Midsummer Combinatorial Workshop XXIII

July 30 - August 3, 2018, Prague

Jan Hubička, Karel Král, Jaroslav Nešetřil (eds.)

The workshop continues the tradition of Prague Combinatorial Workshops held since 1993. Oriented on problems of all fields of graph theory, combinatorics and discrete geometry, it will continue in the spirit and informal working atmosphere of the previous meetings.

The workshop takes place at DIMATIA and the Department of Applied Mathematics of Charles University, Malostranské náměstí 25, Prague 1, which is located in a historic building in the center of Old Prague.

Jan Hubička, Karel Král, Jaroslav Nešetřil

DIMATIA

Contents

Monday

Martin Loebl – Perfect matchings in double-torus grids

Joint work with Andrea Jimenez and Marcos Kiwi

It is known that the number of the perfect matchings of each finite double-torus grid is a linear combination of 16 determinants. It is predicted by the Quantum field theory (Alvarez-Gaume, Vaffa) that exactly 6 of these determinants go to zero when the size of the grid grows. I will introduce the rich world around this claim.

Zdeněk Dvořák – Baker game and approximation algorithms on proper minor-closed classes

Inspired by the Splitter game for nowhere-dense classes and Baker's approach to design of approximation algorithms, we propose a new game, show a strategy winning this game in a constant number of rounds on proper minor-closed classes of graphs (without using the structure theorem), and argue that the existence of such a strategy implies existence of polynomial-time approximation schemes for a number of optimization problems.

Martin Balko – Ramsey numbers and monotone colorings

For positive integers N and $r \geq 2$, an r-monotone coloring of r-tuples from $[N]$ is a 2-coloring by -1 and +1 that is monotone on the lexicographically ordered sequence of r-tuples of every $(r+1)$-tuple from $[N]$. Let $ORS(n;r)$ be the minimum N such that every r-monotone coloring of r tuples from $[N]$ contains n elements with all r-tuples of the same color.

For every $r \geq 3$, it is known that $ORS(n;r)$ is bounded from above by a tower function of height $r-2$ with $\mathcal{O}(n)$ on the top. The Erdős–Szekeres Lemma and the Erdős–Szekeres Theorem imply $ORS(n;2) = (n-1)^2+1$ and $ORS(n;3) = \binom{2n-4}{(n-2)}+1$, respectively. It follows from a result of Eliáš and Matoušek that $ORS(n;4)$ grows as a tower of height 2.

We show that $ORS(n;r)$ grows at least as a tower of height $r-2$ for every $r \geq 3$. This, in particular, solves an open problem posed by Eliáš and Matoušek and by Moshkovitz and Shapira. Using two geometric interpretations of monotone colorings, we show connections between estimating $ORS(n;r)$ and two Ramsey-type problems that have been recently considered by several researchers. We also prove asymptotically tight estimate on the number of r-monotone colorings of on $[N]$.

Pierre Simon – Towards a classification of geometric homogeneous structures

We call a homogeneous structure geometric (or NIP) if it has polynomially many types over finite sets, or equivalently at most exponential growth of the number of finite substructures. Such structures are usually order-like or tree-like. I will present the first step in the classification of the order-like case. As an example, the classification of primitive homogeneous multi-orders, as well as of their reducts, follows at once from it. I will end with a few open problems.

Dušan Knop – Integer Programming in Parameterized Complexity: Three Miniatures

Powerful results from the theory of integer programming have recently led to substantial advances in parameterized complexity. However, our perception is that, except for Lenstra's algorithm for solving integer linear programming in fixed dimension, there is still little understanding in the parameterized complexity community of the strengths and limitations of the available tools.

Specifically, we consider graphs of bounded neighborhood diversity which are in a sense the simplest of dense graphs, and we show several FPT algorithms for Sum Coloring by modeling it as a convex program in fixed dimension, n-fold integer programs, and as ILP with bounded dual treewidth and graver complexity.

Tuesday

Jan Kynčl – The $\mathbb{Z}_2$-genus of Kuratowski minors

Joint work with Radoslav Fulek.

A drawing of a graph on a surface is *independently even* if every pair of nonadjacent edges in the drawing crosses an even number of times. The $\mathbb{Z}_2$-*genus* of a graph G is the minimum g such that G has an independently even drawing on the orientable surface of genus g. An unpublished result by Robertson and Seymour implies that for every t, every graph of sufficiently large genus contains as a minor a projective $t \times t$ grid or one of the following so-called t-*Kuratowski graphs*: $K_{3,t}$, or t copies of K_5 or $K_{3,3}$ sharing at most 2 common vertices. We show that the $\mathbb{Z}_2$-genus of graphs in these families is unbounded in t; in fact, equal to their genus. Together, this implies that the genus of a graph is bounded from above by a function of its $\mathbb{Z}_2$-genus, solving a problem posed by Schaefer and Štefankovič, and giving an approximate version of the Hanani–Tutte theorem on orientable surfaces.

Andreas E. Feldmann – The Parameterized Hardness of the k-Center Problem in Transportation Networks

Joint work with Dániel Marx.

We study the hardness of the k-Center problem on inputs that model transportation networks. For the problem, an edge-weighted graph $G = (V, E)$ and an integer k are given and a center set $C \subseteq V$ needs to be chosen such that $|C| \leq k$. The aim is to minimize the maximum distance of any vertex in the graph to the closest center. This problem arises in many applications of logistics, and thus it is natural to consider inputs that model transportation networks. Such inputs are often assumed to be planar graphs, low doubling metrics, or bounded highway dimension graphs. For each of these models, parameterized approximation algorithms have been shown to exist. We complement these results by proving that the k-Center problem is W[1]-hard on planar graphs of constant doubling dimension, where the parameter is the combination of the number of centers k, the highway dimension h, and even the treewidth t. Moreover, under the Exponential Time Hypothesis there is no $f(k, t, h) \cdot n^{o(t+\sqrt{k+h})}$ time algorithm for any computable function f. Thus it is unlikely that the optimum solution to k-Center can be found efficiently, even when assuming that the input graph abides to all of the above models for transportation networks at once!

