, y]}$ be the set of vertices $u \le_L y$ for which there exists a weak uy-path with length at most r

- then define the **weak r-colouring number** $\mathbf{wcol}_r(G)$ by
  - $\mathrm{wcol}_r(G, L) = \max_{y \in V(G)} \big| W_r[G, L, y] \big|$
  - $\mathrm{wcol}_r(G) = \min_{L} \mathrm{wcol}_r(G, L)$

Some facts about generalised colouring numbers

- studied in some form (in particular $r = 2$) since early 1990's
- introduced in this form by Kierstead & Yang, 2003
- by definition: $\mathrm{scol}_1(G) = \mathrm{wcol}_1(G) = \mathrm{col}(G)$
-
  - obviously: $\mathrm{scol}_r(G) \le \mathrm{wcol}_r(G)$
  - but also: $\mathrm{wcol}_r(G) \le \big(\mathrm{scol}_r(G)\big)^r$

    (Proof: every weak path of length at most r is formed of at most r strong paths of length at most r.)

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  - obviously: $\mathrm{scol}_r(G) \leq \mathrm{wcol}_r(G)$
  - but also: $\mathrm{wcol}_r(G) \leq \left(\mathrm{scol}_r(G)\right)^r$
- $\mathrm{scol}_1(G) \leq \mathrm{scol}_2(G) \leq \ldots \leq \mathrm{scol}_\infty(G) = \text{tree-width}(G) + 1$
- $\mathrm{wcol}_1(G) \leq \mathrm{wcol}_2(G) \leq \ldots \leq \mathrm{wcol}_\infty(G) = \text{tree-depth}(G)$

A structural application

- classes of graphs $\mathcal{G}$ with **bounded expansion** were introduced by Nešetřil & Ossona de Mendez in terms of "densities of shallow minors"
  - generalises bounded tree-width, bounded genus, minor closed, etc., etc.

A structural application

- classes of graphs $\mathcal{G}$ with **bounded expansion** were introduced by Nešetřil & Ossona de Mendez in terms of "densities of shallow minors"

- **equivalent Definition** (Zhu, 2009)

  a class of graphs $\mathcal{G}$ has **bounded expansion**:

  - there exists a function $c : \mathbb{N} \to \mathbb{R}$ such that for every $G \in \mathcal{G}$ and every r we have $\mathrm{scol}_r(G) \leq c(r)$

- we can use the weak colouring numbers $\mathrm{wcol}_r(G)$ as well

Orderings

- for every r,
  $\text{scol}_r(G)$ is defined using some "good" ordering L of $V(G)$:

$$\text{scol}_r(G) = \min_L \text{scol}_r(G, L)$$

Question

- can we use the same ordering L for different r ?

Orderings

- $$\mathrm{scol}_r(G) \;=\; \min_L \;\mathrm{scol}_r(G,L)$$

Question

- can we use the same ordering L for different r ?

<u>**NO**</u>

- for every different r, s and function $f(x)$,
  there exists a graph G such that for any ordering L of $V(G)$:
  - $\mathrm{scol}_r(G,L) = \mathrm{scol}_r(G) \implies \mathrm{scol}_s(G,L) \geq f\big(\mathrm{scol}_s(G)\big)$
  - $\mathrm{scol}_s(G,L) = \mathrm{scol}_s(G) \implies \mathrm{scol}_r(G,L) \geq f\big(\mathrm{scol}_r(G)\big)$

Nevertheless, universal orderings are possible

Theorem (vdH & Kierstead)

- for every graph G, there exists an ordering L^* of $V(G)$, such that for all r we have

$$\mathrm{scol}_r(G, L^*) \;\leq\; (2^r + 1) \cdot \big(\mathrm{scol}_{2r}(G)\big)^{4r}$$

- the dependency on $\mathrm{scol}_{2r}(G)$ is best possible,
  - i.e. we cannot find such an L^* where the bounds on $\mathrm{scol}_r(G, L^*)$ are in terms of $\mathrm{scol}_{2r-1}(G)$

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Corollary

- a class of graphs $\mathcal{G}$ has bounded expansion if and only if
  - there exists a function $c' : \mathbb{N} \to \mathbb{R}$ such that for every $G \in \mathcal{G}$ there exists an ordering L^* of $V(G)$, such that for every r we have $\mathrm{scol}_r(G, L^*) \leq c'(r)$

Ideas of the proof

- the crucial idea of the proof goes back to a proof in the original work of Kierstead & Yang (2003) that introduced generalised colouring numbers
- the main part of that paper actually deals with a game variant of those numbers

The game colouring number

- Alice and a gremlin create an ordering L' of the vertices of a given graph G, as follows
  - they alternately choose the next vertex, starting with the gremlin
  - Alice wants to end up with an ordering L' such that $\mathrm{scol}_r(G, L')$ is "small" (for some given r)

Theorem (Kierstead & Yang, 2003)

- no matter how mischievous the gremlin is, Alice can guarantee the final ordering L' to satisfy:

$$\mathrm{scol}_r(G, L') \le 3\big(\mathrm{wcol}_{2r}(G)\big)^2 \le 3\big(\mathrm{scol}_{2r}(G)\big)^{4r}$$

A first common ordering

- suppose the gremlin is not really mischievous, but has some specific ordering in mind as well

that directly leads to:

