Diameter-Ramsey Sets

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Ramsey Sets

Notation

Given $X \subset \mathbb{R}^d$, $Y \in \mathbb{R}^n$ and $r \in \mathbb{N}$ we say that $Y \xrightarrow{r} X$ if in every r-colouring of Y, there is some monochromatic $X' \subset Y$ which is congruent to X.

Two sets $X \subset \mathbb{R}^d$ and $X' \subset \mathbb{R}^{d'}$ are congruent if there is an isometry $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ such that $X' = f(X)$.

Definition (Erdős–Graham–Montgomery–Rothschild–Spencer–Straus, 1973)

A set $X \subset \mathbb{R}^d$ is called *Ramsey* if for every $r \in \mathbb{N}$, there is some $n \in \mathbb{N}$, such that $\mathbb{R}^n \xrightarrow{r} X$.

What do we know about Ramsey sets?

There has been a lot of work in this area.

- Every Ramsey set must be finite and spherical (EGMRSS).
- Cartesian products of Ramsey sets are Ramsey (EGMRSS).
- Simplices are Ramsey (Frankl–Rödl, 1990).
- Regular polytopes are Ramsey (Kříž, 1991).

The problem of fully characterising Ramsey-sets remains open.

Conjecture (EGMRSS, 1973)

A set is Ramsey if and only if it is finite and spherical.

Conjecture (Leader–Russell–Walters, 2010)

A set is Ramsey if and only if it is contained in a finite transitive set.

Diameter-Ramsey Sets

Definition (Frankl–Pach–Reiher–Rödl, 2017)

A set $X \subset \mathbb{R}^d$ is called *diameter-Ramsey* if for every $r \in \mathbb{N}$, there is some $n \in \mathbb{N}$ and some $Y \subset \mathbb{R}^n$ with $\operatorname{diam}(X) = \operatorname{diam}(Y)$ such that $Y \xrightarrow{r} X$.

FPRR proved the following facts about diameter-Ramsey sets.

- Every diameter-Ramsey set is Ramsey.
- Cartesian products of diameter-Ramsey sets are diameter-Ramsey.
- Acute and right-angled triangles are diameter-Ramsey.
- Almost-regular simplices are diameter-Ramsey.
- Triangles with an angle larger than 150° are not diameter-Ramsey.

Borsuk's Conjecture

Conjecture (Borsuk, 1933)

Every finite set $Y \subset \mathbb{R}^n$ can be partitioned into $n+1$ sets of smaller diameter.

The n-simplex cannot be partitioned into n sets of smaller diameter.

Definition

Given some $X \subset \mathbb{R}^d$, let $f(X, n)$ denote the maximum number $r \in \mathbb{N}$ for which there is some $Y \subset \mathbb{R}^n$ with $\text{diam}(Y) = \text{diam}(X)$ such that $Y \xrightarrow{r} X$.

X is diameter-Ramsey if and only if $f(X, n) \to \infty$ as $n \to \infty$.

$f(X, n)$ describes "how diameter-Ramsey" X is.

Borsuk's conjecture states that $f(X, n) = n$ if $|X| = 2$.

Borsuk's Conjecture

Definition

Given some $X \subset \mathbb{R}^d$, let $f(X, n)$ denote the maximum number $r \in \mathbb{N}$ for which there is some $Y \subset \mathbb{R}^n$ with $\mathrm{diam}(Y) = \mathrm{diam}(X)$ such that $Y \xrightarrow{r} X$.

Conjecture (Borsuk, 1933)

$f(X, n) = n$ if X if $|X| = 2$.

Borsuk's conjecture is false, in fact $f(X, n) \geq 1.2^{\sqrt{n}}$ (Kahn–Kalai, 1993).

For every $d \in \mathbb{N}$, there is some $\varepsilon = \varepsilon(d) > 0$, such that $f(X, n) \geq (1 + \varepsilon)^{\sqrt{n}}$ for the regular d-simplex X (FPRR).

Our Results

Every Ramsey set must be spherical; Every diameter-Ramsey set must in addition have small circumradius.

(The circumradius of a spherical set X is the radius of the smallest sphere containing X.)

Theorem (C.–Frankl N., 2017)

If $X \subset \mathbb{R}^d$ is a finite, spherical set with circumradius strictly larger than $\mathrm{diam}(X)/\sqrt{2}$, then X is not diameter-Ramsey.

Corollary

Triangles with an angle larger than $135°$ are not diameter-Ramsey.

Proof

Theorem

If $X \subset \mathbb{R}^d$ is a finite, spherical set with circumradius strictly larger than $\mathrm{diam}(X)/\sqrt{2}$, then X is not diameter-Ramsey.

To prove the theorem we embed a given set $Y \subset \mathbb{R}^n$ with $\mathrm{diam}(X) = \mathrm{diam}(Y)$ in a ball and then colour the whole ball.

Theorem (Jung's inequality)

Every bounded set $Y \subset \mathbb{R}^n$ can be covered by a closed ball of radius $\sqrt{n/(2n+2)} \cdot \mathrm{diam}(Y) < \mathrm{diam}(Y)/\sqrt{2} < \mathrm{cr}(X)$.

Lemma

For every finite, spherical set $X \subset \mathbb{R}^d$ and every $R < cr(X)$, there is some $r = r(X, R) \in \mathbb{N}$ such that $B_n(R) \not\xrightarrow{r} X$ for every $n \in \mathbb{N}$.

Proof

Lemma

For every finite, spherical set $X \subset \mathbb{R}^d$ and every $R < cr(X)$, there is some $r = r(X, R) \in \mathbb{N}$ such that $B_n(R) \not\xrightarrow{r} X$ for every $n \in \mathbb{N}$.

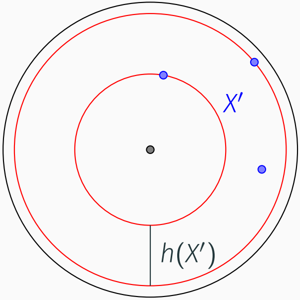
Proof.

For a congruent copy $X' \subset B_n(R)$ of X, define $h(X') := \max_{x,y \in X'} (\|x\| - \|y\|)$.

Since $\mathrm{cr}(X) > R$, we have $h(X') > 0$ for every X'.

Using a compactness argument, we obtain $\min_{X'} h(X') > c > 0$.

Colour $B_n(R)$ concentrically with stripes of length c.



Proof

Lemma

For every finite, spherical set $X \subset \mathbb{R}^d$ and every $R < cr(X)$, there is some $r = r(X, R) \in \mathbb{N}$ such that $B_n(R) \overset{r}{\not\to} X$ for every $n \in \mathbb{N}$.

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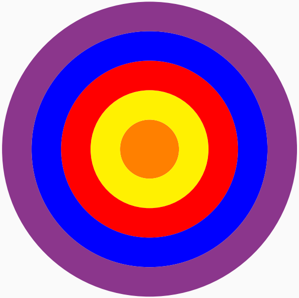

Open Problems

A triangle is diameter-Ramsey if its largest angle is at most 90°.

A triangle is not diameter-Ramsey if its largest angle is larger than 135°.

Question (Frankl P.–Pach–Reiher–Rödl, 2017)

Is there any obtuse triangle which is diameter-Ramsey?

Conjecture (C.–Frankl N., 2017)

A d-simplex is diameter-Ramsey if and only if its circumcentre is contained in its convex hull.

Thanks!