Pavel Hubáček – On Constant-Round Statistical Zero-Knowledge

Based on joint work with Alon Rosen and Margarita Vald.

I will show how to overcome technical challenges when designing constant-round statistical zero-knowledge proofs. Specifically, I will describe an unconditional transformation from any honest-verifier statistical zero-knowledge protocol to a protocol where the statistical zero-knowledge property holds against arbitrary verifiers.

Jan Corsten – Diameter-Ramsey sets

Joint work with Nóra Frankl.

A finite set $A \subset \mathbb{R}^d$ is called *diameter-Ramsey* if for every $r \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ and a finite set $B \subset \mathbb{R}^n$ with $\mathrm{diam}(A) = \mathrm{diam}(B)$ such that whenever B is coloured with r colours, there is a monochromatic set $A' \subset B$ which is congruent to A. We prove that sets of diameter 1 with circumradius larger than $1/\sqrt{2}$ are not diameter-Ramsey. In particular, we obtain that triangles with an angle larger than $135°$ are not diameter-Ramsey, improving a result of P. Frankl, Pach, Reiher and Rödl. Furthermore, we deduce that there are simplices which are almost regular but not diameter-Ramsey.

Zaniar Ghadernezhad – Amenable automorphism groups and convex Ramsey matrices

A group G is amenable if every G-flow has an invariant Borel probability measure. Well-known examples of amenable groups are finite groups, solvable groups and locally compact abelian groups. Kechris, Pestov, and Todorcevic established a very general correspondence which equates a stronger form of amenability, called extreme amenability, of the automorphism group of an ordered Fraïssé structure with the Ramsey property of the collection of its finite substructures. In the same spirit Moore showed a correspondence between the amenability of the automorphism groups of countable structure and a structural Ramsey property called convex Ramsey property, which englobes Følner's existing treatment. The convex Ramsey property can be translated to a combinatorial condition on matrices. We will consider automorphism groups of certain Hrushovski's generic structures and show that they are not amenable using certain matrices and exhibiting a combinatorial/geometrical criterion which forbids amenability.

Lluis Vena – Chromatic number of subgraphs of the kneser graph

Joint work with Bart Litjens, Sven Polak and Bart Sevenster.

Let n, k, r be positive integers with $n \geq kr$ and $r \geq 2$. Consider a circle C with n points $1, \ldots, n$ in clockwise order. The r-stable interlacing graph $IG_{n,k}^{(r)}$ is the graph with vertices corresponding to k-subsets S of $\{1, ..., n\}$ such that any two distinct points in S have distance at least r around the circle, and edges between k-subsets P and Q if they interlace: after removing the points in P from C, the points in Q are in different connected components. We show that the circular chromatic number of $IG^{(r)}_{n,k}$ is equal to n/k, hence the chromatic number is $\lceil n/k \rceil$, and that its circular clique number is also n/k.

Furthermore, we show that the independence number of $IG^{(r)}_{n,k}$ is the binomial coefficient $\binom{n-(r-1)k-1}{k-1}$. This strengthens a result by Talbot, stating that the maximum size of a family of mutually intersecting r-stable k-sets of $\{1, ..., n\}$ is $\binom{n-(r-1)k-1}{k-1}$ (this is analogous to the Erdős-Ko-Rado theorem).

Wednesday

Václav Blažej – Online Ramsey numbers

The online Ramsey theory studies a game between Builder and Painter. They are given an arbitrary graph H and a graph G of an infinite set of independent vertices. On each round, Builder builds a new edge in G and Painter colors it either red or blue. The online Ramsey number of a graph H is the minimum number of rounds Builder needs to force a monochromatic H to appear as a subgraph of G.

In this talk we show strategies for Paths and Cycles. We also introduce upper bounds on strongly induced online Ramsey numbers of Paths, Cycles and two families of trees, one of which has an asymptotic difference to its size-Ramsey counterpart.

Josef Cibulka – Covering Lattice points by subspaces

Joint work with Martin Balko and Pavel Valtr.

Let d and k be integers with $1 \leq k \leq d-1$. The *linear combination* of points $b_1, \dots, b_d \in \mathbb{R}^d$ with coefficients $\alpha_1, \dots, \alpha_d \in \mathbb{R}$ is the point $\alpha_1 b_1 + \dots + \alpha_d b_d$. For linearly independent points $b_1, \dots, b_d \in \mathbb{R}^d$, the d-dimensional *lattice* $\Lambda = \Lambda(b_1, \dots, b_d)$ with *basis* $\{b_1, \dots, b_d\}$ is the set of all linear combinations of the points $b_1, \dots, b_d$ with integer coefficients. A *linear subspace* of $\mathbb{R}^d$ is the set of all linear combinations of a set of points from $\mathbb{R}^d$. The *dimension* k of a linear subspace is the size of the smallest set of points that generate the subspace.

Let Λ be a d-dimensional lattice and let K be a d-dimensional compact convex body symmetric about the origin. We provide estimates for the minimum number of k-dimensional linear subspaces needed to cover all points in $\Lambda \cap K$. In particular, our results imply that the minimum number of k-dimensional linear subspaces needed to cover the d-dimensional $n \times \dots \times n$ grid is at least $\Omega(n^{d(d-k)/(d-1)-\varepsilon})$ and at most $O(n^{d(d-k)/(d-1)})$, where $\varepsilon > 0$ is an arbitrarily small constant. This nearly settles a problem mentioned in the book of Brass, Moser, and Pach [2]. We also find tight bounds for the minimum number of k-dimensional affine subspaces needed to cover $\Lambda \cap K$.

We use these new results to improve the best known lower bound for the maximum number of point-hyperplane incidences by Brass and Knauer [1]. For $d \geq 3$ and $\varepsilon \in (0,1)$, we show that there is an integer $r = r(d, \varepsilon)$ such that for all positive integers n, m the following statement is true. There is a set of n points in $\mathbb{R}^d$ and an arrangement of m hyperplanes in $\mathbb{R}^d$ with no $K_{r,r}$ in their incidence graph and with at least $\Omega\left((mn)^{1-(2d+3)/((d+2)(d+3))-\varepsilon}\right)$ incidences if d is odd and $\Omega\left((mn)^{1-(2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right)$ incidences if d is even. Compared to the previous best bound, this increases the exponent by $\Theta(1/d^2)$. The gap in the exponents in the best known lower and upper bounds remains of size $\Theta(1/d)$.

