Assignment mechanisms

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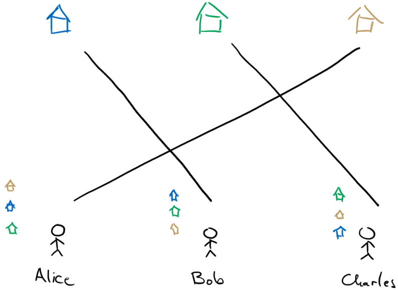
Assignment mechanism

Definition (Assignment mechanism)

Let N be a number of players and indivisible objects (houses). Each player has a list of preferences i.e. strict ordering of houses. An assignment mechanism is a mechanism which assigns (possibly stochastically) each house to a different player.



Probabilistic assignment

$$P = \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \end{pmatrix}$$

For a given profile of preferences $p_{i,j}$ is probability that player i gets house j.

$$\sum_{j=1}^{N} p_{i,j} = 1, \quad \forall i \in \{1, \dots N\},$$

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Mechanism constraints

We want the mechanism to satisfy:

1. Ex-post efficiency,
2. Equal treatment of equals,
3. Strategy proofness.

Ex-post efficiency

No subgroup of players can exchange houses amongst themselves to make everyone happier.
Randomize over efficient assignments.

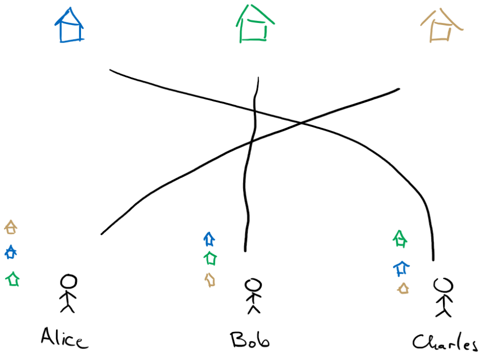

Figure: Example of a non efficient assignment.

Equal treatment of equals

If any two players have the same preferences they have the same chances to receive a given house.

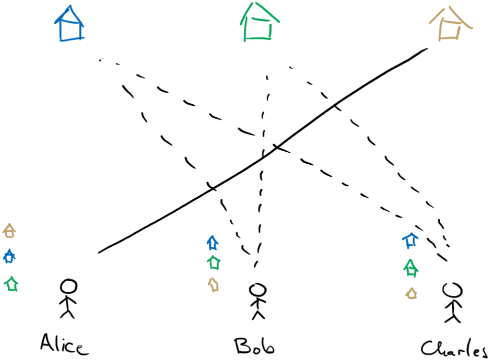

Figure: Example of preferences that cannot be resolved deterministically.

No player may gain by lying about his/her preferences.

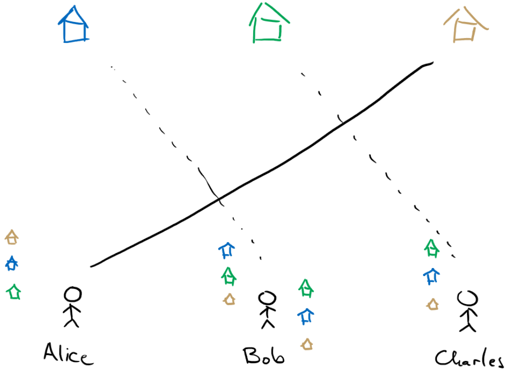

Figure: Example of Bob lying about his preferences.

Strategy proofness

Let K and L be preference profiles that differ only in preferences of agent i. Then

$$\sum_{j=1}^{n} p_{i,j}^{\rho_K}(K) \geq \sum_{j=1}^{n} p_{i,j}^{\rho_K}(L), \ \forall n \in \{1, \dots N\},$$

where p^{ρ_K} are probabilities ordered by preferences of agent i in preference profile K.
This requires that probability distribution obtained by telling the truth stochastically dominates the one obtained from lying.

Random serial dictatorship

Definition (RSD)

RSD is an assignment mechanism with steps:

1. *Choose uniformly randomly one of the* $\frac{1}{N!}$ *possible orders of players.*
2. *First player in the ordering gets the most preferable house.*
3. *For* $1 < r \leq N$ *the* r*-th player gets the most preferable house from the houses left after* $r - 1$ *players get their houses.*

Theorem

RSD satisfies Ex-post efficiency, equal treatments of equals and strategy proofness.

Open question

Conjecture

RSD is the only assignment mechanism for N players satisfying the following:

- *Equal treatment of equals,*
- *Strategy proofness,*
- *Ex-post efficiency.*

The above conjecture was proven to be true for $N < 4$.