Corollary

- let G_1, G_2 be two graphs on the same vertex set V and let r_1, r_2 be two natural numbers
  - then there exists an ordering L^* of V such that

$$\mathrm{scol}_{r_1}(G_1, L^*) \;\le\; 3\big(\mathrm{scol}_{2r_1}(G_1)\big)^{4r_1}$$

and

$$\mathrm{scol}_{r_2}(G_2, L^*) \;\le\; 3\big(\mathrm{scol}_{2r_2}(G_2)\big)^{4r_2}$$

Next step: a common ordering for many graphs

Theorem (vdH & Kierstead)

- let $G_1, \ldots, G_k$ be a collection of graphs on the same set V and let $r_1, \ldots, r_k$ be natural numbers
  - then there exists an ordering L^* of V such that for $i = 1, \ldots, k$: $\mathrm{scol}_{r_i}(G_i, L^*) \leq (k+1)\big(\mathrm{scol}_{2r_i}(G_i)\big)^{4r_i}$

Corollary

- for every graph G and natural number k
  - there exists an ordering L^* of $V(G)$ such that for $r = 1, \ldots, k$: $\mathrm{scol}_r(G, L^*) \leq (k+1)\big(\mathrm{scol}_{2r}(G)\big)^{4r}$

The most general, “weighted”, version

Theorem (vdH & Kierstead)

- let $G_1, \ldots, G_k$ be a collection of graphs on the same set V,
  let $r_1, \ldots, r_k$ be natural numbers,
  and let $a_1, \ldots, a_k$ be natural numbers
  - set $A = a_1 + \cdots + a_k$
  - then there exists an ordering L^* of V such that

    for all $i = 1, \ldots, k$:

$$\mathrm{scol}_{r_i}(G_i, L^*) \le \left(\frac{A}{a_i} + 1\right) \cdot \left(\mathrm{scol}_{2r_i}(G_i)\right)^{4r_i}$$

How to use this general, "weighted", version

- $$\mathrm{scol}_{r_i}(G_i, L^*) \;\le\; \left(\frac{A}{a_i} + 1\right) \cdot \left(\mathrm{scol}_{2r_i}(G_i)\right)^{4r_i}$$

- now set $k = \lfloor \log_2 |V| \rfloor$
- and for $i = 1, \ldots, k$, set $a_i = 2^{k-i}$
  - then: $A = a_1 + \cdots + a_k = 2^k - 1 \le 2^k$, so $\dfrac{A}{a_i} \le 2^i$
- next, for $i = 1, \ldots, k$ take $G_i = G$ and $r_i = i$, and we get:
  $$\mathrm{scol}_i(G, L^*) \;\le\; (2^i + 1) \cdot \left(\mathrm{scol}_{2i}(G)\right)^{4i}$$
- for $i > k$ we have $2^i + 1 > |V|$, so nothing to prove

Algorithmic aspects

- there exists an ordering L^* of V such that for all $i = 1, \ldots, k$:

$$\mathrm{scol}_{r_i}(G_i, L^*) \le \left(\frac{A}{a_i} + 1\right) \cdot \left(\mathrm{scol}_{2r_i}(G_i)\right)^{4r_i}$$

- if orderings L_i with $\mathrm{wcol}_{2r_i}(G_i, L_i) = \mathrm{wcol}_{2r_i}(G_i)$ are given, then L^* can be found in time polynomial in $|V|$ and A

- unfortunately, finding $\mathrm{wcol}_r(G)$ is NP-hard for $r \ge 3$ (Grohe et al., 2015)

- but using results of Dvořák (2013), we can find in polynomial time an ordering L'_i such that $\mathrm{wcol}_{2r_i}(G_i, L'_i)$ "approximates" $\mathrm{wcol}_{2r_i}(G_i)$

Finding universal orderings

Corollary

- let $\mathcal{G}$ be a class with bounded expansion
  - then there exists a function $c' : \mathbb{N} \to \mathbb{R}$ and a polynomial time algorithm
  - that finds for every $G \in \mathcal{G}$:
    - an ordering L^* of $V(G)$
    - such that for every r : $\mathrm{scol}_r(G, L^*) \leq c'(r)$

But what does it really mean . . . ?

Theorem

- a class of graphs $\mathcal{G}$ has bounded expansion if and only if
  - there exists a function $c' : \mathbb{N} \to \mathbb{R}$ such that

    for every $G \in \mathcal{G}$ there exists an ordering L^* of $V(G)$,

    such that for every r we have $\mathrm{scol}_r(G, L^*) \leq c'(r)$

Question

- what (if anything) does this ordering L^* tell us about the structure of the graphs in a class with bounded expansion ?

A more concrete question

Property (Folklore et al.)

- $\mathrm{scol}_1(G) = \mathrm{wcol}_1(G) = \mathrm{col}(G)$ can be found in polynomial time

Theorem (Grohe et al., 2015)

- for $r \geq 3$, finding $\mathrm{scol}_r(G)$ or $\mathrm{wcol}_r(G)$ is NP-hard

Question

- what is the complexity of finding $\mathrm{scol}_2(G)$ or $\mathrm{wcol}_2(G)$?

Thanks for your attention !

Thanks to the organisers

for another wonderful

Midsummer Combinatorial Workshop !

(but please switch off the outdoor heating next year)