Bibliography

[1] Peter Brass and Christian Knauer. On counting point-hyperplane incidences. *Comput. Geom.*, 25(1–2):13–20, 2003.

[2] Peter Brass, William Moser, and János Pach. *Research problems in discrete geometry.* Springer, New York, 2005.

Endre Csóka – Local algorithms on random graphs

Local algorithms on graphs mean constant-time randomized distributed algorithms. The most common objectives of these algorithms are to construct large cuts or independent sets or other locally defined structures on large graphs. It turns out that our best algorithms for most of these problems can be approximated by local algorithms. We give an overview about positive and negative results about what can be constructed by them, including some very recent algorithms and entropic bounds. All these questions have strong connections to sparse graph limit theory and statistical physics.

Matěj Konečný – Completing edge-labelled graphs

Joint work with Hubička and Nešetřil.

We study a variant of the completion (or extension) problem which, given a set L, a class $\mathcal{C}$ of finite **complete** L-edge-labelled graphs and a finite L-edge-labelled graph $\mathbf{G}$, asks whether there is a *completion* $\mathbf{G}' \in \mathcal{C}$ of $\mathbf{G}$, that is, a complete L-edge-labelled graphs on the same vertex set as $\mathbf{G}$ whose labels agree with $\mathbf{G}$ on edges of $\mathbf{G}$. We say that *the completion problem is easy* for a class $\mathcal{C}$ if there is a family $\mathcal{F}$ of L-edge-labelled cycles such that

1. For every finite $S \subseteq L$ there are only finitely many S-edge-labelled graphs in $\mathcal{F}$; and
2. an L-edge-labelled graph $\mathbf{G}$ has a completion in $\mathcal{C}$ if and only if $\mathbf{G} \in \mathrm{Forb}(\mathcal{F})$ (that is, there is no $\mathbf{F} \in \mathcal{F}$ with a homomorphism $\mathbf{F} \to \mathbf{G}$).

We study this problem in context of amalgamation classes of edge-labelled graphs, because by result of Nešetřil [7] we know that every Ramsey class is an amalgamation class. A complementing result of Hubička and Nešetřil [6] says that an amalgamation class for which the completion problem is easy "is not far from a Ramsey class".

We introduce the concept of semigroup-valued metric spaces where the distances come from a partially ordered commutative semigroup $\mathfrak{M}$ and where one can further forbid a family of $\mathfrak{M}$-edge-labelled cycles $\mathcal{F}$. We show that if $\mathcal{M}_{\mathfrak{M}}$ is the class of all finite $\mathfrak{M}$-valued metric spaces and the family $\mathcal{F}$ is nice enough, then there is a family $\mathcal{O}$ of $\mathfrak{M}$-edge-labelled cycles such that an $\mathfrak{M}$-edge-labelled graph $\mathbf{G}$ has a completion in $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ if and only if $\mathbf{G} \in \mathrm{Forb}(\mathcal{O})$.

In particular, we prove that the following gives such a completion (the required niceness of $\mathcal{F}$ ensures that the infimum is defined):

Definition 1 (Shortest path completion)**.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup where $\oplus$ is monotone in $\preceq$ and let $\mathbf{G} = (V, E, \ell)$ with $\ell\colon E \to M$ be an M-edge-labelled graph. Define

$$d(\{x, y\}) = \inf_{\preceq}\{\|\mathbf{P}\| : \mathbf{P} \text{ is a path in } \mathbf{G} \text{ from } x \text{ to } y\},$$

where $\|\mathbf{P}\|$ is the $\oplus$-sum of the labels of $\mathbf{P}$. We say that $\bar{\mathbf{G}} = (V, \binom{V}{2}, d)$ is the *shortest path completion of* $\mathbf{G}$.

One then needs to do some more work to ensure that the completion problem is easy for (a bi-definable expansion of) $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ which in turn allows us to find their Ramsey expansions and prove EPPA.

The concept of semigroup-valued metric spaces is quite robust, they in particular include Sauer's S-metric spaces [8, 6], Conant's generalised metric spaces [4, 5], Braunfeld's Λ-ultrametric spaces [2], Cherlin's metrically homogeneous graphs [3, 1] and other classes. Thus, we get Ramsey expansions and EPPA for all of those, which were sometimes not known before. It also gives rise to the following conjecture:

Conjecture 1. *Let L be finite and let $\mathcal{C}$ be a class of L-edge-labelled graphs such that the completion problem is easy for $\mathcal{C}$. Then $\mathcal{C}$ admits a shortest path completion.*

Bibliography

[1] Andres Aranda, David Bradley-Williams, Jan Hubička, Miltiadis Karamanlis, Michael Kompatscher, Matěj Konečný, and Micheal Pawliuk. Ramsey expansions of metrically homogeneous graphs. Submitted, arXiv:1707.02612, 2017.

[2] Samuel Braunfeld. Ramsey expansions of Λ-ultrametric spaces. arXiv:1710.01193, 2017.

[3] Gregory Cherlin. Forbidden substructures and combinatorial dichotomies: WQO and universality. *Discrete Mathematics*, 311(15):1543–1584, 2011.

[4] Gabriel Conant. Extending partial isometries of generalized metric spaces. *arXiv:1509.04950*, 2015.

[5] Jan Hubička, Matěj Konečný, and Jaroslav Nešetřil. Conant's generalised metric spaces are Ramsey. *arXiv preprint arXiv:1710.04690, accepted to Contributions to Discrete Mathematics*, 2017.

[6] Jan Hubička and Jaroslav Nešetřil. All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms). Submitted, arXiv:1606.07979, 58 pages, 2016.

[7] Jaroslav Nešetril. Ramsey classes and homogeneous structures. *Combinatorics, probability and computing*, 14(1-2):171–189, 2005.

[8] Norbert W. Sauer. Oscillation of Urysohn type spaces. In *Asymptotic geometric analysis*, pages 247–270. Springer, 2013.