Swapped houses

Lemma (entangled elements)

Let K, L be two preference profiles related by swapping $\rho_{i,j}$ and $\rho_{i,j+1}$. Then

$$p_{i,h}(K) = p_{i,h}(L), \qquad \forall h \in \{1, \dots N\} \setminus \{j, j+1\}.$$

For houses $j, j+1$ it holds

$$p_{i,j}^{\rho_K}(K) \geq p_{i,j}^{\rho_K}(L), \;\; p_{i,j+1}^{\rho_K}(K) \leq p_{i,j+1}^{\rho_K}(L).$$

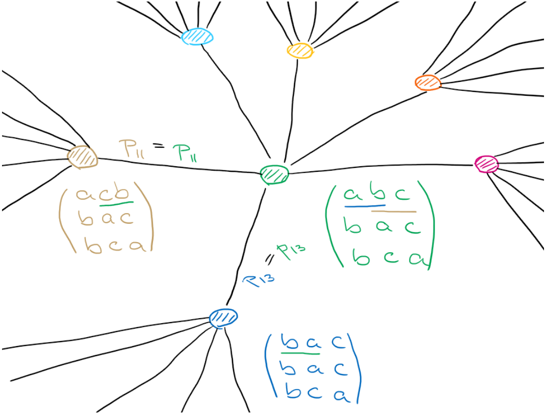

Figure: Part of a graph of neighbouring profiles.

Known (initial) profiles

Lemma (solution for different first preferences)

If all agents have different first preferences, then the solution is uniquely determined.

Lemma ($N-1$ identical preferences)

If all but one agent have the same preferences, then the solution is uniquely determined.

Program

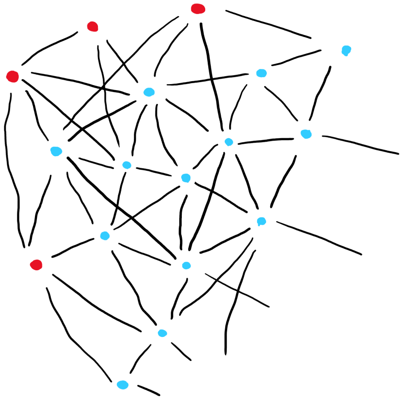

Figure: Illustration of the search algorithm.

Program

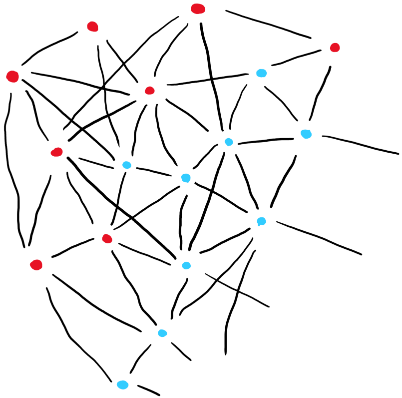

Figure: Illustration of the search algorithm.

Program

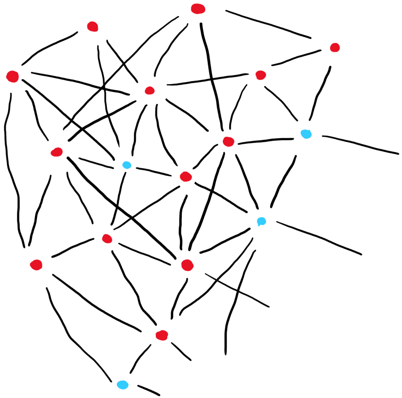

Figure: Illustration of the search algorithm.

Not initial profile

$$Preferences \ldots \begin{pmatrix} a & b & c & d \\ a & b & c & d \\ a & b & d & c \\ a & b & d & c \end{pmatrix}$$

$$Probabilities \ldots \begin{pmatrix} 1/4 & 1/4 & x & 1/2 - x \\ 1/4 & 1/4 & x & 1/2 - x \\ 1/4 & 1/4 & 1/2 - x & x \\ 1/4 & 1/4 & 1/2 - x & x \end{pmatrix}$$

Conclusion

- Program proved the conjecture for $N = 4$ (uniqueness of RSD).
- Uniqueness for some of the profiles for general N.

Future work

- A matrix obtained from SP.
- Rank of the matrix = reformulation of the problem.
- It is needed to use other properties in a clever way to show something about the matrix.

References

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[2] Bogomolnaia, A. and Moulin, H., *A new solution to the random assignment problem*, Journal of Economic Theory 100 (2001) 295-328.

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