Günter Rote – Lattice Paths with States, and Counting Geometric Objects via Production Matrices

Joint work with Andrei Asinowski and Alexander Pilz.

We consider paths in the plane governed by the following rules:

1. There is a finite set of states.

2. For each state q, there is a finite set S_q of allowable

steps $((i,j),q')$. This means that from any point (x,y) in state q, we can move to $(x+i,y+i)$ in state q'. We want to count the number of paths that go from $(0,0)$ in some starting state q_0 to the point $(n,0)$ in state q_1 without going below the x-axis. Under some natural technical conditions, I conjecture that the number of such paths is asymptotic to $C^n/n^{3/2}$, and I will show how to compute the growth constant C.

I will discuss how lattice paths with states can be used to model asymptotic counting problems for some non-crossing geometric structures (such as trees, matchings, triangulations) on certain structured point sets. These problems were recently formulated in terms of so-called production matrices.

Jozsef Solymosi – Rigidity of graphs with given edge lengths

Joint work with Orit E. Raz.

We consider the notion of graph rigidity in $\mathbb{R}^2$. While the problem of determining whether an embedding of a graph G in $\mathbb{R}^2$ is *infinitesimally rigid* is well understood, specifying whether a given embedding of G is *rigid* or not is still a hard task that requires ad hoc arguments. In this work, we show that *every* embedding (not necessarily generic) of a dense enough graph (concretely, a graph with at least $C_0 n^{3/2}\log n$ edges, for some absolute constant $C_0>0$), which satisfies some very mild general position requirements (no three vertices of G are embedded to a common line), must have a subframework of size at least three which is rigid. For the proof we use a connection, established in Raz [2], between the notion of graph rigidity and configurations of lines in $\mathbb{R}^3$. This allows us to use properties of line configurations established in Guth and Katz [1].

We do not know whether our assumption on the number of edges being $\Omega(n^{3/2}\log n)$ is tight, and we provide a construction that shows that requiring $\Omega(n\log n)$ edges is necessary.

Bibliography

[1] L. Guth and N. H. Katz, On the Erdős distinct distances problem in the plane, *Annals Math.* 18 (2015), 155–190.

[2] O. E. Raz, Configurations of lines in space and combinatorial rigidity, *Discrete Comput. Geom.* (special issue), 58 (2017), 986–1009.

Thursday

David Hartman – Equality of MH and HH classes

Joint work with Andrés Aranda.

Homomorphism-homogeneity of relational structures requires every local homomorphism between finite induced substructures extends to surjective homomorphism on the whole structure [1]. Class of structures possessing this type of homogeneity is denoted as MH. In the same way class MH can be defined via using morphism instead of homomorphism for a local mapping between finite substructures. Since the original paper of Cameron and Nešetřil the question of relationship between these two classes for various types of underlying structures is studied. Rusinov and Schweitzer showed equality of these classes for undirected graphs answering thus open problem of Cameron and Nešetřil. Later, equality these classes was shown for of finite L-colored graphs where L is linear order or an anti-chain with additional minimal and maximal element [3]. These structures are derived from ordinary graphs whose edges are further colored with sets of colors both taken from partially ordered set L and corresponding homomorphisms preserve inclusion of colors. We generalized this structure to allow colors for edges, partial ordered set Q, as well as vertices, partial order set P, showing that for equality of classes for countable P, Q-colored graphs necessary as well as sufficient condition is that Q is a linear order [4].

Bibliography

[1] P. J. Cameron, J. Nešetřil, Homomorphism-homogeneous relational structures, Combinatorics, probability and computing 15 (1-2) (2006) 91–103.

[2] M. Rusinov, P. Schweitzer, Homomorphism-homogeneous graphs, Journal of Graph Theory 65 (3) (2010) 253–262.

[3] D. Hartman, J. Hubička, D. Mašulović, Homomorphism-homogeneous L-colored graphs, European Journal of Combinatorics 35 (2014).

[4] A. Aranda, D. Hartman, Morphism extension classes of countable L-colored graphs. Preprint at arXiv:1805.01781.

Yelena Yuditsky – Almost all string graphs are intersection graphs of plane convex sets

Joint work with János Pach and Bruce Reed.

A *string graph* is the intersection graph of a family of continuous arcs in the plane. The intersection graph of a family of plane convex sets is a string graph, but not all string graphs can be obtained in this way. We prove the following structure theorem conjectured by Janson and Uzzell: The vertex set of almost all string graphs on n vertices can be partitioned into five cliques such that some pair of them is not connected by any edge ($n \to \infty$). We also show that every graph with the above property is an intersection graph of plane convex sets. As a corollary, we obtain that almost all string graphs on n vertices are intersection graphs of plane convex sets.

Robert Šámal – A rainbow version of Mantel's Theorem

Joint work with Ron Aharoni, Matt DeVos, Sebastian Gonzales, and Amanda Montejano.

Mantel's Theorem asserts that a simple n vertex graph with more than $\frac{1}{4}n^2$ edges has a triangle (three mutually adjacent vertices). Here we consider a rainbow variant of this problem. We prove that whenever G_1, G_2, G_3 are simple graphs on a common set of n vertices and $|E(G_i)| > (\frac{26-2\sqrt{7}}{81})n^2 \approx 0.2557n^2$ for $1 \le i \le 3$, then there exist distinct vertices v_1, v_2, v_3 so that (working with the indices modulo 3) we have $v_i v_{i+1} \in E(G_i)$ for $1 \le i \le 3$. We provide an example to show this bound is best possible.

José Aliste – Rooted U-polynomials

The U polynomial was introduced by Noble and Welsh in 1999 and it generalizes the chromatic symmetric function of Stanley. It is an open question whether there exist non-isomorphic trees with the same U-polynomial. In this talk, we truncate the U-polynomial and for each k, we find pairs of non-isomorphic trees with the same truncation of the U-polynomial. To construct these trees we introduce and study the rooted U-polynomial for rooted trees and prove several algebraic formulas that allow for simple computations of rooted polynomials, which can later be used to deduce properties of the non-rooted versions U-polynomials of the trees involved.

Dragan Mašulović – Finite big Ramsey degrees in universal structures

Let F be a countable ultrahomogeneous relational structure. A positive integer n is a big Ramsey degree of a finite structure A in $Age(F)$ if n is the least integer such that for every $k > 1$ and every coloring of copies of A in F with k colors there is a copy F' of F sitting inside F such that at most n colors are used to color the copies of A that fall within F'. If such an integer exists we say that A has finite big Ramsey degree in F.

For example, Devlin proved in 1979 that finite chains have finite big Ramsey degrees in Q, Sauer proved in 2006 that finite graphs have finite big Ramsey degrees in the Rado graph, and Dobrinen has just recently proved that finite triangle-free graphs have finite big Ramsey degrees in the Henson graph H_3.

In this talk we consider the context where F is a countable structure universal for a class of finite structures, but not necessarily a Fraisse limit. For each of the following classes of structures: acyclic digraphs, finite permutations, a special class of finite posets with a linear order extending the poset relation, and a special class of metric spaces we show that there exists a countably infinite universal structure S such that every finite structure from the class has finite big Ramsey degree in S.

Gábor Kun – Graphings not local-global approximable by finite graphs

Joint work with Andreas Thom.

We study the sofic approximations of Kazhdan groups. We show that a quite wide class of graphings arising from group actions is not the local-global limit (or the action limit) of finite graphs. This is the first example of such graphings.

Zsolt Tuza – Fractional version of the domination game

Joint work with Csilla Bujtás.

We introduce and study the fractional version of the graph domination game, where the moves are ruled by the condition of fractional domination. Fundamental properties of this new game are proved, including the existence of an optimal strategy for each player.

Preliminaries

In this paper we state results on a competitive optimization game concerning graph domination. We should note at the beginning, however, that the main theorems can be formulated and proved on a higher level of generality, namely for vertex cover in hypergraphs (also called transversal, hitting set, blocking set in different areas of discrete mathematics and computer science) which is also equivalent to set cover by hypegraph duality. The vertex cover formulation has implications not only for the classical version of graph domination, but also for a variant called total domination. Proofs — in the more general setting — will be published in [9].

We deal with finite undirected graphs $G = (V, E)$. A vertex $v \in V$ *dominates* itself and its neighbors; that is, exactly the vertices contained in the *closed neighborhood* $N[v]$ of v. A subset $S \subset V$ is called *dominating set* if every vertex of G is dominated by at least one vertex from S, i.e. $\bigcup_{v \in S} N[v] = V$. The smallest size of a dominating set in G is called the *domination number* and is denoted by $\gamma(G)$.

The *domination game*, introduced in [2], is a competitive optimization version of graph domination. It is played on a graph G by two players, namely Dominator and Staller, who take turns choosing a vertex such that at least one previously undominated vertex becomes dominated.[1] The game is over when all vertices are dominated, and the length of the game is the number of vertices chosen by the players. Dominator wants to finish the game as soon as possible, while Staller wants to delay the end. Assuming that both players play optimally and Dominator starts, the length of the game on G is uniquely determined; it is called the *game domination number* of G and is denoted by $\gamma_g(G)$. Analogously, the *Staller-start game domination number* of G, denoted by $\gamma'_g(G)$, is the length of the game under the same rules when Staller makes the first move.

Below we shall refer to these games as *integer games*, as opposed to their fractional versions which will be introduced.

[1]The condition turns out to be restrictive only for Staller.

Following [2], the domination game has been studied further in many papers, see e.g. [1, 3, 4, 11, 14, 18, 19, 20, 21, 22]. The notion also inspired the introduction of the total domination game on graphs [5, 10, 15, 16, 17], transversal game [6, 7] and domination game on hypergraphs [8].

Fractional domination game

A *fractional dominating function* of $G = (V, E)$ is a function $f : V \mapsto [0, 1]$ if $\sum_{u \in N[v]} f(u) \geq 1$ holds for every vertex v in G. The minimum of the sum $\sum_{v \in V} f(v)$ over all such f is called the *fractional domination number* $\gamma^*(G)$ of G, introduced in [12, 13]. Observe that a dominating set corresponds to a fractional dominating function f where every vertex is assigned to either 0 or 1.

A function $d : V \mapsto [0, 1]$ (omitting the local condition above on the vertices) is called a *partially dominating function*; we denote by $|d|$ the sum $\sum_{v \in V} d(v)$. Given a partially dominating function d, the associated *(domination) load function* is the function ℓ defined on V as

$$\ell(v) = \ell(v, d) = \min\{1, \sum_{u \in N[v]} d(u)\}$$

for every vertex v.

The *fractional domination game* starts with the all-0 load function $\ell(v) \equiv 0$, and is finished when the all-1 function $\ell(v) \equiv 1$ is reached. Dominator and Staller take turns making moves of weight 1 each, except possibly in the last move which may be smaller. A *move* is a sequence $(v_{i_1}, w_1), (v_{i_2}, w_2), \ldots$ of arbitrary length, with its *submoves* (v_{i_k}, w_k) $(k = 1, 2, \ldots)$ where $v_{i_1}, v_{i_2}, \ldots$ are vertices of G; any number of repetitions are allowed. Here $w_1, w_2, \ldots$ are real numbers from $(0, 1]$; it is required that

$$\sum_{k \geq 1} w_k = 1$$

in each move except the last one, whereas $\sum_{k \geq 1} w_k \leq 1$ must hold in the last move.

At the beninning the partially dominating function is $d_0 \equiv 0$ and also the load function is $\ell_0 \equiv 0$. After the i^{th} move $(i = 1, 2, \ldots)$ the values $d_i(v_j)$ are calculated to obtain the new partially dominating function d_i by the rule

$$d_i(v_j) = d_{i-1}(v_j) + \sum_{i_k = j} w_k$$

from which the corresponding load function ℓ_i is also derived. A move $(v_{i_1}, w_1), (v_{i_2}, w_2), \ldots$ is legal if

(∗) For all $k = 1, 2, \ldots$, there exists a vertex $u \in N[v_{i_k}]$ with

$$\ell_{i-1}(u) + \sum_{u \in N[v_{i_s}],\ 1 \leq s \leq k-1} w_s \leq 1 - w_k.$$

That is, in each submove there must exist a vertex whose load increases by exactly the weight in the submove.

The *value of the game* $\mathcal{G}$ is $|\mathcal{G}| = |d_q|$, provided that the all-1 load function is reached after a sequence of legal moves in the q^{th} move; i.e., $\ell_q \equiv 1$.

Analogously to the integer game, also here Dominator wants a small $|\mathcal{G}|$, while Staller wants a large $|\mathcal{G}|$. To define the *fractional game domination number* $\gamma_g^*(G)$, assume that Dominator starts the game on G, and consider the set

$$D_G = \{a : \text{Dominator has a strategy which ensures } |\mathcal{G}| \leq a\}.$$

Now, the fractional game domination number is defined as $\gamma_g^*(G) = \inf D_G$. Assuming that Staller starts the game, the Staller-start fractional game domination number $\gamma_{Sg}^*(G)$ is defined similarly.

General results

As mentioned in the preliminaries, the main results can be stated in the more general framework of vertex cover; here we formulate them for the domination game only.

Optimal strategies. Consider the following set which correspons to D_G from Staller's viewpoint:

$$S_G = \{b : \text{Staller has a strategy which ensures } |\mathcal{G}| \geq b\}.$$

Theorem 1. *For every graph G,*

$$\inf(D_G) = \min(D_G) = \sup(S_G) = \max(S_G) = \gamma_g^*(G)$$

and the corresponding equalities also hold for the Staller-start fractional game domination number $\gamma_{Sg}^*(G)$.

Finite moves. The definition of legal move admits the option that a player splits the value 1 into an infinite number of pieces; e.g., $w_k = 2^{-k}$ may also be feasible (with suitable v_{i_k} respecting $(*)$). It turns out, however, that each player can design an optimal strategy without moves consisting of infinitely many submoves.

Theorem 2. *For every graph G, each player has an optimal strategy such that, in every move, each vertex occurs in at most one submove.*

The Rational Game. One may consider a restricted version of the fractional domination game, in which the players are required to use only *rational* weights w_k in each submove. We define $\gamma_g^{\mathbb{Q}}(G)$ to be $\sup S_G = \inf D_G$ under this extra condition. Also for the Staller-start game, the notation $\gamma_{Sg}^{\mathbb{Q}}(G)$ can be introduced.

Theorem 3. *For every graph G, the equalities $\gamma_g^{\mathbb{Q}}(G) = \gamma_g^*(G)$ and $\gamma_{Sg}^{\mathbb{Q}}(G) = \gamma_{Sg}^*(G)$ are valid.*

Continuation Principle. A monotone property of the fractional domination number is expressed in the following idea, which provides a useful tool in simplifying several arguments. Given G and a load function ℓ, let the fractional game ℓ-domination number be denoted by $\gamma_g^*(G|\ell) = \inf(D|\ell) = \sup(S|\ell)$, where $\inf(D|\ell)$ and $\sup(S|\ell)$ are defined analogously to D_G and S_G, for the game starting with a load function ℓ on G with the move of Dominator.

Theorem 4. *If ℓ_1 and ℓ_2 are load functions on G such that $\ell_1(v) \leq \ell_2(v)$ holds for every $v \in V(G)$, then $\gamma_g^*(G|\ell_1) \geq \gamma_g^*(G|\ell_2)$.*

An immediate consequence is

Theorem 5. *The fractional game domination numbers for the Staller-start and for the Dominator-start games on G may differ by at most 1.*

Paths and cycles

The determination of exact values for $\gamma_g^*(G)$ seems hard, even if G has a very simple structure. We have estimates for paths and cycles in which the difference of the upper and lower bounds does not exceed a small constant; yet the numbers do not coincide. The currently known best bounds for cycles are summarized in Table 1.

	$\leq \gamma_g^*(C_n)$	$\gamma_g^*(C_n) \leq$
$n \equiv 0 \pmod 4$	$\frac{n}{2} - \frac{4}{3} + \frac{4}{3n}$	$\frac{n}{2}$
$n \equiv 1 \pmod 4$	$\frac{n}{2} - \frac{3}{2} + \frac{4}{3n}$	$\frac{n}{2} - \frac{1}{6}$
$n \equiv 2 \pmod 4$	$\frac{n}{2} - 1 + \frac{4}{3n}$	$\frac{n}{2} - \frac{1}{3}$
$n \equiv 3 \pmod 4$	$\frac{n}{2} - \frac{7}{6} + \frac{4}{3n}$	$\frac{n}{2} - \frac{1}{2}$

Table 1: Lower and upper bounds for cycles.

Comparison of domination parametes

Proposition 6. *For any graph G, the following inequalities hold:*

- $\gamma^*(G) \leq \gamma_g^*(G) < 2\gamma^*(G)$
- $\gamma^*(G) \leq \gamma_{Sg}^*(G) < 2\gamma^*(G) + 1$
- *There does not exist any universal constant $c > 0$ with*
$$c\gamma_g(G) \leq \gamma_g^*(G).$$
- *If every block of G is a complete graph (and in particular if G is a tree), then*
$$\frac{\gamma_g(G) + 1}{2} \leq \gamma(G) = \gamma^*(G) \leq \gamma_g^*(G).$$

Some open problems

We close these notes with three conjectures and a further problem.

Conjecture 1. *If each of the first $2k-1$ ($k \geq 1$) moves was integer move, i.e. of the form $(v_{i_1}, 1)$, then Staller has an integer move in the $(2k)^{\text{th}}$ turn, which is optimal in the fractional game.*

This means that fractional moves would be advantageous for Dominator only. If true then this conjecture implies the following weaker one.

Conjecture 2. *For every graph G,* $\gamma_g^*(G) \leq \gamma_g(G)$.

Conjecture 3. *For every isolate-free graph G,* $\gamma_g^*(G) \leq 3n/5$.

Problem 1. *Is there a constant $c < 3/5$ such that $\gamma_g^*(G) \leq c \cdot n$ holds for every isolate-free G?*

Bibliography

[1] B. Brešar, P. Dorbec, S. Klavžar, and G. Košmrlj, Domination game: effect of edge- and vertex-removal. *Discrete Math.* 330 (2014) 1–10.

[2] B. Brešar, S. Klavžar, and D. F. Rall, Domination game and an imagination strategy. *SIAM J. Discrete Math.* 24 (2010) 979–991.

[3] Cs. Bujtás, Domination game on forests. *Discrete Math.* 338 (2015) 2220–2228.

[4] Cs. Bujtás, On the game domination number of graphs with given minimum degree. *Electron. J. Combin.* 22 (2015) #P3.29.

[5] Cs. Bujtás, On the game total domination number. *Graphs Combin.* 34 (2018) 415–425.

[6] Cs. Bujtás, M. A. Henning, and Zs. Tuza, Transversal game on hypergraphs and the $\frac{3}{4}$-conjecture on the total domination game. *SIAM J. Discrete Math.* 30 (2016) 1830–1847.

[7] Cs. Bujtás, M. A. Henning, and Zs. Tuza, Bounds on the game transversal number in hypergraphs. *European J. Combin.* 59 (2017) 34–50.

[8] Cs. Bujtás, B. Patkós, Zs. Tuza, and M. Vizer, Domination game on uniform hypergraphs. arXiv:1710.00298v1 (2017).

[9] Cs. Bujtás and Zs. Tuza, Fractional games: vertex cover and domination. To appear.

[10] P. Dorbec and M. A. Henning, Game total domination for cycles and paths. *Discrete Appl. Math.* 208 (2016) 7–18.

[11] P. Dorbec, G. Košmrlj, and G. Renault, The domination game played on unions of graphs. *Discrete Math.* 338 (2015) 71–79.

[12] M. Farber, Domination, independent domination and duality in strongly chordal graphs. *Discrete Appl. Math.* 7 (1986) 115–130.

[13] S. M. Hedetniemi, S. T. Hedetniemi, and T. V. Wimer, Linear time resource allocation algorithms for trees. Technical report URI-014, department of Mathematics, Clemson University, 1987.

[14] M. A. Henning and W. B. Kinnersley, Domination game: A proof of the 3/5-conjecture for graphs with minimum degree at least two. *SIAM J. Discrete Math.* 30 (2016) 20–35.

[15] M. A. Henning, S. Klavžar, and D. F. Rall, Total version of the domination game. *Graphs Combin.* 31 (2015) 1453–1462.

[16] M. A. Henning, S. Klavžar, and D. F. Rall, The 4/5 upper bound on the game total domination number. *Combinatorica* 31 (2017) 223–251.

[17] M. A. Henning and D. F. Rall, Progress towards the total domination game $\frac{3}{4}$-conjecture. *Discrete Math.* 339 (2016) 2620–2627.

[18] W. B. Kinnersley, D. B. West, and R. Zamani, Extremal problems for game domination number. *SIAM J. Discrete Math.* 27 (2013) 2090–2107.

[19] G. Košmrlj, Domination game on paths and cycles. *Ars Math. Contemp.* 13 (2017) 125–136.

[20] M. J. Nadjafi-Arani, M. Siggers, and H. Soltani, Characterisation of forests with trivial game domination numbers. *J. Comb. Optim.* 32 (2016) 800–811.

[21] S. Schmidt, The 3/5-conjecture for weakly $S(K_{1,3})$-free forests. *Discrete Math.* 339 (2016) 2767–2774.

[22] K. Xu, X. Li, and S. Klavžar, On graphs with largest possible game domination number. *Discrete Math.* 341 (2018) 1768–1777.

Pavel Valtr – The exact chromatic number of the convex segment disjointness graph

Joint work with Ruy Fabila-Monroy, Jakob Jonsson, and David Wood.

Let P be a set of n points in strictly convex position in the plane. Let D_n be the graph whose vertex set is the set of all line segments with endpoints in P, where disjoint segments are adjacent. The chromatic number of this graph was first studied by Araujo, Dumitrescu, Hurtado, Noy, and Urrutia [2005] and then by Dujmovic and Wood [2007]. Improving on their estimates, we prove the following exact formula for the chromatic graph of D_n: $\chi(D_n) = n - [\sqrt{n + 1/4} - 1/2$.

Andrés Aranda – HE and MB-homogeneous graphs

Joint work with

HE-homogeneous structures satisfy that any homomorphism between finite induced substructures extends to a surjective endomorphism. In MB-homogeneous structures, any monomorphism between finite induced substructures extends to a bijective endomorphism. In this talk, I will show that all MB-homogeneous graphs are HE-homogeneous and all "interesting" (i.e., not a union of cliques) HE-homogeneous graphs are MB-homogeneous.

Friday

Hiep Han – Erdős-Rothschild type problems for graph cliques and Schur triples

A K_k-free r-coloring of a graph G is a coloring of its edges with r colors with no monochromatic K_k. Motivated by a question of Erdős and Rothschild we investigate, for given k, r and n, those graphs on n vertices which admit the largest number of K_k-free r-colorings.

For $r = 2$ and $k = 3$ this problem was resolved by Yuster and later on the result was extended to $r = 2, 3$ and arbitrary $k \geq 3$ by Alon et al. For $r \geq 4$ the problem becomes considerably harder and despite vivid interest it is only resolved for $r = 4$ and $k = 3$ and for $r = 4$ and $k = 4$ by Pikhurko and Yilma.

In this talk, we first revisit the result of Alon et al before focusing on K_3-free r-colorings for $r = 5$ and $r = 6$. Further, we investigate the closely related problem concerning sum-free r-colorings of subsets of finite Abelian groups.

Pavel Veselý – Online packet scheduling

Joint work with Marek Chrobak, Łukasz Jeż, and Jiří Sgall.

In the online bounded-delay packet scheduling problem (PacketScheduling),the goal is to schedule transmissions of packets that arrive over time in a network switch to be sent across a link. Each packet has a deadline, representing its urgency, and a weight, representing its priority. Only one packet can be transmitted in any time slot, so, if the system is overloaded, some packets will inevitably miss their deadlines and be dropped. In this scenario, the natural objective is to maximize the total weight of transmitted packets. The problem is inherently online, with the scheduling decisions made without the knowledge of future packet arrivals. The central problem concerning PacketScheduling, that has been a subject of intensive study since 2001, is to determine the optimal competitive ratio of online algorithms, namely the worst-case ratio between the optimum total weight of a schedule (computed by an offline algorithm) and the weight of a schedule computed by a (deterministic) online algorithm.

We solve this open problem by presenting a ϕ-competitive online algorithm for PacketScheduling (where $\phi \approx 1.618$ is the golden ratio), matching the previously established lower bound.

Attila Dankovics – Hamiltonicity and low independence number imply pancyclicity

Given a Hamiltonian graph G with independence number at most k we are looking for the minimum number of vertices $f(k)$ that guarantees that G is pancyclic. The problem of finding $f(k)$ was raised by Erdős who showed that $f(k) \leq 4k^4$, and conjectured that $f(k) = 3D\Theta(k^2)$. Formerly the best known upper bound was $f(k) = 3DO(k^{7/3})$ by Lee and Sudakov. We improve this bound and show that $f(k) = 3DO(k^{11/5})$.

Jana Syrovátková – Maximizing points in prisoners dilemma tournaments

Prisoner's dilemma is a well-known concept. Robert Axelrod initiated the study of tournaments of finite automata playing repeated prisoner's dilemma. Similar tournaments have appeared afterwards. The strategy is written in the form of a finite automaton, the number of rounds is given by a random geometric distribution. In the tournament every automaton plays with every other one. Our goal was to create an automaton that will maximize the score earned in the tournament for a given set of opponents.

In the first stage, we convert the opponent's finite automaton into an oriented graph describing the possible iterations of repeated play. In the second phase, we search the walk in this graph that maximizes the score. To play against one automaton, the best combination consists of a path and a cycle sharing one common vertex. Finding the best automaton can be done in time $\mathcal{O}(n^2)$, where n is the number of states of the opponent's automaton.

When playing against multiple automata, a graph with states from the Cartesian product of the states of the original automata is created. The states are also merged into groups according to the automata responses during the game. Within them, the solution is again the combination of path and cycle. If n is the number of states of the opponent's automaton with the most states and m the product of the number of states of all the automata, the search can be performed in time $\mathcal{O}(mn)$.

Demetres Christofides – Graphs of high girth and high chromatic number - new proofs

A celebrated result of Erdő shows that there exists graphs of arbitrarily large girth and chromatic number. Erdő's construction proceeds in two stages. It first picks a graph from the classical Erdő-Renyi model of random graphs (with an appropriate probability) and it then removes some vertices belonging to short cycles.

In this talk we present a new proof of Erdő's result. The construction is again probabilistic but taken from a model of random Cayley graphs. A difference between this proof and the old one is that in this proof the rendom graph picked will already have the required properties without the need to remove any vertices.

Jan Volec – Fractional colorings vs. Hall ratio

Joint work with J. Balogh, A. Kostochka, S. Norin and A. Blumenthal, B. Lidicky, R. Martin, Y. Pehova, and F. Pfender.

If G is an n-vertex graph with the chromatic number k and the independence number a, then $k \geq n/a$. However, for many graphs G this bound can get very bad, and it is easy to show there is in no function g which would upper-bound the chromatic number by $g(n/a)$.

In this talk, we study a relation between the fractional chromatic number, which can be viewed as a relaxation of the chromatic number, and a hereditary variant of the ratio between the number of vertices and the independence number, called the Hall ratio. By averaging, the Hall ratio is always a lower bound on the fractional chromatic number. In 2009, Johnson asked if the fractional chromatic number can be also upper-bounded by a linear function of the Hall ratio, and in 2016, Harris conjectured that this should be the case. We disprove Harris conjecture, and discuss further problems concerned with the relation between these two parameters.

Jan van den Heuvel – Universal Orderings for Generalised Colouring Numbers

Joint work with H.A. Kierstead.

The generalised colouring numbers were introduced by Kierstead and Yang as a generalisation of the usual colouring number, and have since found important theoretical and algorithmic applications. The definitions of these numbers involve a given distance and linear orderings of the vertices of the graphs. For different distances, the optimal orderings can be completely different. We show that it is possible to find a single uniform ordering that approximates (in a precisely prescribed way) all the generalised colouring numbers of that graph. These uniform orderings also provide a new characterisation of graph classes of bounded expansion, where for every class in the graph we need only find one uniform ordering.

List of participants

Andrea Jiménez
Andrew Goodall
Andrés Aranda
Aneta Šťastná
Antoine Mottet
Attila Dankovics
Caitlin Guccione
Casey Tompkins
Daniel Kráľ
David Bradley-Williams
David Chodounský
Demetres Christofides
Diana Piguet
Dragan Masulovic
Dragan Masulovic
Dusan Knop
Edin Husic
Endre Csóka
Eng Keat Hng
Florent Krzakala
Gabor Kun
Gena Hahn
Goran Umicevic
Gregory L. Cherlin
Günter Rote
Helena Nyklova
Hiep Han
Ida Kantor
Jakub Tětek
Jan Corsten
Jan Hubicka
Jan Kynčl
Jan Volec
Jan van den Heuvel
Jana Syrovátková
Jiří Fiala
Josef Cibulka
José Aliste
Jozef Skokan
Jozsef Solymosi
Karel Král
Lenka Zdeborová
Lluis Vena
Manuel Bodirsky
Martin Balko
Martin Hora
Martin Loebl
Matej Stehlik
Matěj Konečný
Michael Kompatscher
Michael Skotnica
Michael Yang
Nate Ackerman
Ondra Suchý
Ondřej Šplíchal
Oscar Zamora
Parker Hund
Patel Viresh
Pavel Paták
Pavel Valtr
Pavel Veselý
Peter Korcsok
Pierre Simon
Radek Hušek
Rahul Ilango
Reza Naserasr
Robert Woodrow
Robert Šámal
Ryan Rice
Sherry Sarkar
Timothy Chan
Tomáš Valla
Viola Mészáros
Václav Blažej
Václav Končický
Yelena Yuditsky
Zdeněk Dvořák
Zsolt Tuza
Zuzka Patáková